ANALISI MATEMATICA B
19.3.2021 - LEZIONE 67

Fratti semplici

$$f(x) = \frac{1+x^2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\lim_{x\to -1} f(x)\cdot(x+1) = \lim_{x\to -1} \underbrace{\frac{A(x+1)}{x}}_{\to 0} + \underbrace{\frac{B(x+1)}{x^2}}_{\to 0} + \frac{C\cancel{(x+1)}}{\cancel{(x+1)}}$$

$$= C$$

$$\lim_{x\to -1} \frac{1+x^2}{x^2} = \frac{2}{1} = 2 = C.$$

$$\lim_{x\to 0} f(x)\cdot x^2 = \lim_{x\to 0} \underbrace{\frac{A}{\cancel{x}}x^{\cancel{2}}}_{\to 0} + \underbrace{\frac{B}{\cancel{x^2}}\cancel{x^2}}_{\to B} + \underbrace{\frac{C x^2}{x+1}}_{\to 0}$$

$$\lim_{x\to 0} \frac{1+x^2}{x+1} = 1 = B$$

$$\lim_{x\to 0} \left(f(x) - \frac{B}{x^2}\right)\cdot x = \lim_{x\to 0} \underbrace{\frac{A}{\cancel{x}}\cdot\cancel{x}}_{\to A} + \underbrace{\frac{C}{x+1}\cdot x}_{\to 0}$$

$$= A$$

$$\lim_{x\to 0}\left[\frac{1+x^2}{x^2(x+1)}-\frac{1}{x^2}\right]x=\lim_{x\to 0}\frac{\cancel{1}+x^2-(x+\cancel{1})}{x^{\cancel{2}}(x+1)}\cancel{x}$$

$$=\lim_{x\to 0}\frac{x^2-x}{x(x+1)}=\lim_{x\to 0}\frac{x-1}{(x+1)}=-1.$$

$$\begin{array}{c} -1 \\ \shortparallel \\ A \end{array}$$

Integrali impropri

$$f:[a,b)\longrightarrow \mathbb{R}$$

$$\int_a^b f(x)\,dx=\lim_{\beta\to b^-}\int_a^\beta f(x)\,dx$$

- $f\le g \Rightarrow \int_a^b f\le\int_a^b g \quad (a<b)$
- $\int_a^b(\lambda f+\mu g)=\lambda\int_a^b f+\mu\int_a^b g$

