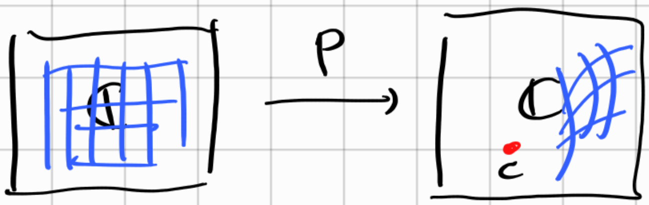
ANALISI MATEMATICA B
LEZIONE 55 – 19.2.2024

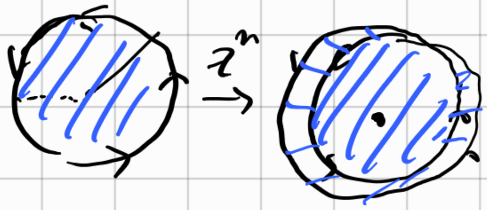
Teorema fondamentale dell'algebra. Sia $P(z) \in \mathbb{C}[z]$ un polinomio a coefficienti complessi $\deg P > 0$. Allora $\exists\, z \in \mathbb{C}$ t.c. $P(z) = 0$.

dim



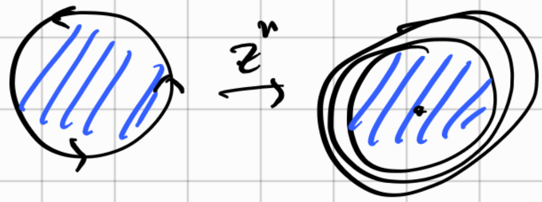
$z \longrightarrow cz + d$

$z \longmapsto z^2$



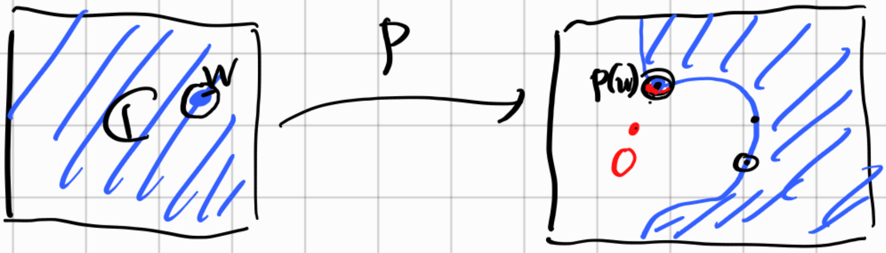
2 strati

$z \longmapsto z^n$



n strati

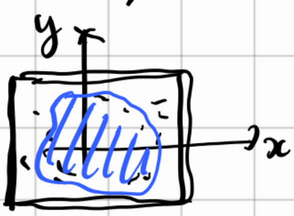
Strategia:



Passo 1 mostrare che $|P| : \mathbb{C} \longrightarrow \mathbb{R}$ ha minimo.

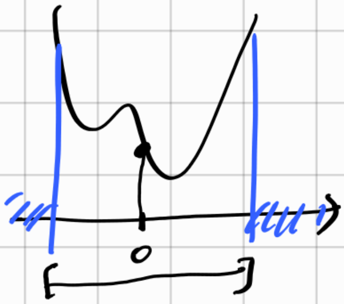
$$z \longmapsto |P(z)|$$

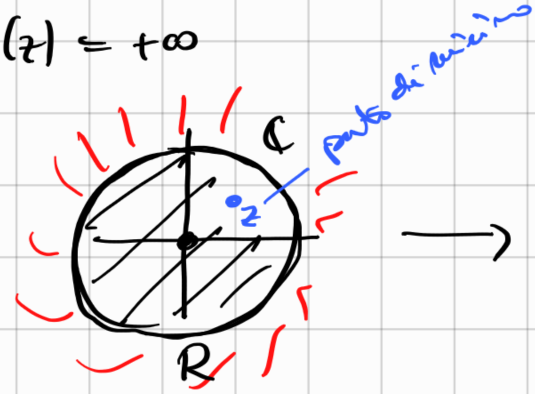
Teorema (Weierstraß) Sia $f : K \to \mathbb{R}$, K chiuso e limitato in $\mathbb{C}$, f continua, allora f ha massimo e minimo.

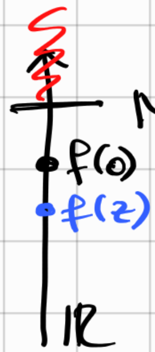


Teorema (Weierstrass, generalizzato) Sia $f:\mathbb{C}\to\mathbb{R}$ continua e coerciva, allora f ha minimo.

