Chapter 2.6
Complex geodesics

In the previous chapters we dealt with two of the main themes of this book, iteration theory and common fixed ponts; now we would like to find a way of attacking the last one. For inspiration, let us turn to our model example, the ball. A careful examination of the proof of Theorem 2.2.29 reveals that a fundamental role is played by the possibility of imbedding any radius $t \mapsto tx$ into an analytic disk $\zeta \mapsto \zeta x$, where $x \in \partial B^n$, $t \in (0,1)$ and $\zeta \in \Delta$. Clearly, if $D \subset\subset \mathbf{C}^n$ is any C^2 domain, $x \in \partial D$ and $\sigma\colon [0,1] \to D$ is a sufficiently well-behaved real-analytic curve tending non-tangentially to x, we can find a holomorphic map $\varphi\colon \Delta \to D$ such that $\varphi(t) = \sigma(t)$ for every $t \in (0,1)$; the point is that in B^n we can find σ and φ strongly connected to the geometry of B^n, whereas generic σ and φ are not.

The leading idea is the (apparently harmless) observation that the map $\zeta \mapsto \zeta x$ is an isometry for the Poincaré and Bergmann distances; hence we can try to find, given a domain $D \subset\subset \mathbf{C}^n$ and a point $x \in \partial D$, a map $\varphi \in \mathrm{Hol}(\Delta, D)$ which is an isometry between the Poincaré distance on Δ and the Kobayashi distance on D (in short, a *complex geodesic*), and such that φ extends continuously to $\partial\Delta$, $\varphi(\partial\Delta) \subset \partial D$, $\varphi(\overline{\Delta})$ is transversal to ∂D and $\varphi(1) = x$. It turns out that in strongly convex domains such a map exists, and it is uniquely determined by its value in 0. Using these complex geodesics, in the next chapter we shall be able to deal with angular derivatives, ending the book.

But let us now describe what we shall do in this chapter. We shall begin by formally introducing the notion of complex geodesic, and studying existence and uniqueness in the unit ball of a norm in $\mathbf{C}^n$, providing a first insight into the general situation. After a digression on Banach spaces theory, we shall address the question of the existence of complex geodesics in convex domains, proving a deep theorem of Royden and Wong: every pair of points of a bounded convex domain is contained in the image of a complex geodesic. As a corollary, we shall show that in a bounded convex domain the Carathéodory and Kobayashi distances coincide.

In the last two sections, we shall describe Lempert's theory of complex geodesics in strongly convex domains, proving that every complex geodesic in a strongly convex domain D extends continuously to the boundary, and that every pair of points of $\overline{D}$ is contained in the image of a (essentially) unique complex geodesic. We shall also deal with the boundary smoothness of complex geodesics in strongly convex C^r domains, and we shall end this chapter proving that in a strongly convex C^3 domain big and small horospheres are one and the same thing, a fact that will be very handy in chapter 2.7.

A final warning for the novice reader: we shall consistently use the basic theory of H^p spaces of the disk. The main facts are collected for easy reference in Theorem 2.6.12; proofs and general framing can be found, for instance, in Hoffman [1962] and Duren [1970]. We shall also freely use standard facts of functional analysis; a good reference book is Brezis [1983].

2.6.1 *Definitions and examples*

Let X be a complex manifold, and take $\varphi \in \mathrm{Hol}(\Delta, X)$. Then φ is a *C-extremal map* with respect to two distinct points z_0, $z_1 \in X$ if there are ζ_0, $\zeta_1 \in \Delta$ such that $\varphi(\zeta_0) = z_0$, $\varphi(\zeta_1) = z_1$ and

$$\omega(\zeta_0, \zeta_1) = c_X(z_0, z_1);$$

φ is an *infinitesimal C-extremal map* with respect to $z_0 \in X$ and a non-zero tangent vector $v \in T_{z_0}X$ if there are $\zeta_0 \in \Delta$ and $\xi \in \mathbf{C}$ such that $\varphi(\zeta_0) = z_0$, $d\varphi_{\zeta_0}(\xi) = v$ and

$$|\xi|_{\zeta_0} = \gamma_X(z_0; v),$$

where $|\xi|_{\zeta_0}$ is the length with respect to the Poincaré metric of ξ considered as tangent vector to Δ at ζ_0. We define analogously the concepts of *K-extremal maps* and *infinitesimal K-extremal maps*, replacing the Carathéodory metric and distance by their Kobayashi relatives. Finally, a *complex geodesic* (respectively, a *complex C-geodesic*) is a map $\varphi \in \mathrm{Hol}(\Delta, X)$ which is K-extremal (C-extremal) with respect to $\varphi(\zeta_0)$ and $\varphi(\zeta_1)$ for any pair of points ζ_0, $\zeta_1 \in \Delta$; an *infinitesimal complex geodesic* (respectively, an *infinitesimal complex C-geodesic*) is a map $\varphi \in \mathrm{Hol}(\Delta, X)$ which is infinitesimal K-extremal (infinitesimal C-extremal) with respect to $\varphi(\zeta_0)$ and $d\varphi_{\zeta_0}(\xi)$ for any $\zeta_0 \in \Delta$ and $\xi \in \mathbf{C}$. In other words, a complex geodesic is an isometry between ω and k_X, and a complex C-geodesic is an isometry between ω and c_X. Let $\varphi \in \mathrm{Hol}(\Delta, X)$ be a (infinitesimal) complex (C-)geodesic; we shall say that φ is *passing through* $z_0 \in X$ if $z_0 \in \varphi(\Delta)$, and that φ is *tangent to* $v \in T_{z_0}X$ if $z_0 = \varphi(z_0)$ and $v = d\varphi_{\zeta_0}(\xi)$ for some $\zeta_0 \in \Delta$ and $\xi \in \mathbf{C}$. The image of a complex geodesic will be sometimes called a *geodesic disk*.