- - - -

Se f è R-integrabile su $[a,b]$

se $\alpha \to a^+$

$$\lim_{\alpha \to a^+} \int_\alpha^b f(x)\,dx = \int_a^b f(x)\,dx.$$

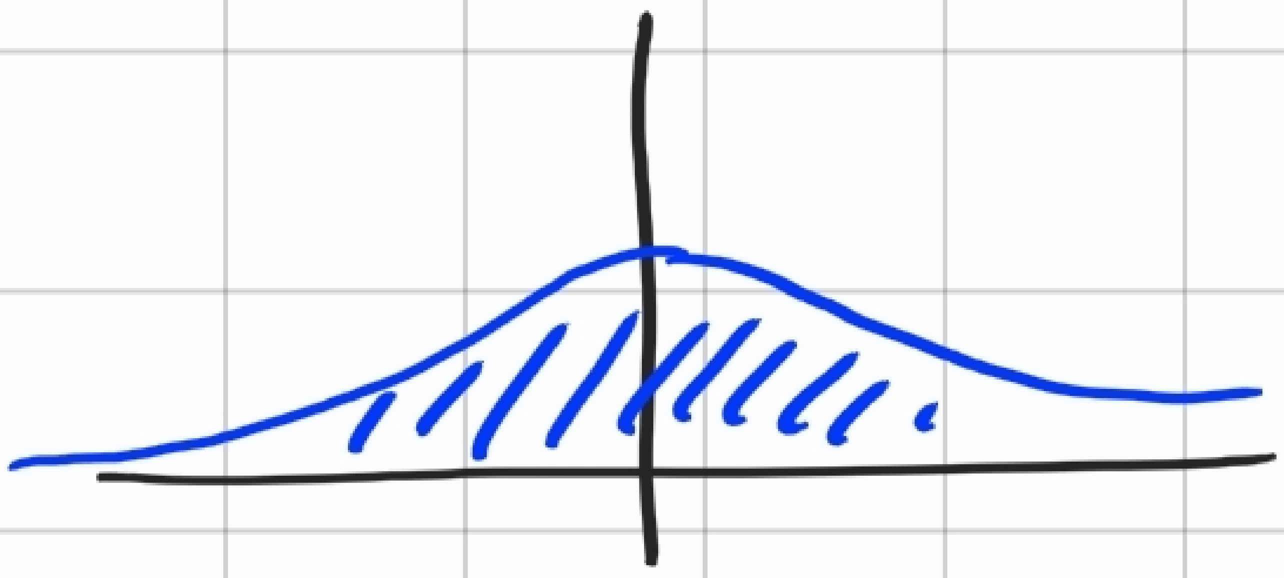
$$\|$$

$\int_a^b f(x)\,dx$ ← in senso improprio

se $f : (a,b] \to \mathbb{R}$

Problema: determinare il carattere di un integrale improprio.

Es $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$



è convergente?

① Se $f \geq 0$ $\quad \int_a^b f$ esiste: $\begin{cases} \text{finito} \geq 0 \\ +\infty \end{cases}$

$f : [a,b) \to \mathbb{R}$ $\qquad \sup_{x<b} F(x)$

$$\int_a^b f(x)\,dx = \lim_{x \to b^-} F(x), \quad F(x) = \int_a^x f(t)\,dt$$

Se $f \geq 0$ F è crescente : $x_1 \leq x_2 \overset{?}{\Rightarrow} F(x_1) \leq F(x_2)$

