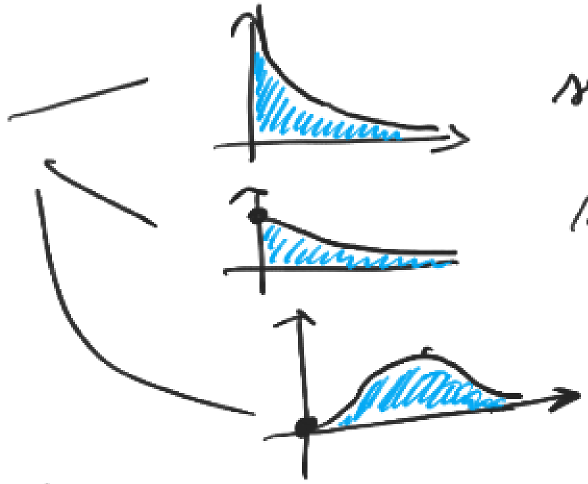
la funzione Gamma di Eulero

LA FUNZIONE Γ DI EULERO

$$\boxed{\Gamma(x) = \int_0^{+\infty} e^{-t}\, t^{x-1}\, dt}$$

Per quali $x \in \mathbb{R}$ l'integrale converge.

$f(t) = e^{-t}\, t^{x-1} \geq 0$



se $x<1$

se $x=1$

se $x>1$

Per $t \to +\infty$

$e^{-t}\, t^{x-1} \ll \frac{1}{t^2}$

l'integrale $\int_1^{+\infty}$ converge $\forall x \in \mathbb{R}$.

Per $t \to 0^+$

$e^{-t}\, t^{x-1} \sim t^{x-1}$ $\quad \int_0^1 t^{x-1}$ converge

$\Updownarrow$

$x-1 > -1$

$x > 0$.

$\Gamma(x)$ è ben definita se $x>0$.

$\Gamma : (0, +\infty) \to \mathbb{R}$.

Alcuni valori:

$$\Gamma(1) = \int_0^{+\infty} e^{-t}\, dt = \left[-e^{-t}\right]_0^{+\infty} = 0 - (-1) = 1.$$

$$\Gamma(2) = \int_0^{+\infty} e^{-t}\, t\, dt = \left[-e^{-t} \cdot t\right]_0^{+\infty} - \int_0^{+\infty} (-e^{-t}) \cdot 1\, dt$$

$$= 0 + \int_0^{+\infty} e^{-t}\, dt = 1.$$

$$\Gamma(x+1) = \int_0 e^{-t}\, t^{x}\, dt = \left[-e^{-t}\cdot t^{x}\right]_0 - \int_0 (-e^{-t})\, x\, t\, dt$$

$$= 0 - 0 + x\int_0^{+\infty} e^{-t}\, t^{x-1}\, dt = x\,\Gamma(x).$$

$x>0$

$$\boxed{\Gamma(x+1) = x\,\Gamma(x)} \qquad \forall x>0$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot 1$$

$$\vdots$$

$$\boxed{\Gamma(n+1) = n!}$$

Per induzione $\underline{\Gamma(n+1)=n!}$

(i) $\Gamma(1) = 1 = 0!$

(ii) se $\Gamma(n+1) = n!$

$$\Gamma(n+2) = (n+1)\cdot\Gamma(n+1)$$
$$= (n+1)\cdot n! = (n+1)!$$

$\Gamma(x+1)$ estende il fattoriale ai numeri reali >-1.

Potremmo definire:

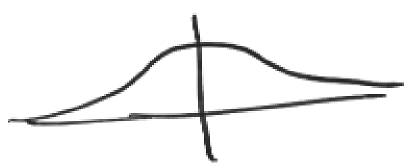
$$x! = \Gamma(x+1) \qquad (\text{per } x>-1).$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^{+\infty} e^{-t}\, t^{-\frac{1}{2}}\, dt = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}}\, dt$$

$$= \int_0^{+\infty} \frac{e^{-y^2}}{\cancel{y}}\, 2\cancel{y}\, dy = 2\int_0^{+\infty} e^{-y^2}\, dy$$

$y = \sqrt{t}$
$t = y^2$
$dt = 2y\,dy$

$$= \int_{-\infty}^{+\infty} e^{-y^2}\, dy$$



$$\left(= \sqrt{\pi}\right) \leftarrow \text{prossimo anno.}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$\vdots$