A first very easy lemma describing the properties of the complex geodesics is:

Lemma 2.6.1: *Let X be a complex manifold. Then*
(i) *every complex geodesic (C-geodesic) is a proper injective map of Δ into X;*
(ii) *every complex C-geodesic is a complex geodesic;*
(iii) *every infinitesimal complex C-geodesic is an infinitesimal complex geodesic.*

Proof: (i) is obvious. If $\varphi \in \mathrm{Hol}(\Delta, X)$ and ζ_1, $\zeta_2 \in \Delta$, then

$$c_X\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) \le k_X\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) \le \omega(\zeta_1, \zeta_2),$$

and (ii) follows. (iii) is proved in the same way, **q.e.d.**

A complex geodesic is essentially given by its image:

Proposition 2.6.2: *Let X be a complex manifold, and φ, $\psi \in \mathrm{Hol}(\Delta, X)$ be two complex (C-)geodesics such that $\varphi(\Delta) = \psi(\Delta)$. Then there is $\gamma \in \mathrm{Aut}(\Delta)$ such that $\psi = \varphi \circ \gamma$.*

Proof: By Lemma 2.6.1.(i), $\gamma = \varphi^{-1} \circ \psi \colon \Delta \to \Delta$ is a well-defined homeomorphism. Moreover, if $d\varphi_{\zeta_0} \neq 0$, then γ is holomorphic in a neighbourhood of $\gamma^{-1}(\zeta_0)$. Since $\{\zeta \in \Delta \mid d\varphi_{\gamma(\zeta)} = 0\}$ is discrete in Δ and γ is bounded, by the Riemann extension theorem γ is holomorphic everywhere, and hence $\gamma \in \mathrm{Aut}(\Delta)$, **q.e.d.**

We shall say that the image $\varphi(\Delta)$ of a complex (C-)geodesic $\varphi\colon\Delta\to X$ determines the map φ *up to parametrization*, and we shall often identify a complex geodesic and its image.

There is a close relationship between C-extremal maps and complex C-geodesics:

Proposition 2.6.3: *Let X be a complex manifold, $\varphi\in\mathrm{Hol}(\Delta,X)$ and $\zeta_0\in\Delta$. Then:*
(i) *If φ is C-extremal with respect to $\varphi(\zeta_0)$ and $\varphi(\zeta_1)$, where $\zeta_1\in\Delta$ is different from ζ_0, then φ is a complex C-geodesic;*
(ii) *If φ is infinitesimal C-extremal with respect to $\varphi(\zeta_0)$ and $v\in T_{\varphi(\zeta_0)}X$, $v\neq 0$, then φ is a complex C-geodesic.*

Proof: (i) By definition there is a sequence $\{f_\nu\}\subset\mathrm{Hol}(X,\Delta)$ such that

$$\lim_{\nu\to\infty}\omega\big(f_\nu\big(\varphi(\zeta_0)\big),f_\nu\big(\varphi(\zeta_1)\big)\big)=\omega(\zeta_0,\zeta_1).$$

Up to a subsequence we can assume that $\{f_\nu\circ\varphi\}$ tends to a map $g\in\mathrm{Hol}(\Delta,\Delta)$ such that

$$\omega\big(g(\zeta_0),g(\zeta_1)\big)=\omega(\zeta_0,\zeta_1).$$

By the Schwarz-Pick lemma, $g\in\mathrm{Aut}(\Delta)$, and therefore for all $\zeta\in\Delta$

$$\omega(\zeta_0,\zeta)\geq c_X\big(\varphi(\zeta_0),\varphi(\zeta)\big)\geq\lim_{\nu\to\infty}\omega\big(f_\nu\big(\varphi(\zeta_0)\big),f_\nu\big(\varphi(\zeta)\big)\big)=\omega\big(g(\zeta_0),g(\zeta)\big)=\omega(\zeta_0,\zeta).$$

So φ is C-extremal with respect to $\varphi(\zeta_0)$ and $\varphi(\zeta)$ for any $\zeta\in\Delta$. Repeating this argument once again we see that φ is a complex C-geodesic.

(ii) Up to parametrization, we can assume $\zeta_0=0$; therefore there is $\xi\in\mathbf{C}$ such that $d\varphi_0(\xi)=v$ and

$$|\xi|=\gamma_X(\varphi(0);v).$$

By definition, there is a sequence $\{f_\nu\}\subset\mathrm{Hol}(X,\Delta)$ such that $f_\nu\big(\varphi(0)\big)=0$ and

$$\lim_{\nu\to\infty}|(df_\nu)_{\varphi(0)}v|=|\xi|.$$

Up to a subsequence we can assume that $\{f_\nu\circ\varphi\}$ tends to a map $g\in\mathrm{Hol}(\Delta,\Delta)$ such that $g(0)=0$ and $|g'(0)|=1$, for $v=d\varphi_0(\xi)$. By Schwarz's lemma, $g\in\mathrm{Aut}(\Delta)$ and therefore for all $\zeta\in\Delta$

$$\omega(0,\zeta)\geq c_X\big(\varphi(0),\varphi(\zeta)\big)\geq\lim_{\nu\to\infty}\omega\big(f_\nu\big(\varphi(0)\big),f_\nu\big(\varphi(\zeta)\big)\big)=\omega\big(g(0),g(\zeta)\big)=\omega(0,\zeta);$$

so φ is C-extremal with respect to two points and, by (i), the assertion follows, **q.e.d.**

Clearly, the first problem we must address is the existence of complex geodesics. Lemma 2.6.1 immediately shows that there are no complex geodesics in compact manifolds, or in Riemann surfaces other than Δ; indeed, as we shall fully appreciate in the next sections, the existence of complex geodesics is not at all a trivial property. Therefore we decided to devote the rest of this section to the study of a model case, the unit balls for a norm in $\mathbf{C}^n$, where the existence is immediate, and where we can also easily investigate the uniqueness of complex geodesics passing through two given points.

