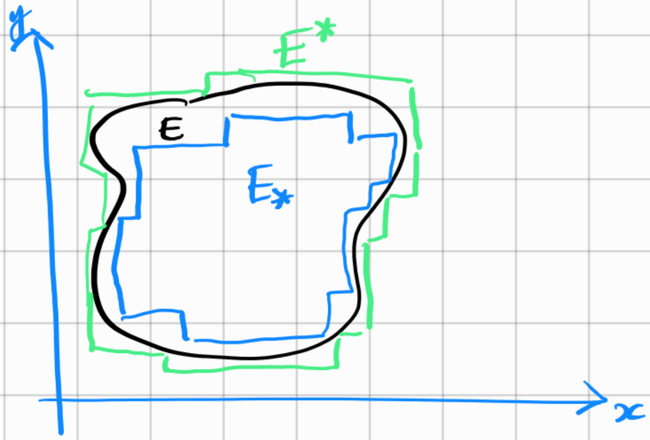
ANALISI MATEMATICA B
LEZIONE 55 - 22.2.2023

MISURA DI PEANO-JORDAN



$E_* \subseteq E \subseteq E^*$

$\underbrace{E_* \subseteq E \subseteq E^*}_{\text{pluri-rettangoli contenuti}}$

$m(E_*) \leq \ ? \ \leq m(E^*)$

$$m^*(E) \doteq \inf\{m(E^*) : E^* \supseteq E,\ E^* \text{ plurirett.}\}$$

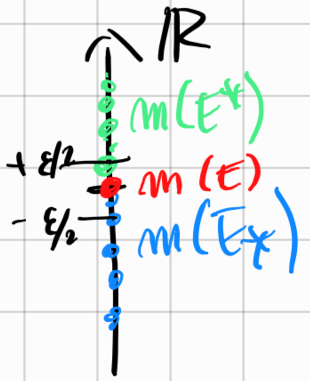
$$m_*(E) \doteq \sup\{m(E_*) : E_* \subseteq E,\ E_* \text{ plurirett.}\}$$

se $m^*(E) = m_*(E)$ diremo che E è Peano-Jordan misurabile e $m(E) = m^*(E) = m_*(E)$.

Teorema Se E, F sono P-J misurabili anche $E \cup F$, $E \cap F$, $E \setminus F$, $F \setminus E$ sono P-J misurabile

$$m(E \cup F) = m(E \setminus F) + m(E \cap F) + m(F \setminus E)$$

$$m(E \cap F) + m(E \cup F) = m(E) + m(F)$$



E è P-J misurabile $\iff \forall \varepsilon > 0\ \exists E^*, E_*$ plurirett. $E_* \subseteq E \subseteq E^*$ tali che $m(E^*) - m(E_*) < \varepsilon$.

idea dim (VERO) $E\setminus F$ è misurabile ?

$E_* \subseteq E \subseteq E^*$
$F_* \subseteq F \subseteq F^*$
$m(E^*) - m(E_*) < \varepsilon$

$m(F^*) - m(F_*) < \varepsilon$

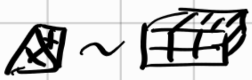
$$E_* \setminus F^* \subseteq E\setminus F \subseteq E^* \setminus F_*$$

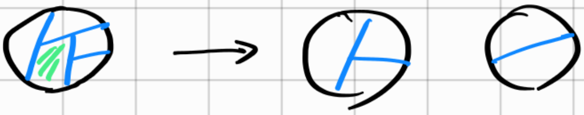
$$(E^* \setminus F_*) \setminus (E_* \setminus F^*) \subseteq (E^* \setminus E_*) \cup (F^* \setminus E_*)$$

$$m\big((E^* \setminus F_*) \setminus (E_* \setminus F^*)\big) \leq m\big((E^* \setminus E_*) \cup (F^* \setminus E_*)\big) \leq 2\varepsilon$$

$\square$

(Paradosso di Dehn. Paradosso di Banach-Tarski)