$$F(x_2) - F(x_1) = \int_a^{x_2} f(t)\,dt - \int_a^{x_1} f(t)\,dt$$

$$= \int_{x_1}^{x_2} f(t)\,dt \geq \int_{x_1}^{x_2} 0\,dt = 0$$

$x_1 \leq x_2$

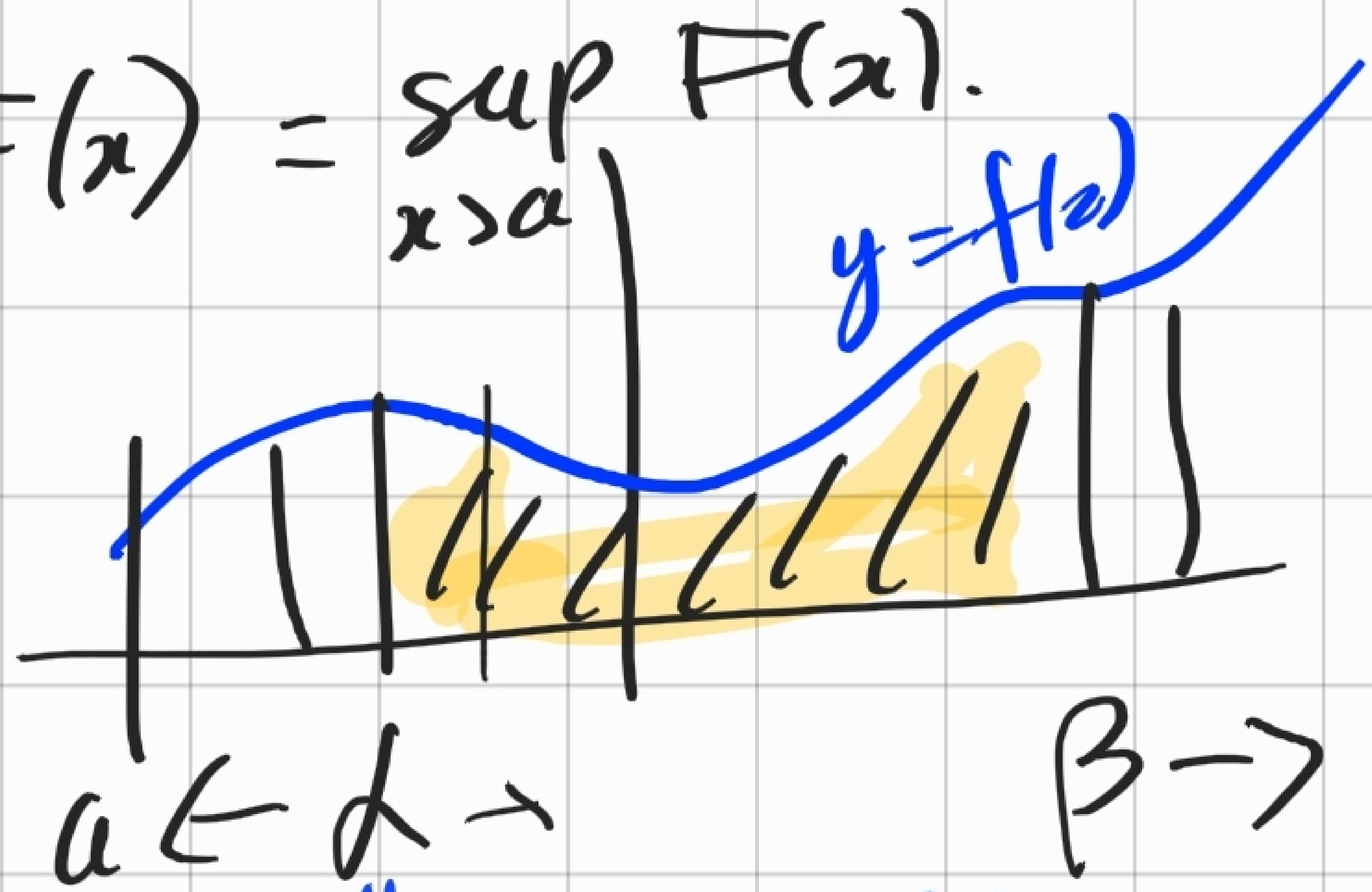
D

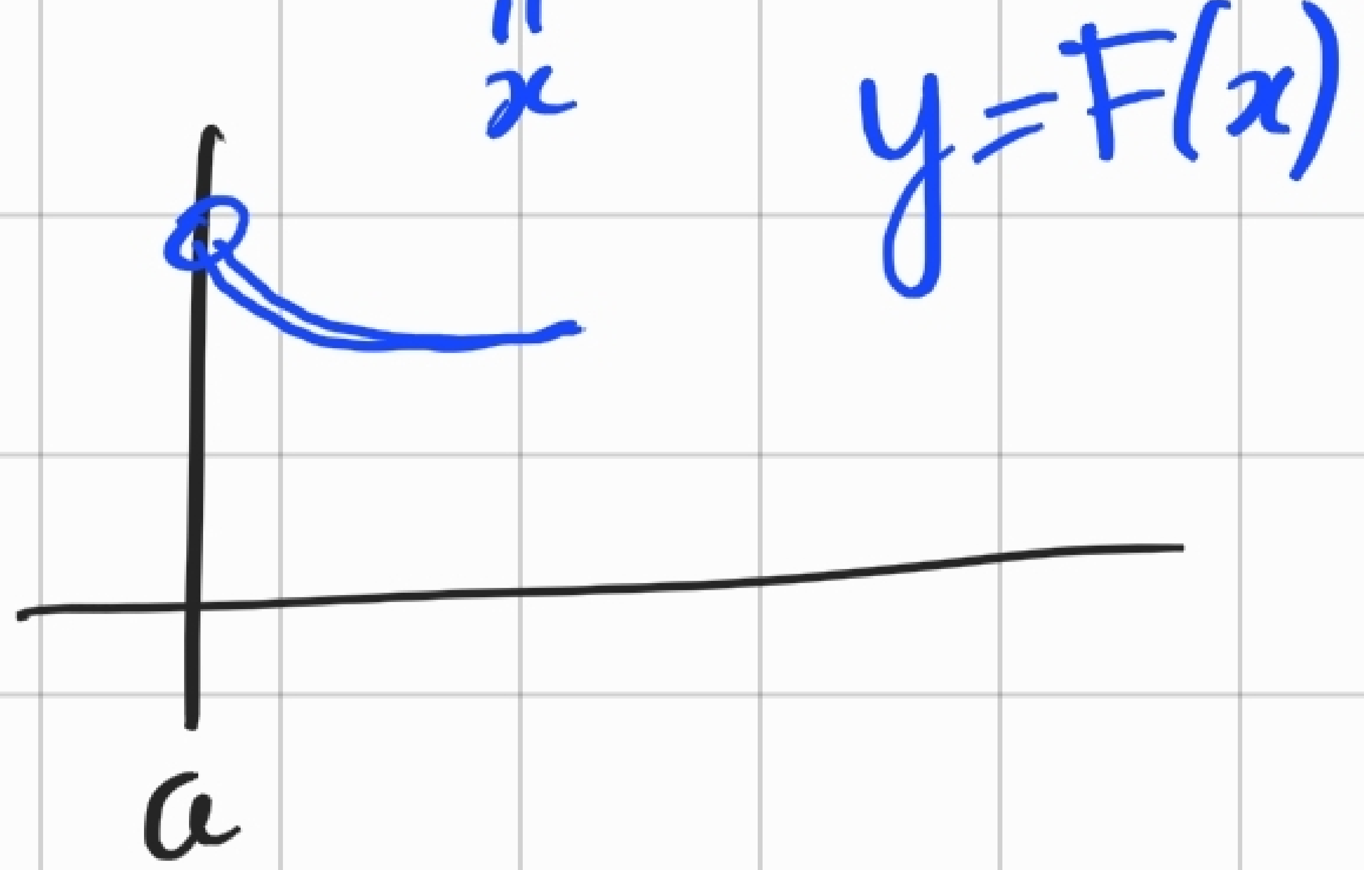
$$f : (a, b] \to \mathbb{R}$$

$$F(x) = \int_x^b f(t)\,dt$$

F è decrescente se $f \geq 0$

$$\int_a^b f(x)\,dx = \lim_{x \to a^+} F(x) = \sup_{x > a} F(x).$$