The existence is immediately provided by

Corollary 2.6.4: *Let $B \subset\subset \mathbf{C}^n$ be the unit ball for a norm $\|\cdot\|\colon \mathbf{C}^n \to \mathbf{R}^+$. Then for any $x \in \partial B$ the map $\varphi(\zeta) = \zeta x$ is a complex C-geodesic.*

Proof: This follows from Propositions 2.3.5 and 2.6.3, **q.e.d.**

In particular, if B is the unit ball for some norm on $\mathbf{C}^n$, then for any $z \in B$ there is always a complex geodesic passing through 0 and z. To study the uniqueness, we need some machinery.

Lemma 2.6.5: *Let $\varphi \in \mathrm{Hol}(\Delta, \Delta)$. Then for all $\zeta \in \Delta$ we have*

$$2\,|\zeta|\,|\varphi(0)| + (1 - |\zeta|)|\varphi(\zeta) - \varphi(0)| \leq 2\,|\zeta|. \tag{2.6.1}$$

Proof: The Schwarz-Pick lemma yields

$$\left|\frac{\varphi(\zeta) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(\zeta)}\right| \leq |\zeta|,$$

for all $\zeta \in \Delta$. Since

$$|1 - \overline{\varphi(0)}\varphi(\zeta)| \leq 1 - |\varphi(0)|^2 + |\varphi(0)|\,|\varphi(\zeta) - \varphi(0)| \leq 2(1 - |\varphi(0)|) + |\varphi(0)|\,|\varphi(\zeta) - \varphi(0)|,$$

then

$$|\varphi(\zeta) - \varphi(0)| \leq 2|\zeta|(1 - |\varphi(0)|) + |\zeta|\,|\varphi(0)|\,|\varphi(\zeta) - \varphi(0)|,$$

and (2.6.1) follows, **q.e.d.**

Lemma 2.6.6: *Let $B \subset\subset \mathbf{C}^n$ be the unit ball for a norm $\|\cdot\|\colon \mathbf{C}^n \to \mathbf{R}^+$ and let $\varphi \in \mathrm{Hol}(\Delta, \mathbf{C}^n)$ be such that $\varphi(\Delta) \subset \overline{B}$. Then*

$$\|\varphi(0) + \lambda\big(\varphi(\zeta) - \varphi(0)\big)\| \leq 1 \tag{2.6.2}$$

for all $\zeta \in \Delta^*$ and $\lambda \in \mathbf{C}$ such that $|\lambda| \leq (1 - |\zeta|)/2|\zeta|$.

Proof: If $n = 1$ the assertion follows from (2.6.1), for $B = \Delta_r$ for some $r > 0$. For $n > 1$ assume, by contradiction, that (2.6.2) does not hold for some $\zeta \in \Delta^*$ and $\lambda \in \mathbf{C}$ such that $|\lambda| \leq (1 - |\zeta|)/2|\zeta|$. But then there is a complex linear form $\Lambda\colon \mathbf{C}^n \to \mathbf{C}$ such that $|\Lambda(z)| \leq \|z\|$ for all $z \in \mathbf{C}^n$ and

$$\Lambda\Big(\varphi(0) + \lambda\big(\varphi(\zeta) - \varphi(0)\big)\Big) = \|\varphi(0) + \lambda\big(\varphi(\zeta) - \varphi(0)\big)\| > 1. \tag{2.6.3}$$

But $\Lambda \circ \varphi$ sends Δ into $\overline{\Delta}$, and so (2.6.3) contradicts (2.6.1), **q.e.d.**

Now we can prove a sort of maximum principle. Let D be a domain in $\mathbf{C}^n$; we shall say that a point $x_0 \in \partial D$ is a *complex extreme point* of $\overline{D}$ if the only vector $y \in \mathbf{C}^n$ such that $x_0 + \Delta y \subset \overline{D}$ is $y = 0$. Then

Corollary 2.6.7: *Let $B \subset\subset \mathbf{C}^n$ be the unit ball for a norm on $\mathbf{C}^n$, and let $\varphi \in \mathrm{Hol}(\Delta, \mathbf{C}^n)$ be such that $\varphi(\Delta) \subset \overline{B}$. Then if $\varphi(\Delta)$ contains a complex extreme point of $\overline{B}$, φ is constant. Conversely, if $x \in \partial B$ is not a complex extreme point, then there is $\varphi \in \mathrm{Hol}(\Delta, \mathbf{C}^n)$ such that $\varphi(\Delta) \subset \partial B$ and $\varphi(0) = x$.*

Proof: If $\varphi(\Delta)$ contains a complex extreme point, that we can assume to be $\varphi(0)$, then (2.6.2) implies that φ is constant. The converse follows from the definitions, recalling Corollary 2.1.11, **q.e.d.**

