ANALISI MATEMATICA B
LEZIONE 51 - 10.2.2021

per $x \to x_0$ $\qquad$ per $x \to 0$

$$f(x) = g(x) + o(g(x)) \quad \Big(\text{ES: } 1-\cos x = \frac{x^2}{2} + o(x^2) \Big)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow \quad 1-\cos x \sim \frac{x^2}{2}$$

$$\frac{f(x)}{g(x)} = \frac{g(x) + o(g(x))}{g(x)} = 1 + \frac{o(g(x))}{g(x)} \longrightarrow 1$$

$$\Downarrow$$

$$f \sim g$$

$$\boxed{\sin(x^2) \sim x^2 \quad \text{per } x \to 0}$$

Formula di Taylor (resto di Peano)

Se P_n è il polinomio di Taylor per f di ordine n centrato in x_0

$$\boxed{f(x) = P_n(x) + o((x-x_0)^n)}$$

Polinomi di Taylor delle fn. elementari

$$\boxed{\begin{matrix} x \to 0 \\ x_0 = 0 \end{matrix}}$$

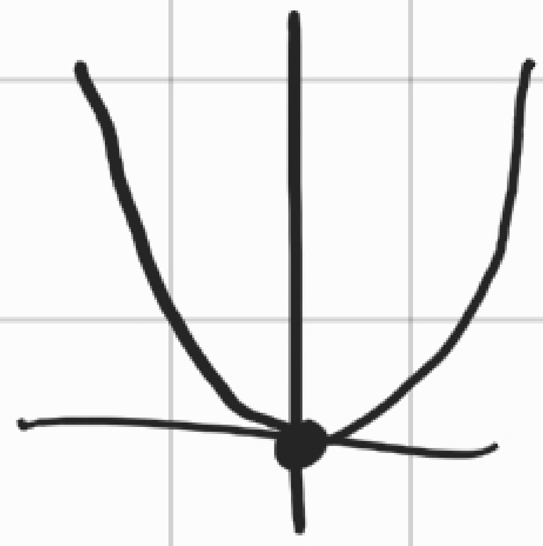
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n})$$

$$\left[\quad (3+x)^\alpha = 3^\alpha \left(1+\frac{x}{3}\right)^\alpha \quad \right.$$



$x^n \quad]$

$$(1+x)^\alpha = \ ?$$

$$f(x) = (1+x)^\alpha \qquad f(0) = 1$$
$$f'(x) = \alpha(1+x)^{\alpha-1} \qquad f'(0) = \alpha$$
$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \qquad f''(0) = \alpha(\alpha-1)$$
$$\vdots$$
$$f^{(k)}(x) = \underbrace{\alpha(\alpha-1)\dots(\alpha-k+1)}_{k \text{ fattori}} \cdot (1+x)^{\alpha-k}$$

Se $\alpha \in \mathbb{N}$ $\downarrow$

$$f^{(k)}(0) = \alpha(\alpha-1)\dots(\alpha-k+1)$$

$$\frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} = \binom{\alpha}{k}$$

