ANALISI MATEMATICA B
LEZIONE 18 - 29.10.2021

Limite

$$f: A \subseteq \mathbb{R} \to \mathbb{R}$$

$f(x) \to \ell$ per $x \to x_0$ $\qquad \ell \in \mathbb{R},\ x_0 \in \mathbb{R}$

se $\tilde{f}(x) = \begin{cases} f(x) & \text{se } x \neq x_0 \\ \ell & \text{se } x = x_0 \end{cases}$

è continua in x_0:

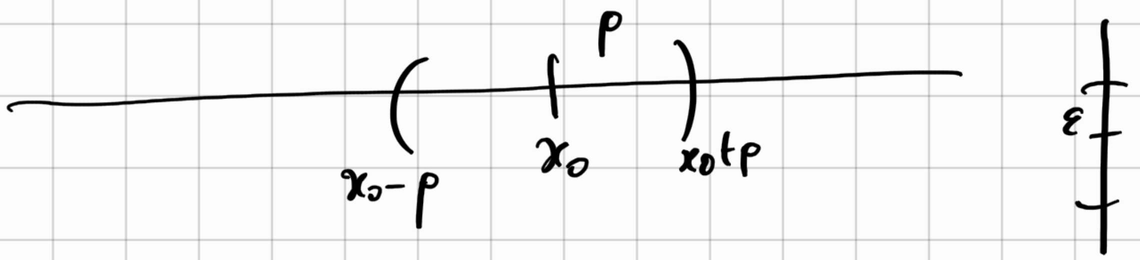
$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A \setminus \{x_0\} : |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

ovvero

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in A \setminus \{x_0\} : x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(\ell)$$

[$B_\rho(x_0) = \{x \in \mathbb{R} : |x - x_0| < \rho\} = \underline{(x_0 - \rho, x_0 + \rho)}$

(palla aperta centrata in x_0, di raggio ρ)



]

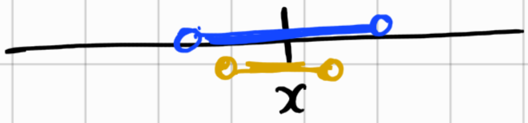
$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x : x \in A \cap B_\delta(x_0) \setminus \{x_0\} \Rightarrow f(x) \in B_\varepsilon(\ell)$$

$$\forall \varepsilon > 0 : \exists \delta > 0 : f(A \cap B_\delta(x_0) \setminus \{x_0\}) \subseteq B_\varepsilon(\ell). \qquad [(\ell - \varepsilon, \ell + \varepsilon)]$$

Definiamo per $x \in \mathbb{R}$

$$\mathcal{B}_x = \{ B_\rho(x) : \rho > 0 \}$$

intorni di x.



basilari

Definizione topologica di limite:

$$\forall U \in \mathcal{U}_\ell : \exists V \in \mathcal{U}_{x_0} : \quad f(A \cap V \setminus \{x_0\}) \subseteq U$$

significa $\quad f(x) \to \ell \quad$ per $x \to x_0$

Definizione topologica di continuità:

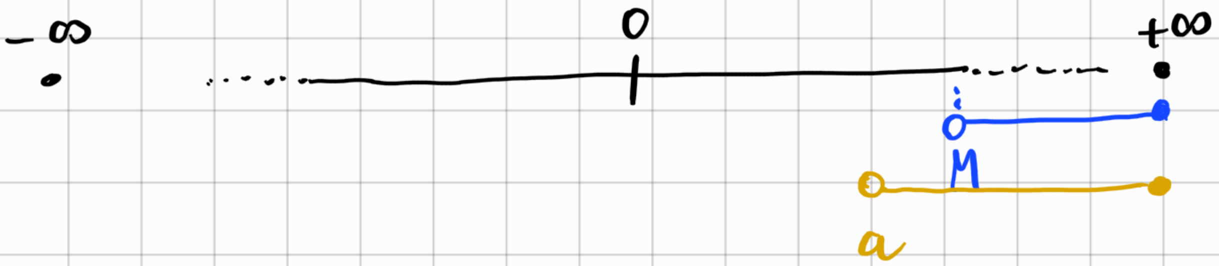
$$\forall U \in \mathcal{U}_{f(x_0)} \ \exists V \in \mathcal{U}_{x_0} : \quad f(A \cap V) \subseteq U.$$

f continua in x_0.