This is what we need for

Proposition 2.6.8: *Let $B \subset\subset \mathbf{C}^n$ be the unit ball for a norm $\|\cdot\|\colon \mathbf{C}^n \to \mathbf{R}^+$, and choose a point $z \in B \setminus \{0\}$. Then up to parametrization $\varphi(\zeta) = \zeta z/\|z\|$ is the unique complex geodesic passing through 0 and z iff $z/\|z\|$ is a complex extreme point of $\overline{B}$.*

Proof: Let $\varphi \in \mathrm{Hol}(\Delta, B)$ be a complex geodesic passing through 0 and z, and assume that $z/\|z\|$ is a complex extreme point of $\overline{B}$. Up to parametrization, we can assume $\varphi(0) = 0$; then, by Proposition 2.3.5, we have $\|\varphi(\zeta)\| = |\zeta|$ for every $\zeta \in \Delta$; in particular, we can also suppose $\varphi(\|z\|) = z$. Define $h\colon \Delta \to \mathbf{C}^n$ by $h(\zeta) = \varphi(\zeta)/\zeta$; clearly, h is holomorphic and $h(\Delta) \subset \partial B$. But $z/\|z\| \in h(\Delta)$; hence, by Corollary 2.6.7, h is constant, and so $\varphi(\zeta) = \zeta z/\|z\|$.

Conversely, assume that $x = z/\|z\|$ is not a complex extreme point, and let $y \in \mathbf{C}^n \setminus \{0\}$ be such that $x + \zeta y \in \overline{B}$ for all $\zeta \in \Delta$. Actually, by Corollary 2.1.11, $x + \zeta y \in \partial B$ for all $\zeta \in \Delta$.

Choose $\lambda \in \Delta$, and define $\varphi_\lambda \in \mathrm{Hol}(\Delta, \mathbf{C}^n)$ by

$$\varphi_\lambda(\zeta) = \zeta \left[x + \lambda \frac{\zeta - \|z\|}{1 - \|z\|\zeta} y \right].$$

Then $\varphi_\lambda(\Delta) \subset B$, $\varphi_\lambda(0) = 0$ and $\varphi_\lambda(\|z\|) = z$; hence, by Propositions 2.3.5 and 2.6.3, every φ_λ is a complex geodesic passing through 0 and z, **q.e.d.**

For instance, if B^n is the euclidean unit ball of $\mathbf{C}^n$, Proposition 2.6.8 determines all the complex geodesics of B^n:

Corollary 2.6.9: *The unique geodesic disk passing through two distinct points z_0 and z_1 of B^n is the one-dimensional affine subset of B^n containing z_0 and z_1.*

Proof: Since the automorphisms of B^n send affine lines into affine lines, we can assume $z_0 = 0$. But then the assertion follows from Proposition 2.6.8, because every point of ∂B^n is a complex extreme point of $\overline{B^n}$, **q.e.d.**

So every pair of points of B^n is contained in a unique geodesic disk, and every complex geodesic in B^n extends continuously to the boundary. As we shall see later on, this situation is typical of strongly convex domains. In weakly convex domains, however, we cannot expect such a nice behavior, as indicated by

Proposition 2.6.10: *A map $\varphi \in \mathrm{Hol}(\Delta, \Delta^n)$ is a complex geodesic iff at least one component of φ is an automorphism of Δ.*

Proof: Write $\varphi = (\varphi_1, \dots, \varphi_n)$ and fix $\zeta_0 \in \Delta^*$. Then, by Proposition 2.6.3 and Corollary 2.3.7, φ is a complex geodesic iff

$$\omega(0, \zeta_0) = \max_{j=1,\dots,n} \omega\big(\varphi_j(0), \varphi_j(\zeta_0)\big),$$

and the assertion follows from the Schwarz-Pick lemma, **q.e.d.**

Therefore, though every pair of distinct points of Δ^n is still contained in a geodesic disk, there is absolutely no uniqueness, and there are complex geodesics that cannot be extended continuously to the boundary.

2.6.2 *An extremal problem*

In the next section we shall prove the existence of complex geodesics in convex domains showing that every K-extremal map is the solution of a linear extremal problem in a suitable complex Banach space. In this section we shall describe this extremal problem in its full generality; we shall also collect a few facts about H^p spaces that we shall need later.

Let X be a complex Banach space with dual X^*. A function $P\colon X \to \mathbf{R}^+$ is a *Minkowski functional* if

(i) $P(x+y) \le P(x) + P(y)$ for all $x, y \in X$;
(ii) $P(\lambda x) = \lambda P(x)$ for all $x \in X$ and $\lambda \ge 0$;
(iii) there exists $c > 0$ such that $c^{-1}\|x\| \le P(x) \le c\|x\|$ for all $x \in X$, where $\|\cdot\|$ is the norm in X.

In particular, $P(0) = 0$ and for every $r > 0$ the set $\{x \in X \mid P(x) < r\}$ is a convex open neighbourhood of 0.

If P is a Minkowski functional on X, the function $P^*\colon X^* \to \mathbf{R}^+$ given by

$$\forall u \in X^* \qquad P^*(u) = \sup_{x \ne 0} \frac{\mathrm{Re}\, u(x)}{P(x)}$$

is called the *dual Minkowski functional*. P^* is a Minkowski functional on X^*, of course.