↳ $\lim_{z\to\infty} f(z) = +\infty$







Visto che $\deg P > 0$ $\quad \lim_{z\to\infty} P(z) = \infty$

$$P(z) = a_0 + a_1 z + \dots + a_n z^n, \quad n = \deg P.$$

$$= \underbrace{z^n\Big(a_n}_{\downarrow\ \infty} + \underbrace{\cancel{\frac{a_{n-1}}{z}}}_{\to 0} + \dots + \underbrace{\cancel{\frac{a_1}{z^{n-1}}}}_{\to 0} + \underbrace{\cancel{\frac{a_0}{z^n}}}_{\to 0}\Big)$$

$$|P(z)| = \underbrace{|z|^n}_{\downarrow\ +\infty} \cdot \underbrace{|a_n + o(1)|}_{\downarrow\ |a_n|} \qquad \text{per } z\to\infty$$

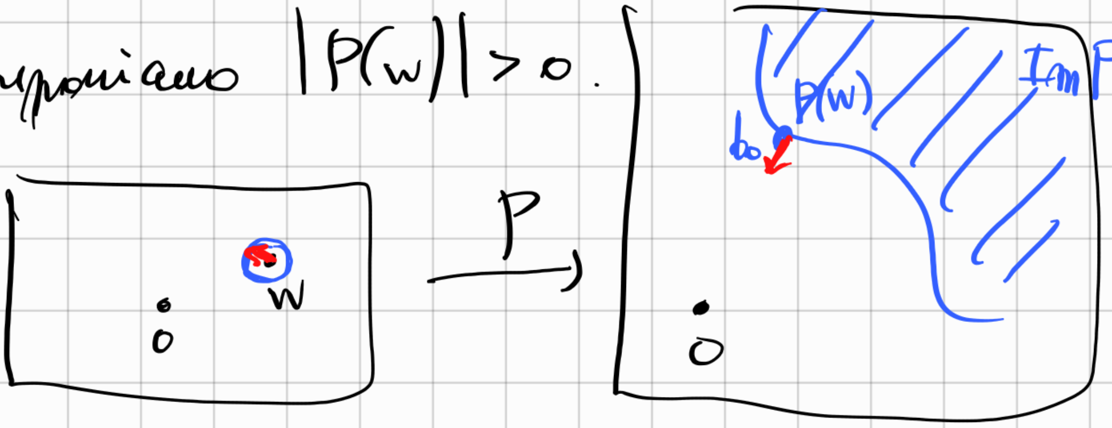
$z \mapsto |P(z)|$ è continua e coerciva

$\Downarrow$

ha minimo.

$\exists\, w\in\mathbb{C}$ t.c. $|P(w)| \le |P(z)| \quad \forall z\in\mathbb{C}$.

Per assurdo supponiamo $|P(w)| > 0$.



$$P(z) = \sum_{k=0}^{n} b_k (z-w)^k \qquad P(w) = b_0$$

Per assurdo $b_0 \neq 0$ (altrimenti $P(w)=0$ e abbiamo finito).

$\deg P = n \Rightarrow b_n \neq 0$.

$$P(z) = b_0 + b_k \underbrace{(z-w)^k} + \sum_{j=k+1}^{n} b_j (z-w)^j$$

k è il minimo indice >0 t.c. $b_k \neq 0$.

$$P(z) = b_0 + \underbrace{b_k \cdot (z-w)^k} \cdot \Big[1 + \underbrace{\sum_{j=k+1}^{n} \frac{b_j}{b_k} (z-w)^{j-k}} \Big]$$

Voglio trovare θ t.c. se $z-w = \rho \cdot e^{i\theta}$

si abbia $|b_k \cdot (z-w)^k| = |b_0| - \rho^k$

$z-w = \rho e^{i\theta} \qquad (z-w)^k = \rho^k \cdot e^{ik\theta}$

$b_k = r_k e^{i\alpha_k} \qquad b_k (z-w)^k = r_k e^{i\alpha_k} \cdot \rho^k e^{ik\theta}$

$b_0 = r_0 e^{i\alpha_0} \qquad = r_k \rho^k e^{i(\alpha_k + k\theta)}$

Scelgo θ in modo che: $\alpha_k + k\theta = \alpha_0 + \pi$.

$$\theta = \frac{\alpha_0 + \pi - \alpha_k}{k}$$

Se $z = w + \rho e^{i\theta}$ $\qquad |b_0 + b_k(z-w)^k| = |b_0| - |b_k| \cdot \rho^k$

$$|P(z)| = \Big| \underbrace{b_0 + b_k(z-w)^k} + \underbrace{\sum_{j=k+1}^{n} b_j \cdot (z-w)^j} \Big|$$

$$\leq |b_0 + b_k(z-w)^k| + \sum_{j=k+1}^{\infty} |b_j| \, |z-w|^j$$

$$= |b_0| - |b_k| \rho^k + \underbrace{\boxed{\sum_{j=k+1}^{\infty} |b_j| \rho^j}}_{= \varepsilon(\rho)}$$

Se scelgo ρ abbastanza piccolo

essendo $\varepsilon(\rho) << \rho^k$ posso trovare ρ

$$\varepsilon(\rho) < |b_k| \rho^k$$

e quindi $\qquad |P(z)| < |b_0| = |P(w)|$

assurdo perché

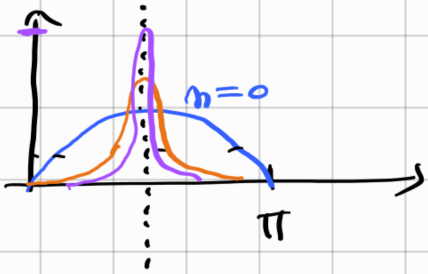
$|P(w)|$ era il minimo

D.

