ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 12 – 6.4.2023

Teorema Se $\Gamma \subseteq L^1([a,b])$ sono equivalenti:

(i) Γ è equi-integrabile

(ii) $$\lim_{M\to+\infty} \sup_{u\in\Gamma} \int_{\{|u|\geq M\}} |u| = 0$$

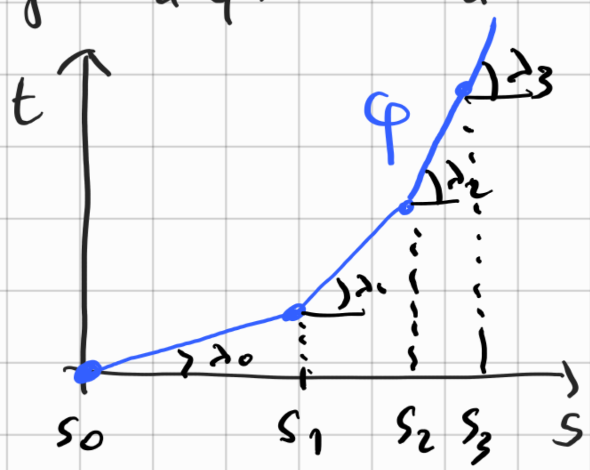
(iii) $\exists\ \varphi : [0,+\infty) \to [0,+\infty)$ convessa, super-lineare: $\displaystyle\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty$

tale $$\int_a^b \varphi(|u|) \leq 1 .$$

dim già visto: $(i) \Rightarrow (ii)$ e $(iii) \Rightarrow (i)$

Vediamo $(ii) \Rightarrow (iii)$.

Dobbiamo definire φ. La costruiamo lineare a tratti:



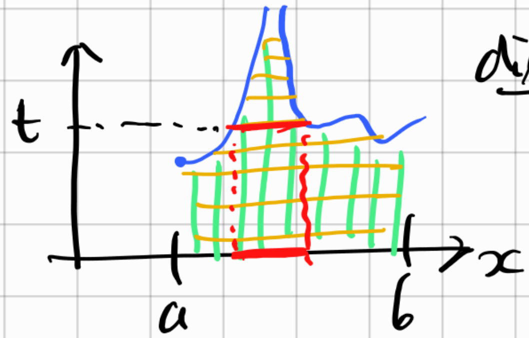
$\varphi'(s) = \lambda_k$ se $s \in [s_k, s_{k+1}]$

$0 = s_0 < s_1 < \dots < s_k < \dots$ $s_k \to +\infty$

$\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_k \leq \dots$ $\lambda_k \to +\infty$

formula di coarea: se $v \geq 0$ $\quad v \in L^1$

$$\int_a^b v(x)\,dx = \int_0^{+\infty} \left|\{x\in[a,b] : v(x)\geq t\}\right| dt$$



dim
$$\int_a^b v(x)\,dx = \int_a^b \int_0^{v(x)} 1\,dt\,dx =$$

$$= \int_0^{+\infty} \int_{\{x : v(x)\geq t\}} 1\,dx\,dt$$

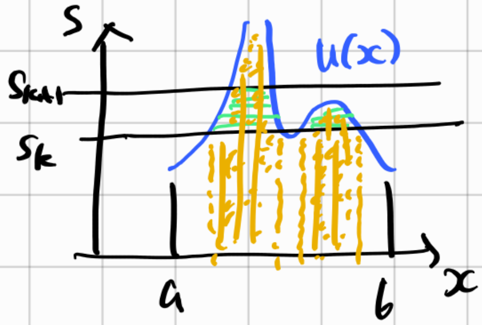
$$= \int_0^{+\infty} \left|\{x : v(x)\geq t\}\right| dt.$$

$$\int_a^b \varphi(|u(x)|)\,dx = \int_0^{+\infty} \left|\{\varphi(|u(x)|)\geq t\}\right| dt =$$

$$\begin{cases} t = \varphi(s) \\ dt = \varphi'(s)\,ds \end{cases}$$

$$= \int_0^{+\infty} \left|\{\varphi(|u(x)|)\geq \varphi(s)\}\right| \varphi'(s)\,ds$$

$$= \int_0^{+\infty} \left|\{|u(x)|\geq s\}\right| \varphi'(s)\,ds = \sum_{k=0}^{+\infty} \lambda_k \boxed{\int_{s_k}^{s_{k+1}} \left|\{|u|\geq s\}\right| ds}$$



$$\leq \sum_{k=0}^{+\infty} \lambda_k \boxed{\int_{\{|u|\geq s_k\}} |u(x)|\,dx} \leq (*)$$

Per ipotesi (ii) per $k>0$ dati λ_k posso scegliere s_k

in modo che $\displaystyle\int_{\{|u| \geq s_k\}} |u(x)|\, dx \leq \frac{1}{\lambda_k^2}$

$$\circledast \leq \lambda_0 \int_a^b |u(x)|\, dx + \sum_{k=1}^{+\infty} \cancel{\lambda_k} \cdot \frac{1}{\lambda_k^{\cancel{2}}} \leq \#$$

[Verifichiamo che $\int_a^b |u|$ è equilimitato.

$$\int_a^b |u| = \underbrace{\int_{\{|u|>M\}} |u|} + \int_{\{|u| \leq M\}} |u| \leq 1 + M \cdot (b-a)$$

[scelgo M in modo che $\displaystyle\int_{\{|u|>M\}} |u| \leq 1$.]]

$$\# \leq \lambda_0 \big(1 + M(b-a)\big) + \sum_{k=1}^{+\infty} \frac{1}{\lambda_k}$$

(scegliamo λ_0 in modo che $\lambda_0 (1+M(b-a)) \leq \frac{1}{2}$

e λ_k in modo che $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} = \frac{1}{2}$)

(es: $\lambda_k = 3^k$)

$$\leq \frac{1}{2} + \frac{1}{2} = 1$$

$\square$