Let Y be a subspace of X, and $x_0 \in X$ a point not in the closure of Y. Set

$$m_P = m_P(x_0, Y) = \inf_{y \in Y} P(x_0 + y),$$

and

$$M_P = M_P(x_0, Y) = \inf\big\{P^*(u) \bigm| u \in Y^\circ \text{ and } \mathrm{Re}\, u(x_0) = 1\big\},$$

where $Y^\circ = \{u \in X^* \mid u|_Y \equiv 0\}$ is the *annihilator* of Y. Since $x_0 \notin \overline{Y}$, it is clear that $m_P > 0$. Then the *extremal problem* associated to the affine subspace $x_0 + Y$ is to find $x \in x_0 + Y$ such that $P(x) = m_P(x_0, Y)$. Any solution of the extremal problem will be called an *extremal element* of $x_0 + Y$. Analogously, the *dual extremal problem* is to find $u \in Y^\circ$ with $\mathrm{Re}\, u(x_0) = 1$ such that $P^*(u) = M_P(x_0, Y)$, and any solution of the dual extremal problem will be called a *dual extremal element* of $x_0 + Y$.

The main fact regarding these extremal problems is the following *duality principle:*

Theorem 2.6.11: *Let X be a complex Banach space, $P\colon X \to \mathbf{R}^+$ a Minkowski functional on X, Y a subspace of X and $x_0 \in X \setminus \overline{Y}$. Then:*
(i) $m_P(x_0, Y)M_P(x_0, Y) = 1$;
(ii) *there is always a dual extremal element of $x_0 + Y$;*
(iii) *if $x \in x_0 + Y$ and $u \in Y^\circ$ are such that $\operatorname{Re} u(x) = P(x)P^*(u) = 1$, then x is an extremal element of $x_0 + Y$ and u is a dual extremal element of $x_0 + Y$.*

Proof: Let $\widetilde{Y} = \mathbf{R}x_0 \oplus Y$ be the linear span over $\mathbf{R}$ of x_0 and Y. Define a $\mathbf{R}$-linear functional $f\colon \widetilde{Y} \to \mathbf{R}$ by setting $f(\lambda x_0 + y) = \lambda$ for all $\lambda \in \mathbf{R}$ and $y \in Y$. Since

$$P(\lambda x_0 + y) = \lambda P(x_0 + y/\lambda) \geq \lambda\, m_P$$

if $\lambda > 0$, and $P(\lambda x_0 + y) \geq 0$ for any $\lambda \in \mathbf{R}$, we conclude that

$$f(x) \leq m_P^{-1} P(x)$$

for all $x \in \widetilde{Y}$. By the Hahn-Banach theorem, we can extend f to an $\mathbf{R}$-linear functional $F\colon X \to \mathbf{R}$ such that $F(x) \leq m_P^{-1}P(x)$ for all $x \in X$. Now define a $\mathbf{C}$-linear functional $u\colon X \to \mathbf{C}$ by

$$u(x) = F(x) - iF(ix).$$

If $x \in X$ then

$$|u(x)| \leq |F(x)| + |F(ix)| \leq 2\, m_P^{-1} c \|x\|,$$

where c is the constant appearing in the definition of P, and so $u \in X^*$. Moreover, if $y \in Y$ then

$$u(y) = F(y) - iF(iy) = f(y) - if(iy) = 0,$$

for Y is a complex subspace of X, and so $u \in Y^\circ$. Furthermore $\operatorname{Re} u(x_0) = f(x_0) = 1$; thus u is a candidate dual extremal element.

By definition of u,

$$M_P \leq P^*(u) \leq m_P^{-1}; \tag{2.6.4}$$

in particular, $m_P M_P \leq 1$. On the other hand, for any $v \in Y^\circ$ with $\operatorname{Re} v(x_0) = 1$ we have

$$\forall y \in Y \qquad P(x_0 + y)P^*(v) \geq \operatorname{Re} v(x_0 + y) = \operatorname{Re} v(x_0) = 1;$$

in particular, $m_P M_P \geq 1$. But then $m_P M_P = 1$, and (2.6.4) implies that u is a dual extremal element of $x_0 + Y$, completing the proof of (i) and (ii).

Finally, let $x \in x_0 + Y$ and $u \in Y^\circ$ be such that $\operatorname{Re} u(x) = P(x)P^*(u) = 1$. Then $\operatorname{Re} u(x_0) = \operatorname{Re} u(x) = 1$, and so $P^*(u) \geq M_P$ and $P(x) \geq m_P$. On the other hand,

$$m_P M_P \leq m_P P^*(u) \leq P(x)P^*(u) = 1 = m_P M_P;$$

hence $P^*(u) = M_P$ and $P(x) = m_P$, **q.e.d.**

For the sake of clarity, we describe now the Banach spaces where we shall apply these techniques; for all the unproved assertions we refer to Duren [1970] and Hoffman [1962]. We shall denote by $\mathcal{M}(\partial\Delta)$ the space of complex Radon measures on $\partial\Delta$; it is the dual of the space $C^0(\partial\Delta)$ endowed with the supremum norm. As usual, there is an isometric immersion of $L^1(\partial\Delta)$ into $\mathcal{M}(\partial\Delta)$: a function $h \in L^1(\partial\Delta)$ is identified with the Radon measure $h\,d\theta/2\pi$ on $\partial\Delta$, that is with the linear functional given by

$$\forall f \in C^0(\partial\Delta) \qquad h(f) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})f(e^{i\theta})\,d\theta, \tag{2.6.5}$$

where $d\theta$ is the Lebesgue measure on $\partial\Delta$.

We shall denote by $H^1(\Delta)$ the space of functions $f \in \mathrm{Hol}(\Delta, \mathbf{C})$ such that

$$\|f\|_1 = \sup_{r\in(0,1)} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta < \infty, \tag{2.6.6}$$

and by $H^\infty(\Delta)$ the space of bounded holomorphic functions on Δ. $H^1(\Delta)$ is a Banach space with the norm defined by (2.6.6), and $H^\infty(\Delta)$ is a Banach space with the supremum norm; clearly, $H^\infty(\Delta)$ is contained in $H^1(\Delta)$.

