ES 9

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx$$

4 gradi di libertà

$$\frac{1}{(1+x^2)^2} = \frac{Ax+B}{1+x^2} + \left(\frac{Cx+D}{1+x^2}\right)'$$

$$= \frac{(Ax+B)(1+x^2) + C(1+x^2) - (Cx+D)\,2x}{(1+x^2)^2}$$

$$= \frac{Ax + B + Ax^3 + Bx^2 + C + Cx^2 - 2Cx^2 - 2Dx}{(1+x^2)^2}$$

$$= \frac{Ax^3 + (B-C)x^2 + (A-2D)x + B + C}{(1+x^2)^2}$$

$$\begin{cases} A = 0 \\ B = C \\ D = 0 \\ B + C = 1 \end{cases} \qquad \begin{cases} A = 0 \\ D = 0 \\ B = \frac{1}{2} \\ C = \frac{1}{2} \end{cases}$$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx + \frac{1}{2}\left[\frac{x}{1+x^2}\right]_0^1$$

$$= \frac{1}{2}\Big[\arctan x\Big]_0^1 + \frac{1}{2}\left[\frac{x}{1+x^2}\right]_0^1$$

$$= \frac{1}{2}\,\frac{\pi}{4} + \frac{1}{2}\,\frac{1}{2} = \frac{\pi}{8} + \frac{1}{4}$$

$$= \frac{\pi + 2}{8} \quad \checkmark$$

Metodi di integrazione

Integrali che si riconducono a funzioni razionali.

Sostituzioni che funzionano sempre:

$$t = e^{\lambda x} \qquad x = \frac{\ln t}{\lambda} \qquad dx = \frac{1}{\lambda t}$$

Se ho:

$$\int R(e^{\lambda x})\,dx \to \int \underbrace{R(t)\,\frac{1}{\lambda t}\,dt}_{\text{è una funzione razionale in } t}$$

se R è razionale

Es : $\int \frac{2\sqrt{e^x} + e^{2x}}{e^x - 4}\, dx = (*)$

$t = e^{\frac{x}{2}} = \sqrt{e^x}$

$t^2 = e^x \qquad x = \ln t^2 = 2\ln t$

$dx = \frac{2}{t}\, dt$

$$(*) = \int \frac{2t + t^4}{t^2 - 4} \cdot \frac{2}{t}\, dt$$

$$= \int \frac{4\cancel{t} + 2t^{\cancel{4}\,3}}{(t^2-4)\cdot \cancel{t}}\, dt$$

$$= 2\int \frac{2 + t^3}{t^2-4}\, dt$$

$t^3 + 2 = (t^2-4)t + 4t + 2$

$$= 2\int t\, dt + 2\int \frac{4t+2}{t^2-4}\, dt$$

$$(*) = t^2 + 2\int \left[\frac{A}{t-2} + \frac{B}{t+2}\right] dt$$

$$\frac{A(t+2)+B(t-2)}{(t-2)(t+2)} = \frac{(A+B)t+2(A-B)}{t^2-4} = \frac{4t+2}{t^2-4}$$

$$\begin{cases} A+B=4 \\ A-B=1 \end{cases} \quad \begin{cases} A=B+1 \\ B=\frac{3}{2} \end{cases}$$

$$\begin{cases} A=5/2 \\ B=3/2 \end{cases}$$

$$(*) = t^2 + 5\int \frac{1}{t-2}\,dt + 3\int \frac{1}{t+2}\,dt$$

$$= t^2 + 5\ln|t-2| + 3\ln|t+2|$$

$$(t=\sqrt{e^x}) \qquad = e^x + 5\ln\left|\sqrt{e^x}-2\right| + 3\ln\left(\sqrt{e^x}+2\right)$$

$$= e^x + \ln\left[\left|\sqrt{e^x}-2\right|^5 \cdot \left(\sqrt{e^x}+2\right)^3\right] \quad \square$$

Funzioni razionali in $\underline{\sin^2 x}$, $\underline{\cos^2 x}$, $\underline{\sin x \cos x}$

$$1+tg^2 x = 1+\left(\frac{\sin x}{\cos x}\right)^2 = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\begin{cases} \cos^2 x = \dfrac{1}{1+tg^2 x} \\ \sin^2 x = tg^2 x \cdot \cos^2 x = \dfrac{tg^2 x}{1+tg^2 x} \\ \sin x \cos x = tg x \cdot \cos^2 x = \dfrac{tg x}{1+tg^2 x} \end{cases}$$

$t = tg x \qquad x = \arctg t$

$$dx = \frac{1}{1+t^2}\, dt$$

$$\int R(\sin^2 x, \cos^2 x, \sin x \cos x)\, dx$$

$$= \int R\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}, \frac{t}{1+t^2}\right) \cdot \frac{1}{1+t^2}\, dt$$

Se R è razionale $\Rightarrow \int$ funzione razionale

$\underline{Es}$ $$\int \frac{1}{\cos x(\sin x + \cos x)}\, dx =$$

$$\int \frac{1}{\sin x \cos x + \cos^2 x}\, dx \qquad t = tg x$$

$$= \int \frac{1}{\frac{t}{1+t^2} + \frac{1}{1+t^2}} \cdot \frac{1}{1+t^2}\, dt$$

$$= \int \frac{1}{t+1}\,dt = \ln|t+1| = \ln|\operatorname{tg} x + 1| \quad \square$$

