integrale del test settimanale

$$\int_0^1 \frac{e^x - 1}{(1+e^{x-1})^2}\,dx =$$

$$e^{x-1} = \frac{e^x}{e}$$

$$y = e^x$$
$$x = \ln y$$
$$dx = \frac{dy}{y}$$

$$= \int_1^e \frac{y-1}{\left(1+\frac{y}{e}\right)^2} \cdot \frac{1}{y}\,dy$$

oppure

$$\left(\frac{A}{1+\frac{y}{e}} + \frac{B}{y} + \frac{C}{\left(1+\frac{y}{e}\right)^2} \right)$$

$$= \int_1^e \left[\frac{A}{1+\frac{y}{e}} + \frac{B}{y} + \left(\frac{C}{1+\frac{y}{e}} \right)' \right] dy$$

$$\frac{A\left(1+\frac{y}{e}\right)\cdot y + B\left(1+\frac{y}{e}\right)^2 - y\cdot\frac{C}{e}}{\left(1+\frac{y}{e}\right)^2 \cdot y}$$

$$\frac{\left(\frac{A}{e}+\frac{B}{e^2}\right)y^2 + \left(A + \frac{2B}{e} - \frac{C}{e}\right)y + B}{\%}$$

$$\begin{cases} A + \frac{B}{e} = 0 \\ \ldots \end{cases} \qquad \begin{cases} A = \frac{1}{e} \\ \ldots \end{cases}$$

$$= \int_1^e \left[\frac{1}{e+y} - \frac{1}{y} - \left(\frac{1+e}{1+\frac{y}{e}} \right)' \right] dy$$

$$= \left[\ln(e+y) - \ln y - \frac{1+e}{1+\frac{y}{e}} \right]_1^e$$

$$= \left(\ln(2e) - \ln e - \frac{1+e}{1+1} \right) - \left(\ln(e+1) - \ln 1 - \frac{1+e}{1+\frac{1}{e}} \right)$$

$$= \ln 2 - \frac{1+e}{2} - \ln(e+1) + \frac{e+e^2}{e+1}$$

$$= \ln 2 - \ln(e+1) - \frac{1+e}{2} + e$$

$$= \ln \frac{2}{e+1} + \frac{e-1}{2} \qquad \text{OK!}$$

III esercizio

$$\int_0^{\pi/3} \frac{1}{\cos x} \, dx \qquad \cos x = \pm\sqrt{1-\sin^2 x}$$

INTEGRANDI CHE SI RICONDUCONO A FUNZIONI RAZIONALI

R razionale

I $\quad R(e^{\lambda x}) \qquad y = e^{\lambda x}$

II $\quad R(\sin^2 x, \sin x \cos x, \cos^2 x)$

$$t = \operatorname{tg} x$$

III $\quad R(\sin x, \cos x) \qquad t = \operatorname{tg} \frac{x}{2}$

IV $\quad R(\sqrt[n]{x})$

$$\int R(\sqrt[n]{x})\, dx = \int \underbrace{R(y)\, n y^{n-1}}_{\text{è razionale}} dy \qquad \begin{cases} y = \sqrt[n]{x} \\ x = y^n \\ dx = n y^{n-1} \end{cases}$$

Esempio

$$\int \frac{\sqrt[4]{x}}{\sqrt{x} + \sqrt[3]{x}}\, dx$$

$x^{1/4}, x^{1/2}, x^{1/3}$ sono potenze di $x^{\frac{1}{12}} = \sqrt[12]{x}$

$$\begin{cases} y = \sqrt[12]{x} \\ x = y^{12} \\ dx = 12 y^{11}\, dy \end{cases}$$

$\|$

$$\int \frac{y^3}{y^6+y^4} \cdot 12y^{11}\,dy = 12\int \frac{y^{10}}{y^2+1}\,dy$$

$$= 12\left[\frac{y^9}{9} - \frac{y^7}{7} + \frac{y^5}{5} - \frac{y^3}{3} + y\right] - 12\int \frac{1}{y^2+1}\,dy$$

$$= 12\left[\frac{\sqrt[12]{x^9}}{9} - \frac{\sqrt[12]{x^7}}{7} + \frac{\sqrt[12]{x^5}}{5} - \frac{\sqrt[12]{x^3}}{3} + \sqrt[12]{x} - \operatorname{arctg}\sqrt[12]{x}\right]$$

$\square$

$$\begin{array}{l|l}
y^{10} & y^2+1 \\ \hline
y^{10}+y^8 & y^8-y^6+y^4-y^2+1 \\ \hline
-y^8 & \\
-y^8-y^6 & \\ \hline
y^6 & \\
y^6+y^4 & \\ \hline
-y^4 & \\
-y^4-y^2 & \\ \hline
y^2 & \\
y^2+1 & \\ \hline
-1 &
\end{array}$$

$$y^{10} = (y^8-y^6+y^4-y^2+1)(y^2+1)-1$$

4

$D\left(e^x+\sin x\right)=e^x+\cos x$

INTEGRALI DA 15''