Let $A^0(\Delta) \subset H^\infty(\Delta)$ be the subset of holomorphic functions on Δ which extend continuously to $\overline{\Delta}$. Every element of $A^0(\Delta)$ is completely determined by its restriction to $\partial\Delta$; hence we can identify $A^0(\Delta)$ with a closed subspace of $C^0(\partial\Delta)$. The classical theorem by F. and M. Riesz tells that this can be done for $H^1(\Delta)$ too, as described in the following theorem, summarizing what we shall need to know about $H^1(\Delta)$:

Theorem 2.6.12: (i) *For any $h \in H^1(\Delta)$ the limit*

$$h^*(e^{i\theta}) = \lim_{r\to 1} h(re^{i\theta}) \tag{2.6.7}$$

exists for almost all $\theta \in \mathbf{R}$, and $h \mapsto h^*$ is an isometric immersion of $H^1(\Delta)$ into $L^1(\partial\Delta)$, and thus onto a closed subspace of $\mathcal{M}(\partial\Delta)$;
(ii) *If $h \in H^1(\Delta)$ then h is the Poisson integral of h^*; in particular,*

$$h(0) = \frac{1}{2\pi}\int_0^{2\pi} h^*(e^{i\theta})\,d\theta;$$

(iii) *If $h \in H^1(\Delta)$ is not identically zero, then $h^* \neq 0$ almost everywhere on $\partial\Delta$;*
(iv) *The annihilator of $A^0(\Delta) \subset C^0(\partial\Delta)$ is $\zeta H^1(\Delta) = \{\zeta f \mid f \in H^1(\Delta)\}$;*
(v) *$h \in H^\infty(\Delta)$ iff $h^* \in L^\infty(\partial\Delta)$.*

Later on we shall identify an element h of $H^1(\Delta)$ with its boundary trace h^* given by (2.6.7); this is possible by Theorem 2.6.12.(i) and (ii). Finally, we shall denote product of

n copies of one of the aforementioned spaces by a subscript n (like $H^1_n(\Delta)$, $A^0_n(\Delta)$, and so on); these are spaces of functions on Δ with range in $\mathbf{C}^n$. The properties already described remain true; the only difference is that (2.6.5) becomes

$$\forall f \in C^0_n(\partial\Delta) \qquad h(f) = \frac{1}{2\pi}\int_0^{2\pi} \langle h(e^{i\theta}), f(e^{i\theta})\rangle \, d\theta, \tag{2.6.8}$$

where $h \in L^1_n(\partial\Delta)$ and $\langle z, w\rangle = \sum\limits_{j=1}^n z_j w_j$ for all z, $w \in \mathbf{C}^n$.

2.6.3 *Convex domains*

The aim of this section is to prove that every pair of distinct points in a bounded convex domain D is contained in a geodesic disk. The idea is that every K-extremal map is a solution of an extremal problem in $H^\infty_n(\Delta)$; using this fact we shall be able to prove that every K-extremal map is a complex geodesic, exactly as happened for C-extremal maps and complex C-geodesics (Proposition 2.6.3).

We begin officially stating, for sake of completeness, the following

Lemma 2.6.13: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain. Then:*
(i) *for every pair of distinct points z_1, $z_2 \in D$ there exists a K-extremal map with respect to z_1 and z_2;*
(ii) *for every point $z_0 \in D$ and non-zero vector $v \in \mathbf{C}^n$ there exists an infinitesimal K-extremal map with respect to z_0 and v.*

Proof: It suffices to notice that in D the Kobayashi distance k_D coincides with the one-disk function δ_D (Proposition 2.3.44), and to recall that D is taut, **q.e.d.**

Now we define a Minkowski functional on $H^\infty_n(\Delta)$. Let D be a bounded convex domain of $\mathbf{C}^n$; we can assume without loss of generality that $0 \in D$. The *Minkowski functional* $p_D\colon \mathbf{C}^n \to \mathbf{R}^+$ of D is given by

$$\forall z \in \mathbf{C}^n \qquad p_D(z) = \inf\{r \mid z \in rD,\ r > 0\}.$$

For every $r > 0$, set

$$D_r = \{z \in \mathbf{C}^n \mid p_D(z) < r\} = rD;$$

clearly, $D_1 = D$, and $p_D(z) = 0$ iff $z = 0$. It is easy to check that p_D is a Minkowski functional on $\mathbf{C}^n$ according to the definition given in the previous section. If D is the unit ball of $\mathbf{C}^n$, then p_D is the euclidean norm; in general, p_D is a norm on $\mathbf{C}^n$ iff D is *balanced,* i.e., iff $\lambda D \subset D$ for all $\lambda \in \overline{\Delta}$.

Now define $P_D\colon H^\infty_n(\Delta) \to \mathbf{R}^+$ by

$$P_D(f) = \sup_{\zeta\in\Delta} p_D\big(f(\zeta)\big). \tag{2.6.9}$$

If D is the unit ball of $\mathbf{C}^n$, P_D is the supremum norm of $H_n^\infty(\Delta)$; in general, it is a Minkowski functional on $H_n^\infty(\Delta)$. We notice that $P_D(f) < 1$ iff $f(\Delta)$ is relatively compact in D.