Es $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$ esiste. Finito o infinito?

Criteri di confronto.

1. Se $\boxed{f \le g}$ $\left(f(x) \le g(x)\ \forall x\right)$

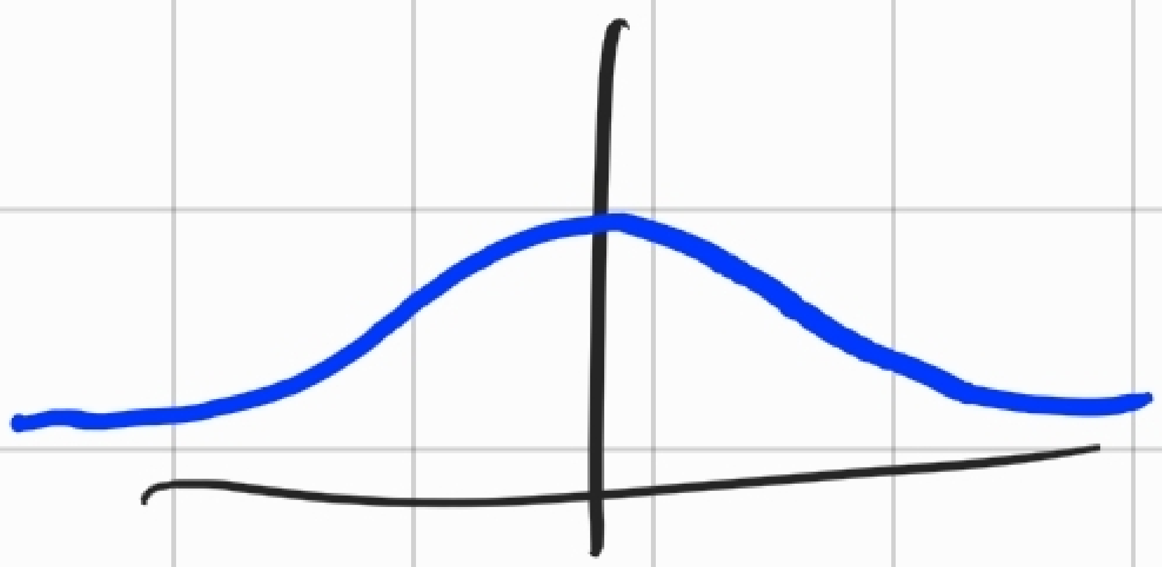
se $a<b$

$$\int_a^b f \le \int_a^b g .$$

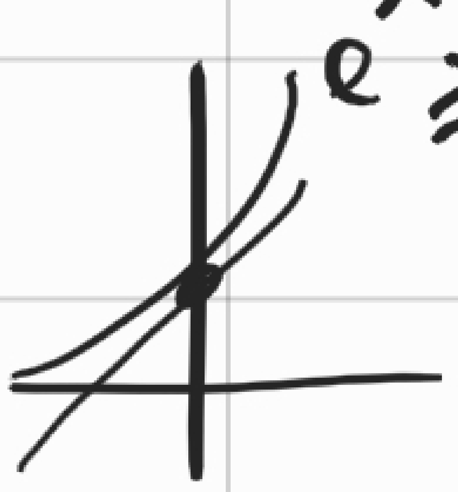
se $\int_a^b g < +\infty \Rightarrow \int_a^b f < +\infty$

se $\int_a^b f = +\infty \Rightarrow \int_a^b g = +\infty$.