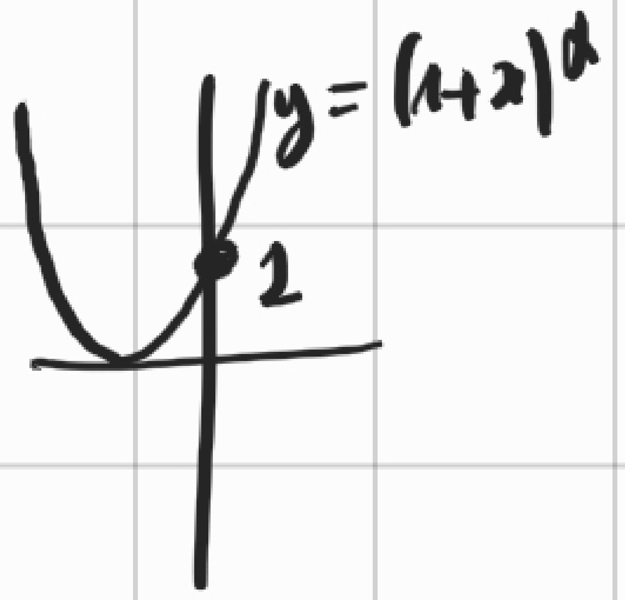
$\uparrow$

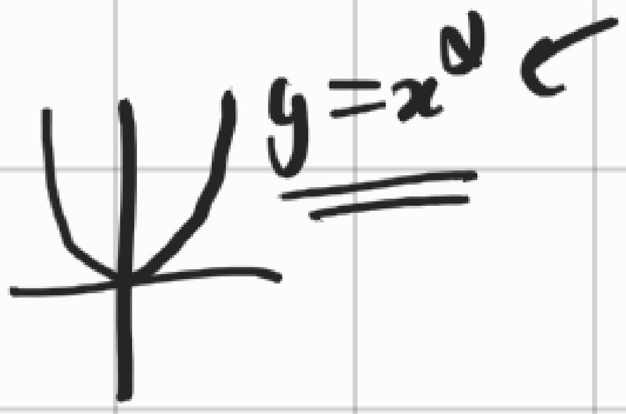
Se $\alpha \in \mathbb{N}$ $\quad (1+x)^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} x^\alpha$

Definizione $\quad$ Per $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$

def nuovo

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$





$$= \frac{\prod_{j=0}^{k-1}(\alpha-j)}{k!}$$

$$\boxed{(1+x)^\alpha = 1+\alpha\cdot x+\binom{\alpha}{2}x^2+\dots+\binom{\alpha}{n}x^n+o(x^n)}$$

$$\binom{\alpha}{0}=1 \qquad \binom{\alpha}{1}=\alpha$$

$$\binom{\alpha}{2}=\frac{\alpha(\alpha-1)}{2}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Es $\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)$

$$\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$$

Es $\frac{1}{1+x} = (1+x)^{-1} = 1-x+x^2-x^3+o(x^3)$

$$\binom{-1}{1}=\frac{-1}{1} \qquad \binom{-1}{2}=\frac{(-1)\cdot(-1-1)}{2!}$$

$$\binom{-1}{3}=\frac{(-1)(-2)(-3)}{3!}=\frac{-3!}{3!}=-1$$

Es $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = \dots\dots$

Es $\sqrt[3]{n+1} - \sqrt[3]{n-1} = ?$ con $n \to +\infty$

$$(*) = \sqrt[3]{n+1} - \sqrt[3]{n-1} = \sqrt[3]{n}\left(\sqrt[3]{1+\frac{1}{n}} - \sqrt[3]{1-\frac{1}{n}}\right)$$

$$\sqrt[3]{1\pm x} = 1 \pm \frac{1}{3}x + o(x) \quad \text{per} \quad x \to 0$$

$$\sqrt[3]{1+\frac{1}{n}} = 1 + \frac{1}{3}\frac{1}{n} + o\left(\frac{1}{n}\right) \quad \text{per} \quad n \to +\infty \quad \frac{1}{n} \to 0$$

$$(*) = \sqrt[3]{n}\cdot\left(\cancel{1} + \frac{1}{3n} + o\left(\frac{1}{n}\right) - \left(\cancel{1} - \frac{1}{3n} + o\left(\frac{1}{n}\right)\right)\right)$$

$$= \sqrt[3]{n}\left(\frac{2}{3n} + o\left(\frac{1}{n}\right)\right) = \frac{2}{3n^{2/3}} + o\left(\frac{1}{n^{2/3}}\right)$$

$$\boxed{\sqrt[3]{n+1} - \sqrt[3]{n-1} \sim \frac{2}{3n^{2/3}}}$$

Per

$$\sqrt[3]{n+1} - \sqrt[3]{n-1} - \frac{2}{3\sqrt[3]{n^2}} \sim c \cdot n^p \; ?$$

trovare c e p.

