ANALISI MATEMATICA B
LEZIONE 22 – 11.11.2022

LIMITI NOTEVOLI:

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

$$\boxed{\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1}$$

$$\boxed{\lim_{x\to 0}\frac{e^x-1}{x} = 1}$$

ES

$$\boxed{\lim_{n\to+\infty}\left(\frac{n^2+n}{n^2+1}\right)^{n+2}}$$

$$\boxed{\frac{n^2+n}{n^2+1} = \frac{\cancel{n^2}\cdot\left(1+\frac{1}{n}\right)}{\cancel{n^2}\cdot\left(1+\frac{1}{n^2}\right)} = \frac{1+\frac{1}{n}}{1+\frac{1}{n^2}} \xrightarrow[n\to+\infty]{} 1}$$

METODO CHE NON MI PIACE

PERCHE'?

$$\lim_{n\to+\infty}\frac{n^2+n}{n^2+1} = \lim_{n\to+\infty}\frac{n^2\left(1+\cancel{\frac{1}{n}}\right)}{n^2\left(1+\cancel{\frac{1}{n^2}}\right)} = \lim_{n\to+\infty}\frac{n^2}{n^2} = 1$$

$$\lim_{n\to+\infty}\frac{\cancel{\frac{1}{n}}-\frac{1}{n^2}}{\frac{1}{n}} \overset{\text{NO}}{=} \lim_{n\to+\infty}\frac{0-\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to+\infty}\frac{-1}{n} = 0$$

$$\frac{\frac{1}{n}-\frac{1}{n^2}}{\frac{1}{n}} = \frac{n-1}{n} = 1-\frac{1}{n} \longrightarrow 1 \quad \text{per } n\to+\infty$$

$$\lim_{n\to+\infty}\left(\frac{n^2+n}{n^2+1}\right)^{n+2} = ?$$

$$f(x)^{g(x)} = e^{\boxed{g(x)\cdot\ln f(x)}}$$

$f(x)>0$

$$\text{"}\boxed{1^\infty} = e^{\infty\cdot\ln 1}\text{"}$$

$$\text{"}0^{+\infty} = e^{\infty\ln 0^+}\text{"} = 0^+ \qquad \text{"}0^0 = e^{0\ln 0}\text{"}$$

∞^∞ ? $\qquad \infty^1$?

METODO 1

$$\left(\frac{n^2+n}{n^2+1}\right)^{n+2} = \left(\frac{n^2+1+n-1}{n^2+1}\right)^{n+2} = \left(1+\frac{n-1}{n^2+1}\right)^{n+2} =$$

$x =$

$$\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e\right]$$

$$= \left[\underbrace{\left(1+\frac{n-1}{n^2+1}\right)^{\frac{n^2+1}{n-1}}}_{\to e}\right]^{\overbrace{\frac{n-1}{n^2+1}\cdot(n+2)}^{\to 1}}$$

$$\longrightarrow e^{1} = e$$

$$\frac{n-1}{n^2+1}\cdot(n+2) = \frac{(n-1)(n+2)}{n^2+1}$$

$$= \frac{\cancel{n}(1-\cancel{\tfrac{1}{n}}^{\,0})\cancel{n}(1+\cancel{\tfrac{2}{n}}^{\,0})}{\cancel{n^2}(1+\cancel{\tfrac{1}{n^2}}^{\,0})} \to 1$$

METODO 2

$$\left(\frac{n^2+n}{n^2+1}\right)^{n+2} = e^{\overbrace{(n+2)\cdot\ln\frac{n^2+n}{n^2+1}}^{\to 1}} \to e^1 = e.$$

$$\boxed{\frac{\ln(1+x)}{x} \to 1 \quad \text{per } x\to 0}$$

$$(n+2)\ln\frac{n^2+n}{n^2+1} = (n+2)\ln\left(1+\frac{n-1}{n^2+1}\right)$$

se $n\to\infty$, $x\to 0$

$$x = \frac{n-1}{n^2+1} \qquad = \underbrace{\frac{\ln\left(1+\frac{n-1}{n^2+1}\right)}{\frac{n-1}{n^2+1}}}_{\to 1}\cdot\underbrace{\frac{(n-1)(n+2)}{n^2+1}}_{\to 1} \to 1$$

$$\cancel{\left(1+\frac{1}{n}\right)^n} = \cancel{e^{n\ln\left(1+\frac{1}{n}\right)}} = \cancel{e^{\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}}} \to \cancel{e^1}$$

$$\lim_{n\to+\infty}\left(\frac{n^2+n}{n^2+1}\right)^{n\sqrt{n}}$$

$$\lim_{n\to+\infty}\left(\frac{n^2+n}{n^2+1}\right)^{n+2} = \lim_{n\to+\infty}\left(\frac{n^2+4}{n^2}\right)^{n+2}$$

PERCHE'?

$$\lim_{n\to+\infty}\left(\frac{n^2+2}{n^2+1}\right)^{n^2+2} \overset{\text{NO}}{=} \lim_{n\to+\infty}\left(\frac{n^2}{n^2}\right)^{n^2+2} = \lim_{n\to+\infty} 1^{n^2+2} = 1$$

$$\left(\frac{n^2+2}{n^2+1}\right)^{n^2+2} = \left(1+\frac{1}{n^2+1}\right)^{n^2+2} = \underbrace{\left(1+\frac{1}{n^2+1}\right)^{n^2+1}}_{e}{}^{\cdot\frac{n^2+2}{n^2+1}} \to e$$

$\frac{n^2+2}{n^2+1} \to 1$

PROPRIETA' FREQUENTI E DEFINITIVE

Sia $P(n)$ un predicato dipendente da $n \to +\infty$

Def $P(n)$ è vero frequentemente (per $n\to+\infty$) se $\forall N\ \exists n > N : P(n)$ è vero

ES n è pari è frequentemente vero

n è un numero primo è frequentemente

$P(n)$ è vero definitivamente se $\exists N : \forall n > N : P(n)$ è vero.