Es $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$



$$e^{-x^2} = \frac{1}{e^{x^2}} \le \frac{1}{1+x^2}$$



$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx \le \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi < +\infty$$

$$= \left[\operatorname{arctg}\right]_{-\infty}^{+\infty}$$

confronto asintotico

$f, g : [a, b) \to \mathbb{R}$ $\qquad$ $\boxed{f, g > 0}$

se $\underbrace{f << g}$ per $x \to b$ $\qquad$ $\boxed{\frac{f(x)}{g(x)} \to 0 \quad x \to b}$

se $\int_a^b g < +\infty \Rightarrow \int_a^b f < +\infty$

se $\int_a^b f = +\infty \Rightarrow \int_a^b g = +\infty$.

dim $\quad$ se $\frac{f(x)}{g(x)} \to 0$ per $x \to b^-$

$\exists\ c < b \quad \forall x \geq c \quad f(x) \leq g(x)$.

il risultato segue dal confronto puntuale su $[c, b)$.

su $[a, c]$ f e g sono limitate e R-integrabili

$\int_a^c f \in \mathbb{R} \qquad \int_a^c g \in \mathbb{R}$

Es $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{+\infty} e^{-x^2}\,dx$

per $x \to +\infty$ $\quad e^{-x^2} << \frac{1}{x^2}$ $\qquad (e^x >> x)$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{+\infty} = 1 < +\infty$$

$$\Rightarrow \int_1^{+\infty} e^{-x^2}\,dx < +\infty.$$

Confronto asintotico 2: $[a,b)$

se $f \sim g$ $\qquad \frac{f}{g} \to 1$

allora $\int_a^b f$ e $\int_a^b g$ hanno lo stesso carattere

dim

$$\frac{f}{g} \to 1$$

$$\frac{1}{2} g \leq f \leq 2g \quad \ldots\ldots \quad \square$$

Esempio $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$??

Esempio $\int_1^{+\infty} \frac{1}{x^2+\ln x}\,dx$ è convergente?

Sì perché $\frac{1}{x^2+\ln x} \sim \frac{1}{x^2}$ per $x \to +\infty$

Integrali con cui confrontarsi:

A $+\infty$: $\int_1^{+\infty} \frac{1}{x^p}\,dx$ $\left[\sum_{k=1}^{+\infty} \frac{1}{k^p}\right]$

$$= \begin{cases} \left[\frac{x^{1-p}}{1-p}\right]_1^{+\infty} & \text{se } p \neq 1 \\ \left[\ln x\right]_1^{+\infty} & \text{se } p = 1 \end{cases}$$

se $p > 1$ l'integrale converge

se $p \leq 1$ l'integrale diverge $(+\infty)$

Al finito: (in $x_0 = 0$)

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \left[\frac{x^{1-p}}{1-p}\right]_0^1 & \text{se } p \neq 1 \\ \left[\ln x\right]_0^1 & \text{se } p = 1 \end{cases}$$