$\int \frac{dy}{y}=\ln y$

(1) $\int \frac{e^x+\cos x}{e^x+\sin x}\,dx$

LABELLA

$\ln\left(e^x+\sin x\right)$

(2) $\int \frac{2}{\sqrt{x}-\sqrt{x+2}}\,dx$

TEOFILO

$-\frac{2}{3}x^{3/2}-\frac{2}{3}(x+2)^{3/2}$

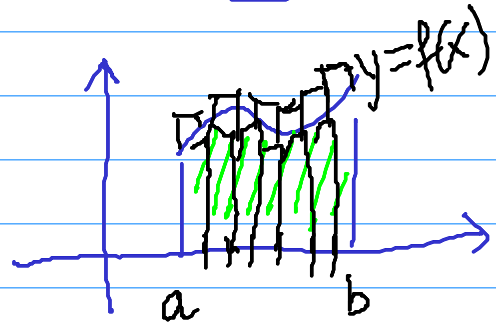
(3) $\int \frac{\ln \ln x}{x}\,dx$

PELLITTERI

$(\ln x)\cdot\ln\ln x-\ln x$

$y=\ln x \longrightarrow \int \ln y\,dy=y\ln y-y$

INTEGRALI IMPROPRI (o GENERALIZZATO)

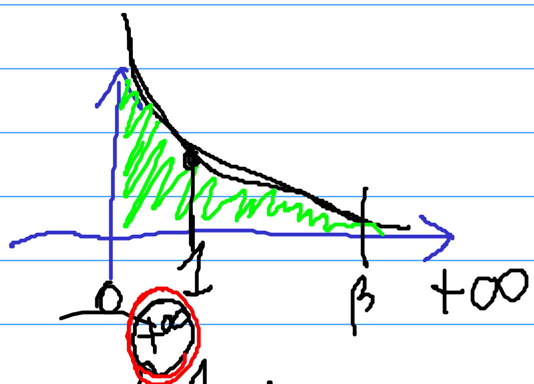


$$\int_a^b f(x)\,dx$$

INTEGRALE di RIEMANN

$[a,b]$ chiuso e limitato

f limitata



Es $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ $\qquad \displaystyle\int_0^1 \frac{1}{x^2}\,dx$ analogamente su $(a,b]$

Def Sia $f : [a,b) \to \mathbb{R}$, f localmente Riemann-integrabile

integrale improprio

$$(*) \quad \boxed{\int_a^b f(x)\,dx = \lim_{\beta \to b^-} \int_a^\beta f(x)\,dx}$$

def $f : A \to \mathbb{R}$ si dice essere localmente Riemann integrabile se f è Riemann integrabile e limitata su ogni intervallo chiuso e limitato $[\alpha,\beta] \subseteq A$.

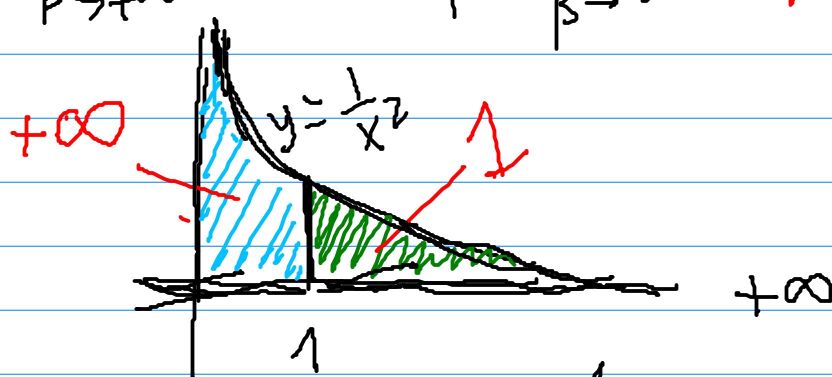
(Ad esempio basta che f sia continua)

def. Diremo che f è integrabile in senso improprio se l'integrale improprio (*) esiste ed è finito. (l'integrale converge)

L'integrale improprio può esistere e non essere finito. In tal caso diremo che l'integrale diverge.

Es
$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{\beta\to+\infty} \int_1^{\beta} \frac{1}{x^2}\,dx$$

$$= \lim_{\beta\to+\infty} \left[-\frac{1}{x}\right]_1^{\beta} = \lim_{\beta\to+\infty} \left[\overset{0}{-\frac{1}{\beta}} + \frac{1}{1}\right] = 1$$



Es
$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\alpha\to0^+} \int_\alpha^1 \frac{1}{x^2}\,dx = \lim_{\alpha\to0^+} \left[-\frac{1}{x}\right]_\alpha^1$$

$$= \lim_{\alpha\to0^+} \left(-1 + \frac{1}{\alpha}\right) = +\infty$$