$n^2 - n > 1000$ è definitivamente vero.

Proprietà

def. $P(n)$ ($n\to+\infty$) $\Rightarrow$ freq. $P(n)$ ($n\to+\infty$)

def. $P(x)$ ($x\to x_0$) se $\exists B\in\mathcal{B}_{x_0} : \forall x\in B\setminus\{x_0\}$ vale $P(x)$.

freq $P(x)$ ($x\to x_0$) se $\forall B\in\mathcal{B}_{x_0}\ \exists x\in B\setminus\{x_0\}$ t.c. $P(x)$.

non def. $P(n) \iff$ freq. non $P(n)$
non freq. $P(n) \iff$ def. non $P(n)$

Esempi
$\lim_{n\to+\infty} a_n = \ell \iff \forall \varepsilon>0$: definitivamente $|a_n-\ell|<\varepsilon$.

non $(a_n \to \ell) \iff \exists \varepsilon>0$: frequentemente $|a_n-\ell| \geq \varepsilon$.

Teoremi di confronto

1. confronto puntuale se per $x \to x_0$ $\left.\begin{array}{l} f(x) \leq g(x) \\ f(x) \to \ell \\ g(x) \to m \end{array}\right\} \Rightarrow \ell \leq m$.

dim $f(x)-g(x) \leq 0$ per la permanenza del segno

$f(x)-g(x) \to \ell - m \leq 0$. $\ell \leq m$.

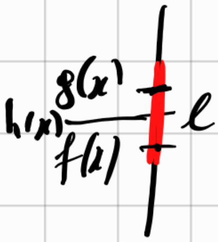
2. confronto all'infinito se $x \to x_0$ $\left.\begin{array}{l} f(x) \leq g(x) \\ \text{e } f(x) \to +\infty \end{array}\right\} \Rightarrow g(x) \to +\infty$

dim non $(g \to +\infty)$: $\exists M$: $g(x) \leq M$ frequentemente
$\Rightarrow \exists M$: $f(x) \leq M$ frequentemente
$\Rightarrow$ non $(f \to +\infty)$.

$\boxed{g \to +\infty \iff \forall M : \operatorname{def}_{x\to x_0} g(x) > M}$

3. (2 carabinieri) $x \to x_0$ $\left\{\begin{array}{l} f(x) \leq h(x) \leq g(x) \\ f(x) \to \ell \\ g(x) \to \ell \end{array}\right. \Rightarrow h(x) \to \ell$

$\forall U$ intorno di ℓ (basilare) $\quad f(x) \in U$ definitivamente
$g(x) \in U$ definitivamente



$$f(x) \le h(x) \le g(x)$$

h è intermedio $\Rightarrow h(x) \in U$.

ORDINI DI INFINITO

def $\quad f(x) << g(x)$ per $x \to x_0$ se $\dfrac{f(x)}{g(x)} \to 0$ per $x \to x_0$

molto più piccolo

ES $\quad x^2 << x$ per $x \to 0$ perché $\dfrac{x^2}{x} = x \to 0$.

ES $\quad x^2 >> x$ per $x \to +\infty$ perché $\dfrac{x}{x^2} = \dfrac{1}{x} \to 0$

$f(x) >> g(x)$ se $g(x) << f(x)$.

molto più grande.

Teo Per $n \to +\infty$ $\quad \forall \alpha > 0, \ \forall a > 1$

$$n^\alpha << a^n << n! << n^n$$

Per $x \to +\infty$

$$\log_a x << x^\alpha << a^x$$

ES $$\lim_{n \to +\infty} \frac{\ln(n^2+2) + \sqrt{n} + 2^{-n}}{\sqrt{n + \ln n}} = 1$$

$$\frac{\ln(n^2+2)+\sqrt{n}+2^{-n}}{\sqrt{n+\ln n}} = \frac{\sqrt{n}\left(\frac{\ln(n^2+2)}{\sqrt{n}}+1+\frac{2^{-n}}{\sqrt{n}}\right)}{\sqrt{n\left(1+\frac{\ln n}{n}\right)}} \longrightarrow 1$$

$\ln(n^2+2) \to +\infty$, $\sqrt{n} \to +\infty$, $2^{-n} \to 0$; $n \to +\infty$, $\ln n \to +\infty$

$$\frac{\ln(n^2+2)}{\sqrt{n}} = \frac{\ln\left(n^2\left(1+\frac{2}{n^2}\right)\right)}{\sqrt{n}} = \frac{2\ln n+\ln(\dots)}{\sqrt{n}} = \frac{2\ln n\left(1+\frac{\ln(\dots)}{2\ln\dots}\right)}{\sqrt{n}} \to 0$$

$\ln n \ll \sqrt{n}$

NO

$$\ln(2^n+n^2) \ll \sqrt{n}$$

$$\ln\left(2^n\left(1+\frac{n^2}{2^n}\right)\right) = n\ln 2+\ln\left(1+\frac{n^2}{2^n}\right)$$

$n\ln 2 \gg \sqrt{n}$

$n \gg \sqrt{n}$ $\qquad \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \to 0$