se $p \geq 1$ diverge $+\infty$

se $p < 1$ converge

Lo stesso su $(a, b]$

$$\int_a^b \frac{1}{(x-a)^p}\,dx$$

stessi risultati

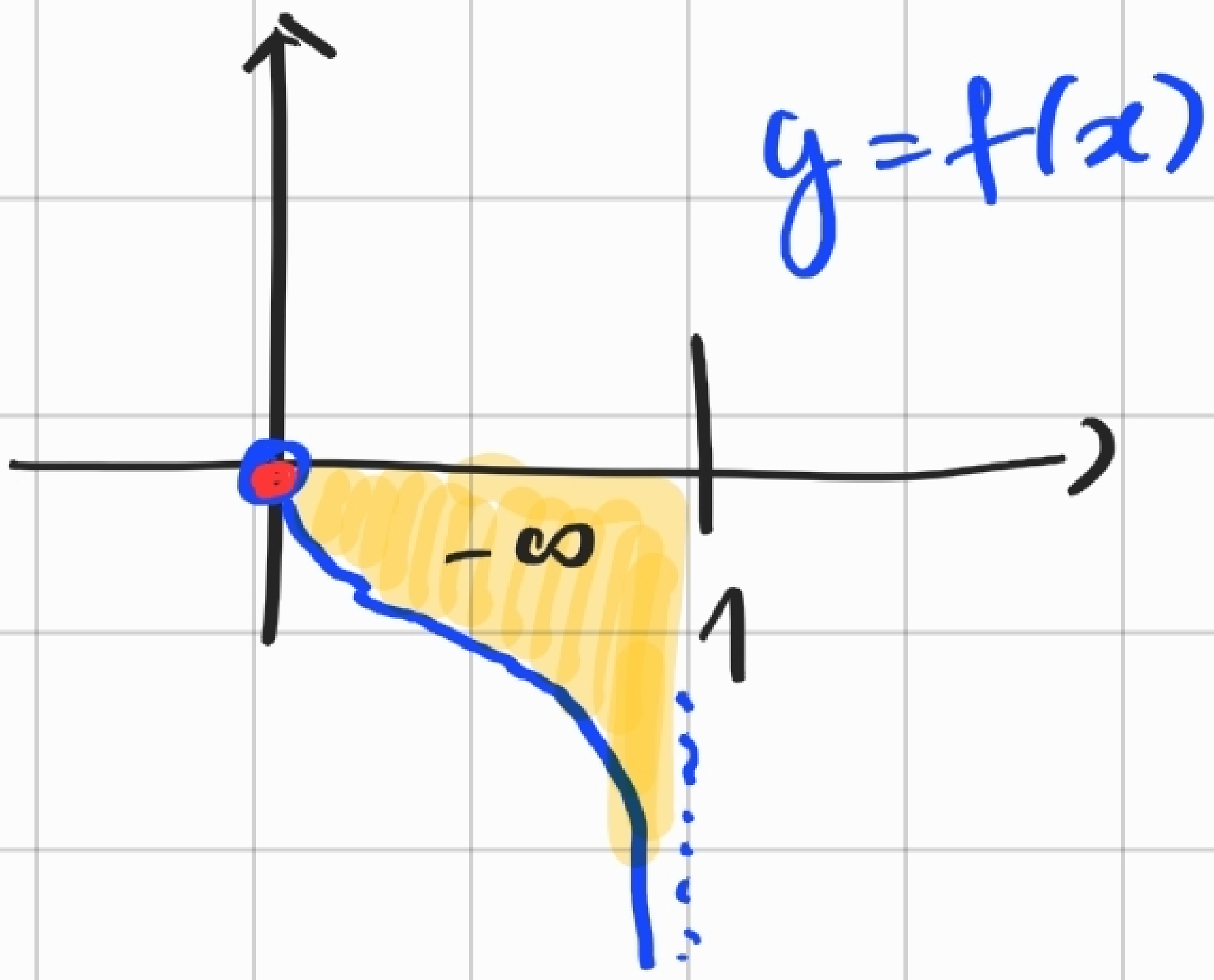
Esempio $$\int_0^1 \frac{1}{\ln x}\,dx$$

