ANALISI MATEMATICA B
LEZIONE 29 — 26.11.2021

ES 19.12.2019 (I compitino)

$$\sum_k \frac{1-x^k \cdot \ln k}{k \cdot \ln(x^2+k)}$$ per quali $x \in \mathbb{R}$ converge?

Condizione necessaria: $a_k = \dfrac{1-x^k \cdot \ln k}{k \cdot \ln(x^2+k)}$

$\lim a_k = ?$

Se $|x|>1$ $\quad \lim_{k\to+\infty} |a_k| \overset{?}{=} +\infty$

$|x_k| \to +\infty \Downarrow \frac{1}{x_k} \to 0$

$$a_k = \frac{x^k \ln k \cdot \left(\frac{1}{x^k \ln k} - 1\right)}{k \ln(x^2+k)}$$

$\to 1$

$$\ln(x^2+k) \overset{?}{\sim} \ln k$$

$$\frac{\ln(x^2+k)}{\ln k} = \frac{\ln\left[k\cdot\left(\frac{x^2}{k}+1\right)\right]}{\ln k} = \frac{\ln k + \ln\left(1+\frac{x^2}{k}\right)}{\ln k}$$

$$= 1 + \frac{\ln\left(1+\frac{x^2}{k}\right)}{\ln k} \longrightarrow 1$$

$$\left|\frac{x^k}{k}\right| \to +\infty \qquad |a_k| \to +\infty$$

la serie non converge.

Se $|x| < 1$

$$a_k = \frac{1 - x^k \cdot \ln k}{k \cdot \ln(x^2 + k)}$$

$$a_k \sim \frac{1}{k \ln k}$$ (quindi $a_k > 0$ definitivamente)

$x^k \ln k \to 0$ se $|x| < 1$

$\left[\ln k \ll \left(\frac{1}{|x|}\right)^k \right]$

$\sum a_k$ ha lo stesso carattere di $\sum \frac{1}{k \cdot \ln k} = +\infty$

la serie diverge.

Se $x = 1$

$$\sum \frac{1 - \ln k}{k \ln(1+k)} = -\sum \frac{(\ln k) - 1}{k \cdot \ln(k+1)}$$

$(\ln k) - 1 \sim \ln k \qquad k \ln(k+1) \sim k \ln k$

$$\frac{(\ln k) - 1}{k \cdot \ln(k+1)} \sim \frac{\ln k}{k \ln k} = \frac{1}{k}$$

Ma $\sum \frac{1}{k} = +\infty$ la nostra serie diverge a $-\infty$.

Se $x = -1$

$$\sum \frac{1 - (-1)^k \cdot \ln k}{k \cdot \ln(k+1)}$$

$$= -\sum \frac{(-1)^k \left[\ln k - (-1)^k \right]}{k \cdot \ln(k+1)}$$

$$= -\sum (-1)^k \underbrace{\frac{\ln k}{k \ln(k+1)}}_{\sim \frac{1}{k}} \left[1 - \frac{(-1)^k}{\ln k} \right]$$

$\frac{(-1)^k}{\ln k} \to 0$

Criterio di Leibniz $b_n \to 0$, b_n decrescente

allora $\sum (-1)^n b_n$ è convergente.

$$b_k = \frac{\ln k}{k \ln(k+1)} \left[1 - \cancel{\frac{(-1)^k}{\ln k}}\right] \to 0$$

$\frac{\ln k}{\ln(k+1)} \to 1$, $k \to +\infty$, $\frac{(-1)^k}{\ln k} \to 0$

$b_k \sim \frac{1}{k}$ $\overset{NO}{\Longrightarrow}$ $\sum (-1)^k b_k$ ha lo stesso carattere di $\sum \frac{(-1)^k}{k}$

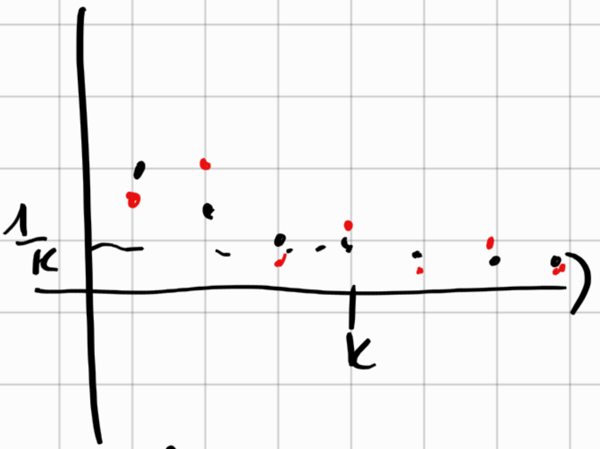
applico Leibniz..

$\overset{NO}{\Downarrow}$

$\frac{1}{k}$ decrescente

quindi b_k decrescente

Leibniz ...



$$\frac{\ln k}{k \ln(k+1)} \left[1 - \frac{(-1)^k}{\ln k}\right]$$

$$= \underbrace{\frac{\ln k}{k \ln(k+1)}}_{①} - \underbrace{\frac{(-1)^k}{k \ln(k+1)}}_{②}$$

① $\sim \frac{1}{k}$ $\qquad \sum \frac{1}{k} = +\infty$ $\qquad \sum ①$ diverge

$$\sum (2) = \sum \frac{(-1)^k}{k\ln(k+1)}$$

converge per leibniz

$$b_k = \frac{1}{k \ln(k+1)} \to 0$$

$$b_{k+1} = \frac{1}{(k+1)\ln(k+2)} < \frac{1}{k\ln(k+2)}$$

$$< \frac{1}{k\ln(k+1)} = b_k$$

b_k decrescente

$$\underbrace{\sum (c_k - d_k)} = +\infty \Leftarrow \begin{cases} \text{se } \sum c_k = +\infty \\ \text{se } \sum d_k \text{ converge} \end{cases}$$