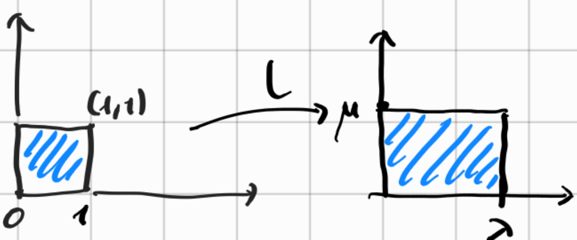




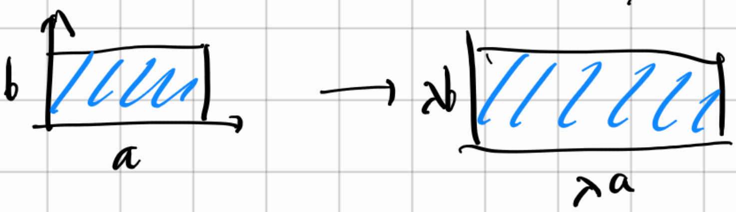
Teorema $L : \mathbb{R}^n \to \mathbb{R}^n$ lineare, $E \subseteq \mathbb{R}^n$ P-J misurabile
anche $L(E)$ è misurabile e:

$$m(L(E)) = |\det L| \cdot m(E).$$

idea dim. Caso 1 $E = [0,1]^n$, $L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.



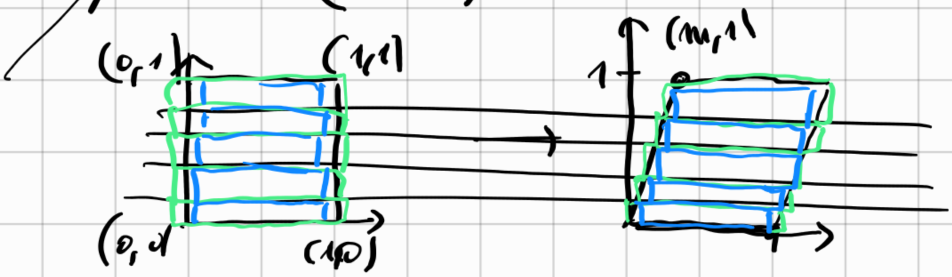
$m(E) = 1$ $\qquad$ $m(\tilde{E}) = \lambda \cdot \mu = |\det L|$



__Oss__

- Basta dimostrare la formula quando E è un rettangolo.
  - Anzi basta dimostrarlo per E un quadrato
  - Anzi basta " " $E = [0,1]^n$.

__Caso 2__ $L = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ $\quad m(L(E)) = 1$.

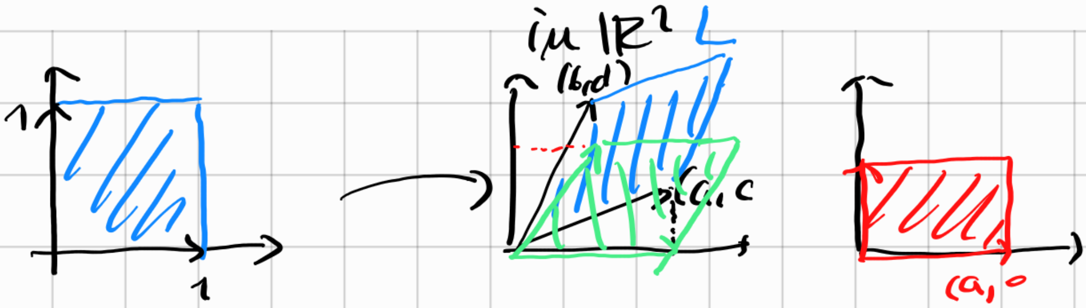


Vale lo stesso $S = \begin{array}{c} \\ i \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & \\ & 1 & m & 0 \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}$, con m in colonna j. $A = \begin{pmatrix} \text{———} \end{pmatrix}$

$$S_n \cdots S_2 S_1 L S_{n+1} S_{n+2} \cdots S = \Delta = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

riduzione completa di Gauss.

$$m(L(E)) = m(\Delta(E)) = |\lambda_1 \cdots \lambda_n|$$
$$= |\det L| \qquad \square$$