Es $f(x) = \dfrac{1}{\cos x} = \dfrac{1}{\underbrace{1 - \frac{x^2}{2} + o(x^2)}} = (*)$

$$\left[\ (1+y)^{\alpha} = 1+\alpha y + o(y) \quad \text{per } y \to 0\ \right]$$

$$(*) = (1+y)^{-1} = 1 - \left(-\frac{x^2}{2} + o(x^2)\right) + o(x^2) =$$

$$\left[\ \underbrace{y = -\frac{x^2}{2} + o(x^2)} \qquad o(y) = o(x^2)\ \right]$$

$$= 1 + \frac{x^2}{2} + o(x^2) \qquad \text{per } x \to 0$$

$$(1+y)^{-1} = 1 - y + y^2 + o(y^2)$$

$$\frac{1}{\cos x} = \frac{1}{1-\frac{x^2}{2}+o(x^2)} = 1 - \left(-\frac{x^2}{2} + o(x^2)\right) + \cancel{\left(-\frac{x^2}{2}+o(x^2)\right)^2 + o(x^4)}$$

$$y = -\frac{x^2}{2} + o(x^2)$$

$$\frac{1}{\cos x} = \frac{1}{1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)} = 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) +$$

$$+ \left(-\frac{x^2}{2} + o(x^2)\right)^2 + o(x^4)$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + o(x^4) = \underline{\underline{1 + \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^4)}}$$

... proseguiamo con la tabella ...

$$\ln(1+x) = ?$$

$$f(x) = \ln(1+x) = P_n(x) + o(x^n)$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} = \underbrace{1 - x + x^2 - \dots + (-1)^n x^n}_{} + o(x^n)$$

$$P_n'(x) = 1 - x + x^2 + \dots (-1)^{n-1} x^{n-1}$$

$P_n(x) = ?$ anti-derivata (o primitiva)

$$P_n(x) = \sum_{k=0}^{n} a_k \cdot x^k$$

$$P_n'(x) = \sum_{k=1}^{n} \underbrace{k \cdot a_k}_{} \cdot x^{k-1}$$

$$\frac{x^3}{3} \xrightarrow{D} x^2$$

$$P_n'(x) = 1 - x + x^2 + \dots (-1)^{n-1} x^{n-1}$$

$$P_n(x) = a_0 + x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1} \frac{x^n}{n}$$

$$a_0 = \ln(1+0) = 0$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

$$f(x) = \operatorname{arctg} x = ?$$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + o(x^{2n})$$

$$\operatorname{arctg} x = 0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+1})$$

(Esercizio per) Se $f(x)$ è dispari $f'(x)$ è pari
Se $f(x)$ è pari $f'(x)$ è dispari

$f(x) = \arcsin x \Rightarrow ?$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} =$$

$$(1+y)^{-\frac{1}{2}} = 1 - \frac{y}{2} + \frac{3}{8}y^2 - \frac{1\cdot 3\cdot 5}{8\cdot 3!}y^3$$

$$\binom{-\frac{1}{2}}{0} = 1 \qquad \binom{-\frac{1}{2}}{2} = \frac{-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)}{2!} = \frac{3}{8}$$

$$\binom{-\frac{1}{2}}{1} = -\frac{1}{2}$$

$$\binom{-\frac{1}{2}}{3} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} = \frac{-5\cdot 3\cdot 1}{(3\cdot 2)\cdot(2\cdot 2)\cdot(1\cdot 2)}$$

$$= \frac{-5\cdot 3\cdot 1}{6\cdot 4\cdot 2}$$

$$\vdots$$

$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} = \frac{(-1)^n(2n-1)!!}{2^n n!} \Leftarrow$$