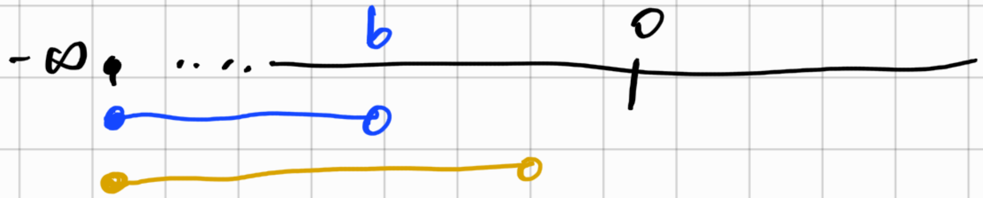
RETTA REALE ESTESA $\overline{\mathbb{R}}$

$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

$$\mathcal{B}_{+\infty} = \{ (a, +\infty] : \quad a \in \mathbb{R} \}$$



$$\mathcal{B}_{-\infty} := \{ [-\infty, b) : b \in \mathbb{R} \}$$



In generale in $\overline{\mathbb{R}}$: $\ell \in \overline{\mathbb{R}}$, $x_0 \in \overline{\mathbb{R}}$

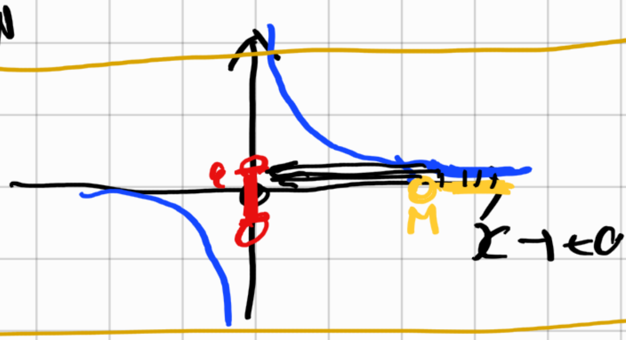
$$f(x) \to \ell \quad \text{per} \quad x \to x_0$$

se $\boxed{\forall U \in \mathcal{U}_\ell : \exists V \in \mathcal{U}_{x_0} : f(V \setminus \{x_0\}) \subseteq U.}$

<u>Esempio</u>

$$\boxed{\frac{1}{x} \to 0 \quad \text{per} \quad x \to +\infty}$$

$A = \mathbb{R} \setminus \{0\}$



$$\boxed{\forall \varepsilon > 0 : \exists M : \forall x \in A : x > M \Rightarrow |f(x) - \cancel{0}| < \varepsilon}$$

arbitrariamente piccolo

sufficientemente grande.

questo è vero, infatti:

Fissato $\varepsilon > 0$ devo trovare M : $\forall\, x > M$

$$\left|\frac{1}{x}\right| \overset{?}{<} \varepsilon$$

① imponiamo $M > 0$: $x > M > 0$

$$\left|\frac{1}{x}\right| = \frac{1}{x} \overset{?}{<} \varepsilon \iff x > \frac{1}{\varepsilon}$$

② Prendiamo $M \geq \frac{1}{\varepsilon}$.

Basta prendere $M = \frac{1}{\varepsilon}$:

$$\forall \varepsilon > 0 \ \exists M = \frac{1}{\varepsilon} : x > M \Rightarrow 0 < \frac{1}{x} < \varepsilon . \ \square$$

Esempio 2

$$\frac{1}{x^2} \to \underset{\ell}{\overset{}{\underset{\shortparallel}{+\infty}}} \quad \text{per} \quad x \to \underset{x_0}{\underset{\shortparallel}{0}}$$

$$\forall M \in \mathbb{R} : \exists \delta > 0 : 0 < |x - x_0| < \delta \Rightarrow f(x) > M$$

$\underbrace{\frac{1}{x^2} \overset{?}{>} M}$ se $M \leq 0$ non c'è problema.