$f(x) = \frac{1}{\ln x}$ $\qquad f : (0,1) \to \mathbb{R}$

$f < 0$

$\int -f$ esiste $\Rightarrow \int f$ esiste

per $x \to 0$ $\ln x \to -\infty$ $\qquad \frac{1}{\ln x} \to 0$



f și extinde prin continuitate în $x=0$

$$\Rightarrow \int_0^{1/2} f(x)\,dx \quad \text{e convergentă}$$

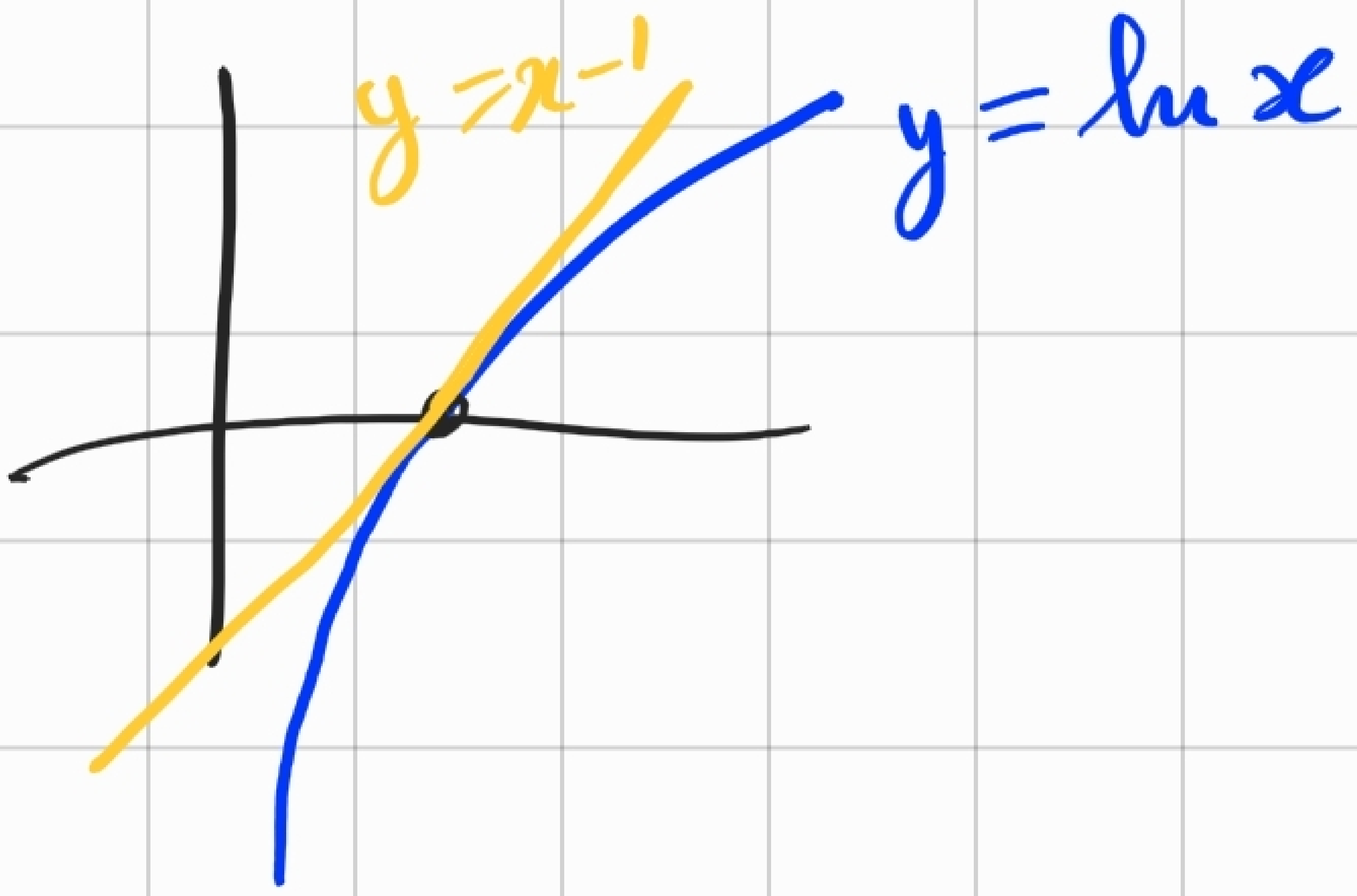
$$\int_{1/2}^{1} f(x)\,dx \quad \text{e convergentă ?}$$

pt $\underline{\underline{x \to 1}}$ $\quad \dfrac{1}{\ln x} \sim \,?$

$$\ln(1+t) = t + o(t) \quad \text{pt } t \to 0$$

$$\ln(1+t) \sim t \quad \text{pt } t \to 0$$

$$\ln x \sim x-1 \quad \text{pt } x \to 1$$



$$\frac{1}{\ln x} \sim \frac{1}{x-1} \quad \text{pt } x \to 1$$

$$\int_{\frac{1}{2}}^{1} \frac{1}{x-1}\, dx = -\infty$$

$$\int \frac{1}{(x-1)^p}\, dx$$

$p = 1$

$$\int_{\frac{1}{2}}^{1} \frac{1}{\ln x}\, dx = -\infty$$

$$\int_0^1 \frac{1}{\ln x}\, dx = \underbrace{\boxed{\int_0^{1/2} \frac{1}{\ln x}\, dx}}_{\text{finito}} + \underbrace{\boxed{\int_{1/2}^{1} \frac{1}{\ln x}\, dx}}_{-\infty}$$

$$= -\infty.$$

Altri integrali di confronto:

$$\int \frac{1}{x \ln^p x} = \int \frac{1}{y^p}\, dy$$

$y = \ln x$

$dy = \frac{1}{x}\, dx$

$$\boxed{\sum \frac{1}{k \ln^p k}}$$

se $x \to +\infty$ $\quad \ln x \to +\infty \quad y \to +\infty$

$\int$ converge $\iff p > 1$

se $x \to 0$ $\ln x \to -\infty$ $y \to -\infty$

$\int$ converge $\Leftrightarrow$ $p > 1$.