$$(2n-1)!! = \underbrace{(2n-1)(2n-3)(2n-5)\cdots\cdots}_{n \text{ fattori}} 1$$

$$(2n)!! = \underbrace{2n(2n-2)(2n-4)\cdots 2}_{n \text{ fattori}}$$

$$= 2^n \cdot n!$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \dots + \frac{(2n-1)!!}{(2n)!!}x^{2n} + o(x^{2n})$$

$$1$$

$$\arcsin x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \dots + \frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1} + o(x^{2n+1})$$

$$\arccos x = \frac{\pi}{2} - \arcsin x = \dots$$

$$f(x) = \operatorname{tg} x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5).$$

$$f'(x) = 1 + \operatorname{tg}^2 x \qquad f'(0) = 1$$

$$f''(x) = 2\operatorname{tg} x(1+\operatorname{tg}^2 x) \qquad f''(0) = 0$$

$$f'''(x) = \underline{2(1+\operatorname{tg}^2 x)}^2 + 2\operatorname{tg} x \, 2\operatorname{tg} x(1+\operatorname{tg}^2 x)$$

$$t = \operatorname{tg} x \qquad = 2(1+t^2)^2 + 4t^2(1+t^2) \qquad f'''(0) = 2$$

$$f^{IV}(x) = 4(1+t^2)2t + 8t(1+t^2) + \underline{4t^2\, 2t} \qquad f^{IV}(0) = 0$$

$$\underline{f^{V}(0) = 4 \cdot 2 + 8 = 16}$$

$$\frac{\cancel{16}^{\,2}}{5 \cdot 4 \cdot 3 \cdot \cancel{2}}$$

Teorema (de L'Hospital) $\left[\frac{0}{0}\right]$

$f, g: I \setminus \{x_0\} \to \mathbb{R}$ derivabili, I intervallo, $g'(x) \neq 0$

x_0 p.to di acc. per I. $x_0 \in [-\infty, +\infty]$

$$\lim_{x \to x_0} f(x) = 0, \quad \lim_{x \to x_0} g(x) = 0$$

$$\Downarrow$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \boxed{\lim_{x \to x_0} \frac{f'(x)}{g'(x)}} = \ell$$

se quest'ultimo limite esiste.

dim 1. $x_0 \in \mathbb{R}$. $\quad I = [x_0, b]$

f, g derivabili su $(x_0, b]$

possono essere estese per continuità

$$\tilde{f}(x) = \begin{cases} f(x) & \text{se } x \neq x_0 \\ 0 & \text{se } x = x_0 \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & \text{se } x \neq x_0 \\ 0 & \text{se } x = x_0 \end{cases}$$

$\tilde{f}, \tilde{g}$ sono continue perché $f(x), g(x) \to 0$ per $x \to x_0$

Cauchy $[x_0, x]$ $\quad \exists\, x_1 \in (x_0, x)$

$$\frac{f(x)}{g(x)} = \frac{\tilde{f}(x) - \overset{=0}{\tilde{f}(x_0)}}{\tilde{g}(x) - \underset{=0}{\tilde{g}(x_0)}} = \frac{\tilde{f}'(x_1)}{\tilde{g}'(x_1)} = \frac{f'(x_1)}{g'(x_1)} \to \ell$$

se $x \to x_0$ $\quad x_1 \in (x_0, x)$

$x_1 \to x_0$

$$\Rightarrow \frac{f(x)}{g(x)} \to \ell \qquad \square$$

se $x_0 = +\infty$ $\quad x \to +\infty \quad y = \frac{1}{x} \quad \underline{y \to 0^+}$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{y\to 0^+} \frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)} \overset{\hat{H}}{=} \overset{\text{x esiste}}{\lim_{y\to 0^+}} \frac{f'\left(\frac{1}{y}\right)\cancel{\left(-\frac{1}{y^2}\right)}}{g'\left(\frac{1}{y}\right)\cancel{\left(-\frac{1}{y^2}\right)}}$$

$$= \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \ell \qquad \square$$

Hospital $\left|\frac{\cancel{*}}{\infty}\right.$ è molto più complicato.

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vedi gli appunti.