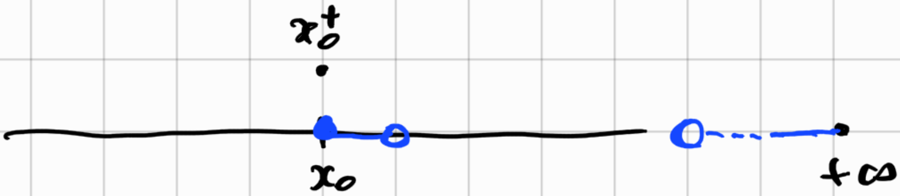
se $M > 0$ $\quad x^2 < \frac{1}{M} \iff |x| < \sqrt{\frac{1}{M}}$

Scelto $\delta = \sqrt{\frac{1}{M}}$ si ha che se $|x| < \delta \Rightarrow x^2 < \frac{1}{M}$

$$\Downarrow$$

$$\frac{1}{x^2} > M \ \square .$$

Se $x_0 \in \mathbb{R}$



$$\mathcal{B}_{x_0^+} = \left\{ [x_0, x_0+\varepsilon) : \varepsilon > 0 \right\}$$

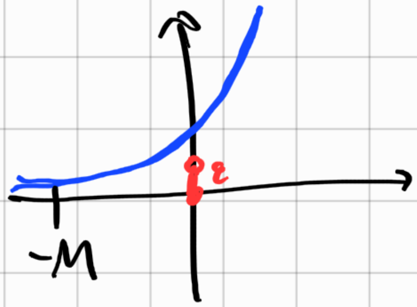
$$\mathcal{B}_{x_0^-} = \left\{ (x_0-\varepsilon, x_0] : \varepsilon > 0 \right\}.$$

[Oss " $+\infty = \infty^-$ "
" $-\infty = \infty^+$ "]

Es 3 $\frac{1}{x} \to +\infty$ per $x \to 0^+$

$$\forall M \in \mathbb{R}\ \exists \delta > 0 : 0 < x < \delta \Rightarrow f(x) > M.$$

Es 4 $2^x \to 0^+$ per $x \to -\infty$



$$\forall \varepsilon > 0\ \exists M \in \mathbb{R} : x < -M \Rightarrow 0 < 2^x < \varepsilon$$

$f(x) \to \ell$ per $x \to \bar{x}$

$$\ell = \begin{cases} y_0 \in \mathbb{R} \\ y_0^+ \\ y_0^- \\ +\infty \\ -\infty \end{cases} \qquad \bar{x} = \begin{cases} x_0 \in \mathbb{R} \\ x_0^+ \\ x_0^- \\ +\infty \\ -\infty \end{cases}$$

• $\underbrace{f(x) \to l}$ per $x \to x_0$

operatore limite

• $\lim\limits_{x\to x_0} f(x) = l$

__Unicità__ del __limite__

• $2^x \to 0$ per $x \to -\infty$

• $2^x \to 0^+$ per $x \to -\infty$

$$\boxed{\lim_{x\to-\infty} 2^x = 0}$$

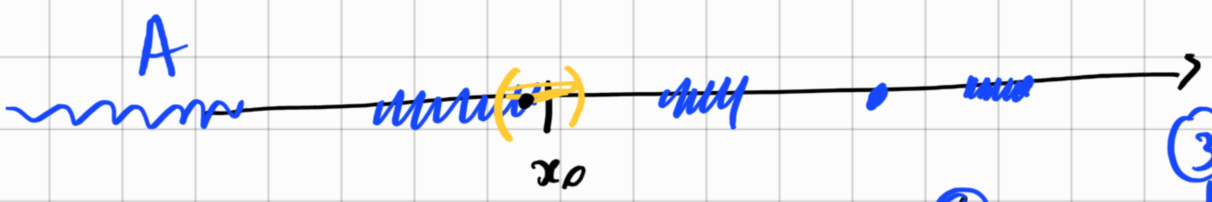
$$\lim_{x\to -1} \sqrt{x} = ?$$

$$\boxed{\sqrt{x} \to 42 \quad \text{per } x \to -1}$$

__Def__ Dato $A \subseteq \overline{\mathbb{R}}$, $x_0 \in \overline{\mathbb{R}}$ diremo che

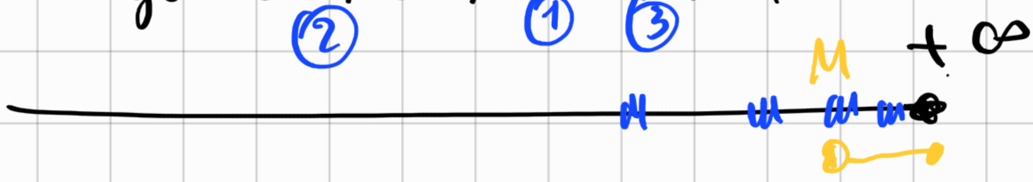
x_0 è __punto di accumulazione__ di A se

$$\underline{\forall U \in \mathcal{B}_{x_0}} : \underbrace{U \cap A \setminus \{x_0\}} \neq \emptyset$$