Analogously, on $C_n^0(\partial\Delta)$ we can define a Minkowski functional by

$$\forall f \in C_n^0(\partial\Delta) \qquad P_D(f) = \sup_{\tau\in\partial\Delta} p_D\bigl(f(\tau)\bigr). \tag{2.6.10}$$

Now, since $A_n^0(\Delta)$ is both a subspace of $H_n^\infty(\Delta)$ and a subspace of $C_n^0(\partial\Delta)$, it seems that we have defined two Minkowski functionals on $A_n^0(\Delta)$. Fortunately, this is not the case:

Lemma 2.6.14: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain containing the origin. Then for every $f \in A_n^0(\Delta)$ we have*

$$\sup_{\zeta\in\Delta} p_D\bigl(f(\zeta)\bigr) = \sup_{\tau\in\partial\Delta} p_D\bigl(f(\tau)\bigr). \tag{2.6.11}$$

Proof: Call r_1 the left-hand side of (2.6.11), and r_2 its right-hand side. Clearly $r_1 \ge r_2$; to prove the equality, it suffices to show that $f(\Delta) \subset \overline{D_{r_2}}$. Suppose it is not true; then we can find $\zeta_0 \in \Delta$ such that $f(\zeta_0) \notin \overline{D_{r_2}}$. Then there are a point $x_0 \in \partial D_{r_2}$ — for instance, the point of ∂D_{r_2} closest to $f(\zeta_0)$ — and a linear functional $\lambda\colon \mathbf{C}^n \to \mathbf{C}$ such that

$$\forall z \in \overline{D_{r_2}} \qquad \operatorname{Re}\lambda(z) \le \operatorname{Re}\lambda(x_0) < \operatorname{Re}\lambda\bigl(f(\zeta_0)\bigr).$$

Define $g \in \operatorname{Hol}(\mathbf{C}^n, \mathbf{C})$ by $g(z) = \exp[\lambda(z - x_0)]$; then $|g \circ f| \le 1$ on $\partial\Delta$ and $|g \circ f(\zeta_0)| > 1$, and this is impossible by the maximum principle, **q.e.d.**

So (2.6.9) and (2.6.10) define the same Minkowski functional on $A_n^0(\Delta)$, as claimed.

Let $\mathcal{M}_n(\partial\Delta)$ be the space of complex vector valued Radon measures on $\partial\Delta$, the dual space of $C_n^0(\partial\Delta)$. The dual Minkowski functional P_D^* on $\mathcal{M}_n(\partial\Delta)$ is given by

$$P_D^*(\mu) = \sup\left\{ \frac{1}{P_D(f)} \operatorname{Re} \int_{\partial\Delta} \langle f, d\mu\rangle \;\middle|\; f \in C_n^0(\partial\Delta),\ f \ne 0 \right\},$$

for every $\mu \in C_n^0(\partial\Delta)$, where

$$\int_{\partial\Delta} \langle f, d\mu\rangle = \sum_{j=1}^n \int_{\partial\Delta} f_j \, d\mu_j.$$

Therefore if we imbed $H_n^1(\Delta)$ in $\mathcal{M}_n(\partial\Delta)$ as usual, for every $h \in H_n^1(\Delta)$ we have

$$P_D^*(h) = \sup\left\{ \frac{1}{2\pi P_D(f)} \operatorname{Re} \int_0^{2\pi} \langle h(e^{i\theta}), f(e^{i\theta})\rangle \, d\theta \;\middle|\; f \in C_n^0(\partial\Delta),\ f \ne 0 \right\}.$$

Now define the *dual Minkowski functional* $p_D^*\colon \mathbf{C}^n \to \mathbf{R}^+$ of D by

$$\forall z \in \mathbf{C}^n \qquad p_D^*(z) = \sup_{w\neq 0} \frac{\mathrm{Re}\langle z, w\rangle}{p_D(w)}; \tag{2.6.12}$$

p_D^* is the Minkowski functional dual to p_D if we identify $(\mathbf{C}^n)^*$ with $\mathbf{C}^n$ by means of the bilinear form $\langle\cdot,\cdot\rangle$. Then it is not difficult to check that

$$\forall h \in H_n^1(\Delta) \qquad P_D^*(h) = \frac{1}{2\pi}\int_0^{2\pi} p_D^*\bigl(h(e^{i\theta})\bigr)\, d\theta. \tag{2.6.13}$$

Now we can introduce the affine subspaces of $H_n^\infty(\Delta)$ we shall work with.

A *divisor* $\mathcal{D}$ of *total degree* $\deg\mathcal{D}$ on Δ is a set of points $\zeta_1,\ldots,\zeta_k \in \Delta$ and integers $d_1,\ldots,d_k \in \mathbf{Z}$ such that $d_1+\cdots+d_k = \deg\mathcal{D}$. We shall write

$$\mathcal{D} = [\zeta_1]^{d_1}\cdots[\zeta_k]^{d_k},$$

and d_j is called the *degree* of ζ_j. Later on we shall need only *positive divisors*, that is divisors with only positive degrees; therefore from now on a divisor will always be positive.

We associate to any divisor $\mathcal{D} = [\zeta_1]^{d_1}\cdots[\zeta_k]^{d_k}$ the function $\gamma_{\mathcal{D}} \in A^0(\Delta)$ given by

$$\gamma_{\mathcal{D}}(\zeta) = \prod_{j=1}^{k}\left(\frac{\zeta-\zeta_j}{1-\overline{\zeta_j}\zeta}\right)^{d_j}.$$

The zeroes of $\gamma_{\mathcal{D}}$ are exactly $\zeta_1,\ldots,\zeta_k$ with multiplicity $d_1,\ldots,d_k$ respectively.

