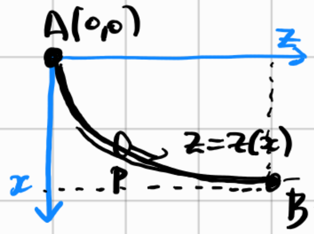
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 1

LUN 14-16 | aula 01 | • ALDO PRATELLI | ricevimento
GIO 11-13 | | • emanuele.paolini@unipi.it | MAR 15

PROBLEMA della BRACHISTOCRONA



Percorso "più veloce" per andare da A a B seguendo la gravità.

$$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gx$$

$$v = \sqrt{2gx}$$

$x = x(t) \qquad z = z(t) \qquad z = z(x) \qquad z'(x) = \frac{dz}{dx}$

$$\sqrt{2gx} = v = \sqrt{\dot{x}^2+\dot{z}^2} = |\dot{x}|\cdot\sqrt{1+\left|\frac{\dot{z}}{\dot{x}}\right|^2} = \frac{dx}{dt}\sqrt{1+(z'(x))^2}$$

$$T = \int_0^T dt \underset{z=z(x)}{=} \int_0^{x_B} \frac{1}{\frac{dx}{dt}}\,dx = \int_0^{x_B}\frac{\sqrt{1+(z'(x))^2}}{\sqrt{2gx}}\,dx$$

Formulazione analitica: trovare la funzione $u=u(x)$ $(z=u(x))$ tale che 1) $u(0)=0$, $u(x_B)=z_B$

2) è minimo: $\mathcal{F}(u) = \int_0^{x_B}\frac{\sqrt{1+(u'(x))^2}}{\sqrt{2gx}}\,dx$.

Problema classico del calcolo delle variazioni

Lagrangiana

minimizzare $\boxed{\mathcal{F}(u) = \int_a^b F(x,u(x),u'(x))\,dx}$

l'incognita

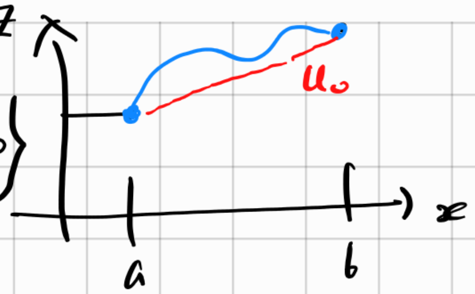
dove $u\in X$ insieme di funzioni.

Es: $X = \{u\in C^1([a,b]) : u(a)=z_A,\ u(b)=z_B\} \subseteq C^1([a,b])$

Oss X è un sottospazio affine di $C^1([a,b])$

$$X = u_0 + C_0^1([a,b])$$

$$L = \{u \in C^1 : u(a) = u(b) = 0\}$$



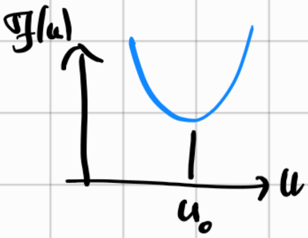
$$F = F(x, z, p) \qquad z = u(x), \quad p = u'(x)$$

ad es. $F \in C^2$

Brachistocrona: $F(x,z,p) = \dfrac{\sqrt{1+p^2}}{\sqrt{2gx}}$, $a=0$, $b = x_B$, $z_a = 0$, $z_b = z_B$.

Come trovo il minimo?

$u_0 \in X \subseteq C^1$, $\varphi \in C_0^1 \left[= T_{u_0}(X)\right]$



$$t \mapsto \mathcal{J}(\underbrace{\boxed{u_0 + t\cdot\varphi}}_{v \in X}) = \int_a^b F(x, \underbrace{u_0(x) + t\varphi(x)}_{z}, \underbrace{u_0'(x) + t\varphi'(x)}_{p})\, dx$$

Se u_0 è minimo per $\mathcal{J}$ allora $\left[\dfrac{d}{dt}\mathcal{J}(u_0 + t\varphi)\right]_{t=0} = 0 \quad \forall \varphi$

$F \in C^1$

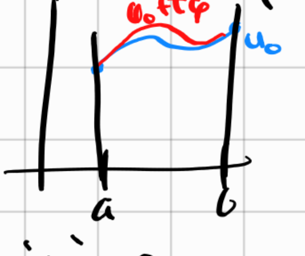
$$\left[\frac{d}{dt}\int_a^b (\dots)\, dx\right]_{t=0} = \left[\int_a^b \frac{d}{dt}(\dots)\, dx\right]_{t=0} = \left[\int_a^b \left[F_z(x, u_0(x) + t\varphi(x), \dots)\varphi(x) + \dots\right] dx\right]_{t=0}$$

$$= \int_a^b \left[F_z(x, u_0(x), u_0'(x))\cdot\varphi(x) + F_p(x, u_0(x), u_0'(x))\cdot\varphi'(x)\right] dx$$

$u_0 \in C^2$, $F \in C^2$

$$= \int_a^b \left[F_z(x, u_0, u_0') - \frac{d}{dx}\left(F_p(x, u_0(x), u_0'(x))\right)\right]\varphi(x)\, dx + \cancel{\left[F_p \cdot \varphi\right]_a^b}$$

$\varphi(a) = \varphi(b) = 0$



$$= \int_a^b \left[F_z - \frac{d}{dx}(F_p)\right]\cdot\varphi\, dx = 0$$

Se u_0 è minimo

Teo Sia $u_0 \in C^1([a,b])$, sia $F \in C^1$, F definita in un intorno della curva $(x, u_0(x), u_0'(x))$, Se u_0 è minimo (locale) di $\mathcal{F}$.