Funzioni razionali in $\sin x$, $\cos x$

$$\cos x = \cos\left(\frac{x}{2}+\frac{x}{2}\right) = \cos^2\frac{x}{2} - \operatorname{sen}^2\frac{x}{2}$$

$$t = \operatorname{tg}\frac{x}{2} \qquad = \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$$

$$= \frac{1-t^2}{1+t^2}$$

$$\sin x = \sin\left(\frac{x}{2}+\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{t}{1+t^2}$$

$$\frac{x}{2} = \operatorname{arctg} t \qquad x = 2\operatorname{arctg} t$$

$$dx = \frac{2}{1+t^2}\,dt$$

ES

$$\int \frac{1}{\sin x}\,dx = \int \frac{1}{\cancel{2}\frac{t}{\cancel{1+t^2}}} \cdot \frac{\cancel{2}}{\cancel{1+t^2}}\,dt$$

$$= \int \frac{1}{t}\,dt = \ln|t| = \ln\left|\operatorname{tg}\frac{x}{2}\right| \quad \square$$

Integrali di funzioni razionali con radicali

$$\int R\left(\sqrt[n]{x}\right)dx$$

$$t=\sqrt[n]{x} \qquad x=t^n$$
$$dx = n\,t^{n-1}$$

ES $$\int \frac{\sqrt[4]{x}}{\sqrt[2]{x}+\sqrt[3]{x}}\,dx$$

$$t=\sqrt[12]{x}$$
$$x=t^{12} \qquad \sqrt[4]{x}=t^3$$
$$dx=12\,t^{11}\,dt \qquad \sqrt[2]{x}=t^6$$
$$\sqrt[3]{x}=t^4$$

$$=\int \frac{t^3}{t^6+t^4}\cdot 12\,t^{11}\,dt$$

$$=12\int \frac{t^{10}}{t^2+1}\,dt$$

$$\begin{array}{l|l} t^{10} & t^2+1 \\ \hline t^{10}+t^8 & t^8-t^6+t^4-t^2+1 \\ -t^8 & \\ -t^8-t^6 & \\ t^6 & \\ t^6+t^4 & \\ -t^4 & \\ -t^4-t^2 & \\ t^2 & \\ t^2+1 & \\ -1 & \end{array}$$

$$=12\int\left[t^8-t^6+t^4-t^2+1-\frac{1}{t^2+1}\right]dt$$

$$= \frac{12}{9} t^9 - \frac{12}{7} t^7 + \frac{12 t^5}{5} - \frac{12 t^3}{3} + 12 t - 12 \operatorname{arctg} t$$

$$= \frac{4}{3} \sqrt[12]{x^9} - \frac{12}{7} \sqrt[12]{x^7} + \frac{12}{5} \sqrt[12]{x^5} - 4 \sqrt[12]{x^3}$$

$$+ 12 \sqrt[12]{x} - 12 \operatorname{arctg} \sqrt[12]{x} \qquad \square$$

Pagani-Salsa

3 integrali da risolvere mentalmente in non più di 15'':

1. $\displaystyle\int \frac{e^x + \cos x}{e^x + \sin x}\, dx = \ln |e^x + \sin x|$ — Niccolò

2. $\displaystyle\int \frac{1}{\sqrt{x+2} - \sqrt{x}}\, dx = \frac{1}{2} \left[\frac{(x+2)^{3/2}}{\frac{3}{2}} + \frac{x^{3/2}}{\frac{3}{2}} \right]$ — Davide

3. $\displaystyle\int \frac{\ln \ln x}{x}\, dx = \ln x \cdot \ln \ln x - \ln x$ — Daria

$$\int \ln y \, dy = y \ln y - y$$

funzioni che non hanno una primitiva che si scrive tramite composizione di funzioni elementari. (Th. Liouville)

funzione speciale

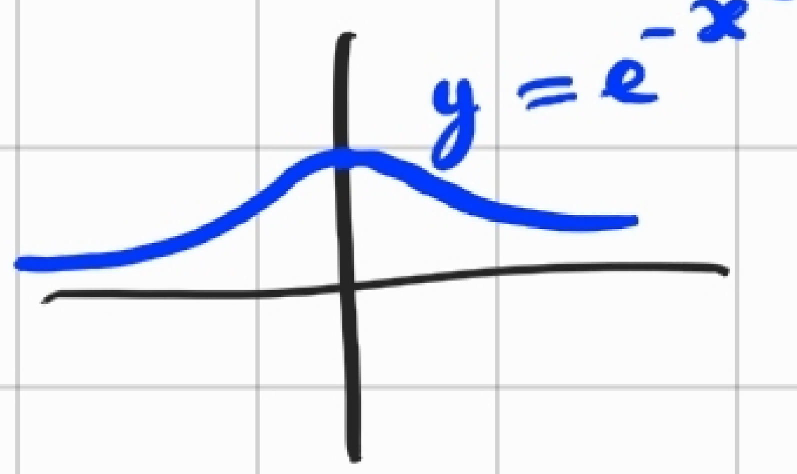
ES 1 $\displaystyle\int \frac{1}{\ln x}\,dx \ni \mathrm{li}\, x$

ES 2 $\displaystyle\frac{2}{\sqrt{\pi}}\int e^{-x^2}\,dx \ni \mathrm{erf}\, x$

$$F(x) = \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

$$F'(x) = \frac{2}{\sqrt{\pi}}\, e^{-x^2}.$$

$$F(0) = 0, \quad F'(x) = \frac{2}{\sqrt{\pi}}\, e^{-x^2}$$



ES 3 $\displaystyle\int \frac{e^x}{x}\,dx \longrightarrow \mathrm{ei}\, x$

ES 4 $\displaystyle\int \frac{\sin x}{x}\,dx \longrightarrow \mathrm{Si}\, x$

ES 5 $\displaystyle\int \sin\left(\frac{\pi x^2}{2}\right)dx \longrightarrow S\, x$