$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \tilde{L} = \begin{pmatrix} a & b \\ \underbrace{c + \cancel{ma}} & d + mb \end{pmatrix} \sim \tilde{\tilde{L}} = \begin{pmatrix} a + \cancel{k\cdot 0} & b + \cancel{k(d+mb)} \\ 0 & d+mb \end{pmatrix}$$

$$c + ma = 0 \qquad m = -\frac{c}{a}$$

$$m\left(\tilde{L}\left([0,1]^2\right)\right) = a \cdot (d + mb)$$

$$= a \cdot \left(d - \frac{c}{a} b\right) = ad - bc$$

$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Se L è una isometria

$$|Lv| = |v|$$

L si rappresenta con una matrice ortogonale

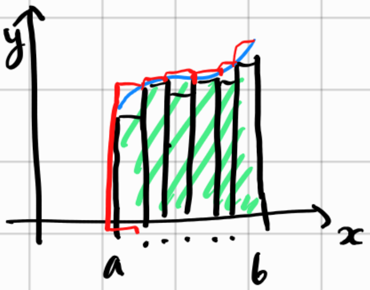
$$L^t \cdot L = Id$$

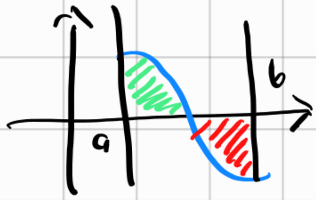
$$|\det L| = 1.$$

$\to$ le isometrie preservano l'area

INTEGRALE di RIEMANN

Idea geometrica $f:[a,b]\to\mathbb{R},\ f\geq 0$

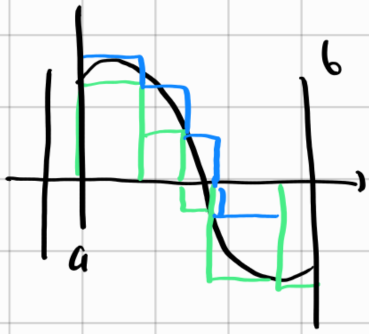




$$\int_a^b f(x)\,dx = m\left(\{(x,y):\ x\in[a,b],\ 0\leq y\leq f(x)\}\right)$$

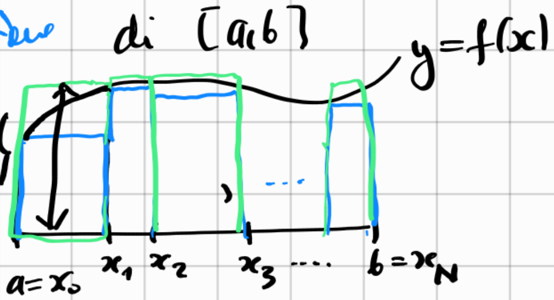
In generale $\int_a^b f(x)\,dx = \quad - \quad$

Def Se $P\subseteq[a,b]$ è un insieme finito con $a\in P$, $b\in P$ diremo che P è una suddivisione di $[a,b]$



$$P=\{x_0, x_1, \dots, x_N\}$$

$$a=x_0<x_1<\dots<x_N=b$$



Data $f:[a,b]\to\mathbb{R}$ LIMITATA, $\forall$ data P definisco:

$$S^*(f,P) = \sum_{k=1}^{N}(x_k-x_{k-1})\cdot\sup f([x_{k-1},x_k])$$

$$S_*(f,P) = \sum_{k=1}^{N}(x_k-x_{k-1})\cdot\inf f([x_{k-1},x_k])$$

$$S^*(f) = \inf\{S^*(f,P):\ P \text{ suddivisione di } [a,b]\}$$

$$S_*(f) = \sup\{S_*(f,P):\ P \text{ suddivisione di } [a,b]\}$$

Se $S^*(f)=S_*(f)$ diremo che f

è Riemann-integrabile su $[a,b]$ e scriveremo:

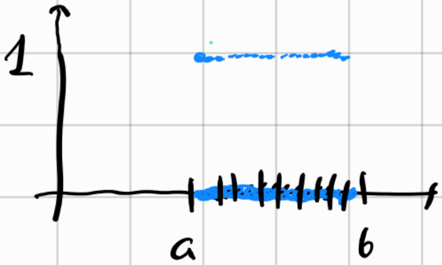
integrale di f: $\int_a^b f = S^*(f) = S_*(f)$
su $[a,b]$

$$\int_a^b f(x)\cdot dx$$

$\leftarrow$ ricordo $\sum f(x)\cdot \Delta x$.

<u>Esempio</u> Esiste f non R-integrabile.

$f:[a,b]\to \mathbb{R}$



$$f(x)=\begin{cases} 1 & \text{se } x\in\mathbb{Q} \\ 0 & \text{se } x\notin\mathbb{Q}\end{cases}$$

Qualunque sia P

$$S^*(f,P)=\sum_{k=1}^{N}(x_k-x_{k-1})\cdot 1 = b-a.$$

$$S_*(f,P)=\sum_{k=1}^{N}(x_k-x_{k-1})\cdot 0 = 0$$