Allora
$$\delta \mathcal{F}(u_0, \varphi) \doteq \frac{\partial \mathcal{F}}{\partial \varphi}(u_0) = \frac{d}{dt}\left[\mathcal{F}(u_0 + t\varphi)\right]_{t=0} = 0$$

$\forall \varphi \in C^1([a,b])$.

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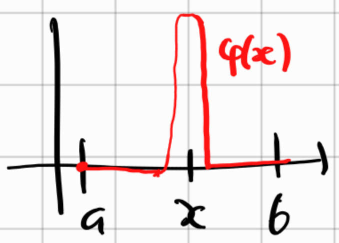
dim Se $\mathcal{F}$ ha minimo in u_0 allora $\mathcal{F}(u_0 + t\varphi)$ ha minimo per $t=0$ $\forall \varphi$
$\Rightarrow$ $\delta \mathcal{F}(u_0, \varphi) = 0$ $\forall \varphi$.

Lemma (fondamentale del CdV)
Se $g \in C^0([a,b])$
$$\int_a^b g(x) \cdot \varphi(x)\, dx = 0 \qquad \forall \varphi \in C_c^\infty((a,b))$$

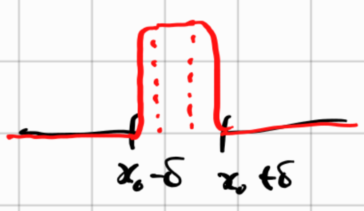
Allora $g \equiv 0$

dim

Supponiamo $\exists\, x_0$ t.c. $g(x_0) \neq 0$ $\quad x_0 \in (a,b)$
$g(x_0) > 0$



Se $g(x) > 0$ quando $\varphi(x) \neq 0$
$\int g(x)\, \varphi(x)\, dx > 0$
$(x_0 - \delta, x_0 + \delta)$
$g(x) > 0$ in un intorno di x_0
(per la permanenza del segno)



$$\varphi(x) = \begin{cases} 0 & \text{se } x \notin (x_0 - \delta, x_0 + \delta) \\ 1 & \text{se } x \in (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}) \\ C^\infty & \text{altrove.} \end{cases}$$

$\square$

Teorema (Eulero-Lagrange) Se $u \in C^2$, $F \in C^2$, F definito in un intorno della curva $(x, u(x), u'(x))$

se $\underbrace{\delta \mathcal{F}(u, \varphi) = 0 \quad \forall \varphi \in C^1_0}$

def u è estremale debole

Allora vale l'Equazione di Eulero-Lagrange:

(EL) $$\boxed{F_z(x, u(x), u'(x)) = \frac{d}{dx} F_p(x, u(x), u'(x))} \quad \forall x \in [a, b].$$

Torniamo alla Brachistocrona: $F(x, z, p) = \dfrac{\sqrt{1+p^2}}{\sqrt{2gx}}$

$$F_z = 0 \underset{EL}{=} \frac{d}{dx} \frac{u'(x)}{\sqrt{2gx}\sqrt{1+(u'(x))^2}} \qquad F_p = \frac{p}{\sqrt{2gx}\sqrt{1+p^2}}$$

$$u' = C \cdot \sqrt{2gx}\sqrt{1+u'(x)^2}$$

$$(u')^2 = C^2 \cdot 2g\, x\,(1+u'^2)$$

$$(1 - 2C^2 g x) u'^2 = 2C^2 g\, x$$

$$u' = \sqrt{\frac{2C^2 g\, x}{1 - 2C^2 g\, x}}$$

$$u = \int \frac{\sqrt{2C^2 g}\sqrt{x}}{\sqrt{1 - 2C^2 g\, x}}\, dx = \int \frac{\sqrt{2C^2 g} \cdot x}{\sqrt{x - 2C^2 g\, x^2}}\, dx$$

$$= \int \frac{\sqrt{2C^2 g}\; x}{\sqrt{\frac{1}{8C^2 g} - \left(\sqrt{2C^2 g}\, x - \frac{1}{2\sqrt{2C^2 g}}\right)^2}}\, dx = 4C^2 g \int \frac{x}{\sqrt{1 - (4C^2 g x - 1)^2}}\, dx$$

$$= \int \frac{4c^2 g x - 1 + 1}{\sqrt{1-(4c^2 g x - 1)^2}} dx$$

$$\boxed{R = \frac{1}{4c^2 g}}$$

$u(0) = 0$

$$= \int \frac{\left(\frac{x}{R} - 1\right) + 1}{\sqrt{1 - \left(1 - \frac{x}{R}\right)^2}} dx = -R \cdot \sqrt{1 - \left(\frac{x}{R} - 1\right)^2} + R \arccos\left(1 - \frac{x}{R}\right) + 0$$

$$= R \arccos\left(\frac{R-x}{R}\right) - \sqrt{R^2 - (x-R)^2}$$