Teorema $\pi \notin \mathbb{Q}$.

dim
$$I_n = \int_0^\pi \overbrace{x^n \cdot (\pi - x)^n \cdot \underbrace{\sin x}}^{f_n(x)} \cdot dx$$

$$f_n(\pi - x) = f_n(x)$$



$$\max f_n = f_n\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^{2n} \to +\infty$$

$$0 < I_n \le \left(\frac{\pi}{2}\right)^{2n} \pi = \frac{\pi^{2n+1}}{2^{2n}}$$

Calcoliamo I_n.

$$I_0 = \int_0^\pi \sin x \, dx = 2$$

per parti

$$I_1 = \int_0^\pi x(\pi - x) \cdot \sin x \, dx = \cancel{\left[x(\pi-x)(-\cos x)\right]_0^\pi} - \int_0^\pi (\pi - 2x) \cdot (-\cos x) dx$$

deriva, integro

$$\left(x(\pi - x) = \pi x - x^2 \right)$$

$$= \pi \int_0^\pi \cancel{\cos x} \, dx - \int_0^\pi 2x \cos x \, dx = - \cancel{\left[2x \sin x\right]_0^\pi} + \int_0^\pi 2 \sin x \, dx$$

deriva, integro

$$= 4.$$

⋮

$$I_n = \int_0^\pi x^n (\pi - x)^n \sin x \, dx = \cancel{\left[\cdots\right]_0^\pi} + \int_0^\pi n x^{n-1} (\pi - x)^{n-1} (\pi - 2x) \cos x \, dx$$

deriva, integro, deriva, integro

$$D\left(x^n \cdot (\pi - x)^n\right) = n x^{n-1} (\pi - x)^n - x^n \cdot n (\pi - x)^{n-1}$$
$$= n x^{n-1} (\pi - x)^{n-1} (\pi - x - x)$$

$$D^2\left(x^n (\pi - x)^n\right) = n(n-1) x^{n-2} (\pi - x)^n - n^2 x^{n-1} (\pi - x)^{n-1}$$
$$- n^2 x^{n-1} (\pi - x)^{n-1} + n(n-1) x^n (\pi - x)^{n-2}$$

$$= x^{n-2}(\pi-x)^{n-2}\cdot\left[n(n-1)\underbrace{(\pi-x)^2}-2n^2\underbrace{x\cdot(\pi-x)}+n(n-1)\underbrace{x^2}\right]=$$

$$I_n = \dots = \cancel{\left[(\dots)\sin x\right]_0^\pi} - \int x^{n-2}(\pi-x)^{n-2}\left[\text{———}\right]\sin x\,dx$$

$$(\pi-x)^2+x^2 = \pi^2-2\pi x+2x^2$$
$$= \pi^2-2x(\pi-x)$$

$$\boxed{\overset{?}{=} A\,x(\pi-x)+B}$$

$$I_n = -\int_0^\pi x^{n-2}(\pi-x)^{n-2}\left\{(n^2-n)\left[\underline{\pi^2}-\underbrace{2x(\pi-x)}\right]-\underbrace{2n^2x(\pi-x)}\right\}\sin x.$$

$$= \underbrace{2n^2 I_{n-1}} - (n^2-n)\underline{\pi^2}\, I_{n-2} + \underbrace{2(n^2-n)\, I_{n-1}}$$

$$I_n = (4n^2-2n)\, I_{n-1} - (n^2-n)\pi^2\, I_{n-2}$$

Supponiamo per assurdo $\pi\in\mathbb{Q}$ $\quad \pi = \frac{p}{q}$ $\quad p\in\mathbb{N}$, $q\in\mathbb{N}$

$a_n = \frac{q^{2n}}{n!}\cdot I_n$ ..... si vede che $\boxed{a_n\in\mathbb{Z}.}$

$$0 < I_n \le \frac{\pi^{2n+1}}{4^n}$$

$$\vdots$$

$$0 < a_n \le \frac{q^{2n}}{n!}\cdot\frac{\pi^{2n+1}}{4^n} \longrightarrow 0$$

assurdo perché $a_n\in\mathbb{Z}$. $\qquad n! \gg C^n$