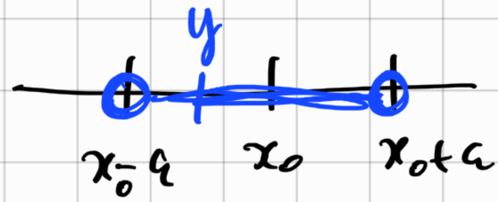


se $x_0 \in \mathbb{R}$: $\forall \varepsilon > 0 : \exists\, \overset{1}{y \in A} : \overset{3}{0 <} |y - x_0| \overset{2}{< \varepsilon}$

$$y \in \underset{2}{(x_0-\varepsilon, x_0+\varepsilon)} \underset{1}{\cap A} \underset{3}{\setminus\{x_0\}} \neq \emptyset$$



$y \in (x_0-\varepsilon,\, x_0+\varepsilon)$ $\Leftrightarrow$ $x_0-\varepsilon < y < x_0+\varepsilon$ $\Leftrightarrow$ $|y-x_0|<\varepsilon$



Es $A=\mathbb{Q}$: $\forall x_0 \in \overline{\mathbb{R}}$ x_0 è punto di accumulazione.

Es $A=(0,+\infty)$ i punti di accumulazione di A sono: $[0,+\infty]$

Es $A=\{-1\}\cup(0,1]$

i punti di accumulazione di A sono: $[0,1]$

se $x>1$ se prendo $\varepsilon = x-1$

Es

0 è punto di accumulazione.

Teorema Sia $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, x_0 punto di accumulazione di A.

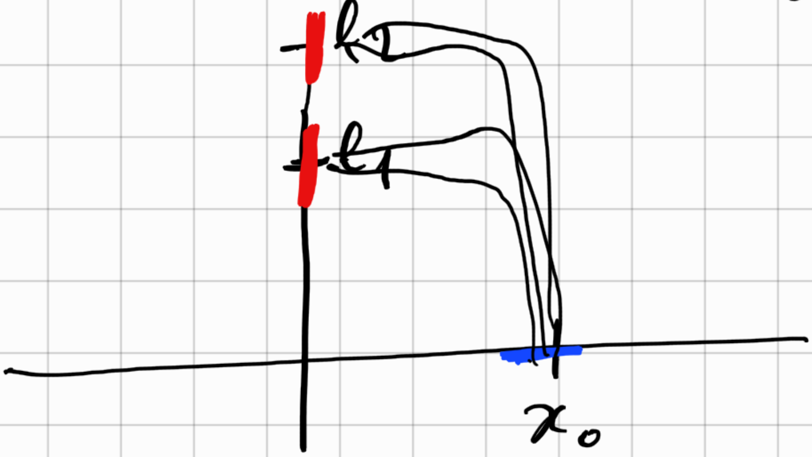
Allora se $f(x) \to l \in \overline{\mathbb{R}}$ per $x \to x_0$

l è unico.

Lo stesso vale se $x_0 \in \mathbb{R}$ per $x \to x_0^+$ o $x \to x_0^-$ quando x_0^+, x_0^- è punto di accumulazione di A

(x_0^+ è punto di acc. di A: x_0 è punto di acc. di $A \cap [x_0, +\infty)$ $(-\infty, x_0]$)

dim



per assurdo $\begin{cases} f(x) \to l_1 \\ f(x) \to l_2 \end{cases}$ per $x \to x_0$

$l_1 \neq l_2$ $\quad l_1 < l_2$.

$\exists\, U_1 \in \mathcal{U}_{l_1}$, $U_2 \in \mathcal{U}_{l_2}$ t.c. $U_1 \cap U_2 = \emptyset$

ma $\exists\ V_1, V_2 \in \mathcal{U}_{x_0}$ t.c.

$$f(A \cap V_1 \setminus \{x_0\}) \subseteq U_1$$

$$f(A \cap V_2 \setminus \{x_0\}) \subseteq U_2$$

$V = V_1 \cap V_2$ è un intorno di x_0

$$f(A \cap V \setminus \{x_0\}) \subseteq \underset{\emptyset}{\underset{\shortparallel}{U_1 \cap U_2}}$$

$$A \cap V \setminus \{x_0\} = \emptyset$$

$\Longrightarrow$ x_0 non sarebbe pto di accumulazione.

$\square$