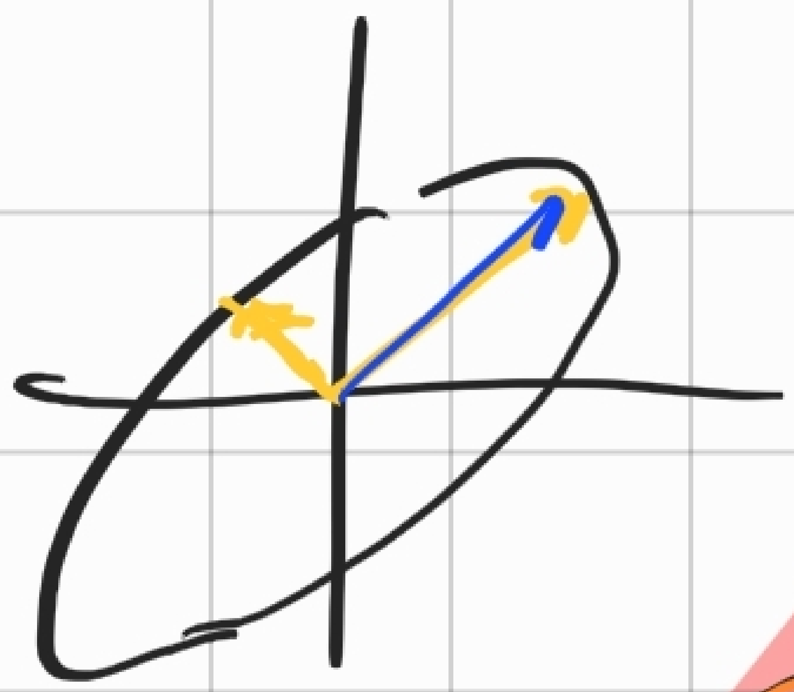
$$\|(x_1, x_2)\|_p = \sqrt[p]{|x_1|^p + |x_2|^p}$$

se fosse euclideo

$$\|(x_1, x_2)\|_p^2 = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle$$

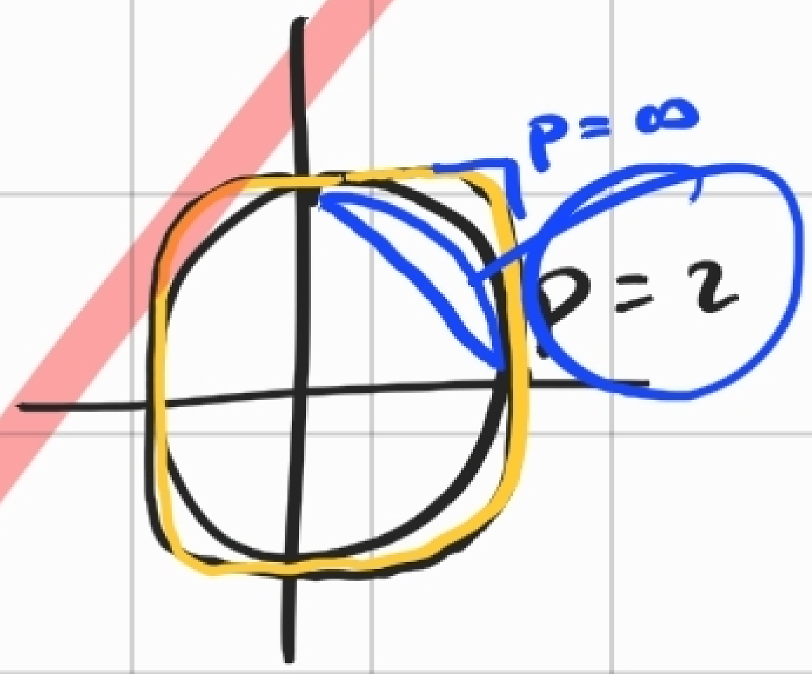
$$= (x_1 x_2) \cdot M \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$v^t M v = 1$$

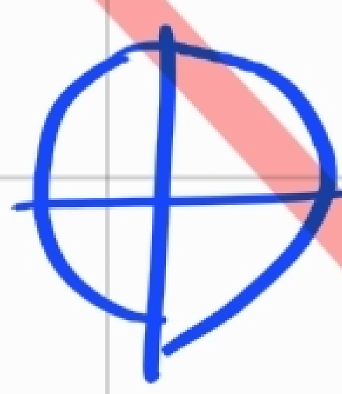


$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$\|\cdot\|_2$



$x^2 + y^2 = 1$

$|x|^p + |y|^p = 1$

è un cerchio $\Rightarrow$ $p = 2$.