La nostra serie diverge anche per $x = -1$

Esempio $\sum \frac{(-1)^k}{k\ln(k+1)} <$

$$\sum_{k \text{ pari}} \frac{(-1)^k}{k\ln(k+1)} = \sum_k \frac{1}{2k\ln(2k+2)} = +\infty$$

$$\sum_{k \text{ dispari}} \frac{(-1)^k}{k} = \sum_k \frac{-1}{(2k+1)\ln(2k+3)} = -\infty$$

??

I compitino 18/12/2020

Per quali $x \in \mathbb{R}$ la serie $\sum x^n \left(\frac{2n}{2n+1}\right)^{n^2}$ converge?

SUCCESSIONI RICORSIVE (definite per induzione)

(o SISTEMI DINAMICI DISCRETI)

$$\underset{t=0}{\boxed{\text{stato}}} \xrightarrow{f} \underset{t=1}{\boxed{}} \xrightarrow{f} \underset{t=2}{\boxed{}} \longrightarrow \dots$$

Es lo stato è dato da una sola variabile

a_k

$$\begin{cases} a_0 = \alpha \\ a_{n+1} = f(a_n) \end{cases}$$

$\Downarrow \quad \alpha$

$$a_{n+1} = \lambda \cdot a_n$$

$$f(x) = \lambda \cdot x$$

$$a_0 = \alpha$$
$$a_1 = \lambda \cdot a_0 = \lambda \cdot \alpha$$
$$a_2 = \lambda \cdot a_1 = \lambda^2 \cdot \alpha$$
$$\vdots$$
$$a_n = \lambda^n \cdot \alpha$$

$\lambda > 1 \quad a_n \to +\infty$

$\lambda < 1 \quad a_n \to 0$

$\lambda = 1 \quad a_n = \alpha$

Esempio   Algoritmo di Erone per calcolare $\sqrt{p}$

$x = \sqrt{p}$ $x^2 = p$ $x \cdot x = p$ $p > 1$

se $x > \sqrt{p} \iff \frac{p}{x} < \sqrt{p}$

$$x \cdot \frac{p}{x} = p$$

Idea: faccio la media: $\boxed{f(x) = \frac{x + \frac{p}{x}}{2}}$

$$\begin{cases} a_0 = p \\ a_{n+1} = \frac{a_n + \frac{p}{a_n}}{2} \end{cases}$$

definisce a_n

$$\boxed{\lim_{n \to +\infty} a_n \overset{?}{=} \sqrt{p}}$$

Osservazione   se $a_n \to \ell \Rightarrow a_{n+1} \to \ell$

$$a_{n+1} = \frac{a_n + \frac{p}{a_n}}{2}$$

$a_n \to \ell$ pto fisso

$$\Rightarrow \boxed{\ell = \frac{\ell + \frac{p}{\ell}}{2}}$$

$f(a_n) \to f(\ell)$ $(\ell = f(\ell))$

$$2\ell = \ell + \frac{p}{\ell}$$

$$2\ell^2 = \ell^2 + p$$

$$\ell^2 = p \qquad \ell = \pm\sqrt{p}.$$

Si può escludere $\ell = -\sqrt{p}$?

Sì perché:

$a_0 = p > 0$ $\quad a_n > 0 \Rightarrow \frac{p}{a_n} > 0$

$\Downarrow$

$a_n + \frac{p}{a_n} > 0$

[l'intervallo $(0, +\infty)$ è invariante]

$a_{n+1} = \dfrac{a_n + \frac{p}{a_n}}{2} > 0$

per induzione $a_n > 0 \ \forall n$.

$\lim a_n \geq 0$ (se il limite esiste).

Se a_n converge allora $a_n \to \sqrt{p}$.

Può essere $a_n \to +\infty$? a priori non lo escludo.

Come faccio a dimostrare che a_n converge?
Basta dimostrare che a_n decrescente.

① $a_{n+1} \overset{?}{\leq} a_n$

$$\frac{a_n + \frac{p}{a_n}}{2} \overset{?}{\leq} a_n \qquad a_n + \frac{p}{a_n} \overset{?}{\leq} 2a_n$$

$$a_n^2 + p \overset{?}{\leq} 2a_n^2$$

$$a_n^2 \overset{?}{\geq} p \qquad (a_n \overset{?}{\geq} \sqrt{p})$$

② $\underbrace{a_n \overset{?}{\geq} \sqrt{p}}$

$a_0 \geq \sqrt{p}$ vero $p \geq \sqrt{p}$ $(p>1)$

$\underbrace{a_n \geq \sqrt{p}} \overset{?}{\Rightarrow} \underbrace{a_{n+1} \geq \sqrt{p}}$

$$a_{n+1} = \frac{a_n + \frac{p}{a_n}}{2} \overset{?}{\geq} \sqrt{p}$$

$$a_n + \frac{p}{a_n} \geq 2\sqrt{p}$$

$$a_n^2 - 2\sqrt{p}\, a_n + p \geq 0$$

$$(a_n - \sqrt{p})^2 \geq 0$$ vero qualunque sia a_n.

a_n converge $\Rightarrow$ a_n converge a $\sqrt{p}$.