A *set of data* $\mathbf{D}$ associated to a divisor $\mathcal{D} = [\zeta_1]^{d_1}\cdots[\zeta_k]^{d_k}$ is a subset of $\mathbf{C}^n$ composed by $\deg\mathcal{D}$ elements:

$$\mathbf{D} = \bigl\{a_{\mu,\nu_\mu} \in \mathbf{C}^n \bigm| 1 \le \mu \le k,\ 0 \le \nu_\mu \le d_\mu - 1\bigr\}.$$

We shall be interested in the following spaces of holomorphic maps:

$$L(\mathcal{D},\mathbf{D}) = \bigl\{f \in H_n^\infty(\Delta) \bigm| f^{(\nu_\mu)}(\zeta_\mu) = a_{\mu,\nu_\mu} \text{ for } 1 \le \mu \le k,\ 0 \le \nu_\mu \le d_\mu - 1\bigr\}.$$

In other words, $L(\mathcal{D},\mathbf{D})$ contains the bounded holomorphic maps with prescripted values and derivatives at the points of $\mathcal{D}$. Note that if we denote by $\mathbf{0}$ the set of data containing just the origin repeated $\deg\mathcal{D}$ times, then

$$L(\mathcal{D},\mathbf{0}) = \gamma_{\mathcal{D}} H_n^\infty(\Delta) = \{\gamma_{\mathcal{D}} f \mid f \in H_n^\infty(\Delta)\}. \tag{2.6.14}$$

In particular, $L(\mathcal{D},\mathbf{0})$ is a closed linear subspace of $H_n^\infty(\Delta)$ and, if $\mathbf{D}$ is a set of data associated to $\mathcal{D}$, then $L(\mathcal{D},\mathbf{D})$ is a closed affine subspace of $H_n^\infty(\Delta)$, for

$$L(\mathcal{D},\mathbf{D}) = f_0 + L(\mathcal{D},\mathbf{0}),$$

where f_0 is any element of $L(\mathcal{D},\mathbf{D})$.

Now fix a bounded convex domain D of $\mathbf{C}^n$ containing the origin, a positive divisor $\mathcal{D}$ on Δ, a set of data $\mathbf{D}$ associated to $\mathcal{D}$, and set

$$m_D(\mathcal{D},\mathbf{D}) = \inf\bigl\{P_D(f) \bigm| f \in L(\mathcal{D},\mathbf{D})\bigr\}.$$

Then the *extremal problem* associated to $L(\mathcal{D},\mathbf{D})$ is to find $f \in L(\mathcal{D},\mathbf{D})$ such that

$$P_D(f) = m_D(\mathcal{D},\mathbf{D});$$

a solution of the extremal problem will be called an *extremal map* of $L(\mathcal{D},\mathbf{D})$.

As announced, the K-extremal maps are solutions of an extremal problem of this kind. This is a corollary of the following

Theorem 2.6.15: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain containing the origin. Choose two distinct points z_1, $z_2 \in D$ with $p_D(z_1) \geq p_D(z_2)$, a non-zero vector $v \in \mathbf{C}^n$, and two distinct points ζ_1, $\zeta_2 \in \Delta$. Set $\mathcal{D}_1 = [\zeta_1][\zeta_2]$, $\mathbf{D}_1 = \{z_1, z_2\}$, $\mathcal{D}_2 = [\zeta_1]^2$ and $\mathbf{D}_2 = \{z_1, v\}$. Then*
(i) for $r \in \big(p_D(z_1), +\infty\big)$ the functions $r \mapsto k_{D_r}(z_1, z_2)$ and $r \mapsto \kappa_{D_r}(z_1; v)$ are strictly decreasing;
(ii) $m_D(\mathcal{D}_1, \mathbf{D}_1)$ is the unique $r > p_D(z_1)$ such that $k_{D_r}(z_1, z_2) = \omega(\zeta_1, \zeta_2)$; in particular, $k_D(z_1, z_2) = \omega(\zeta_1, \zeta_2)$ iff $m_D(\mathcal{D}_1, \mathbf{D}_1) = 1$;
(iii) $m_D(\mathcal{D}_2, \mathbf{D}_2)$ is the unique $r > p_D(z_1)$ such that $\kappa_{D_r}(z_1; v) = \kappa_\Delta(\zeta_1; 1)$; in particular, $\kappa_D(z_1; v) = \kappa_\Delta(\zeta_1; 1)$ iff $m_D(\mathcal{D}_2, \mathbf{D}_2) = 1$.

Proof: (i) It is clear that the two functions are not increasing. Now take $r_1 > r_2 > p_D(z_1)$, and choose $\psi \in \mathrm{Hol}(\Delta, D_{r_2})$ and $\eta > 0$ so that $\psi(0) = z_1$, $\psi(\eta) = z_2$ and

$$\omega(0, \eta) = k_{D_{r_2}}(z_1, z_2);$$

ψ exists by Lemma 2.6.13.

Clearly, $\psi(\Delta)$ is relatively compact in D_{r_1}. For every $s < 1$ define $\psi_s\colon \Delta_{1/s} \to D_{r_1}$ by $\psi_s(\zeta) = \psi(s\zeta)$. We have $\psi_s(0) = z_1$ and $z_2 - \psi_s(\eta) \to 0$ as $s \to 1$. Let $\phi_s\colon \Delta_{1/s} \to \mathbf{C}^n$ be given by

$$\phi_s(\zeta) = \psi_s(\zeta) + \frac{z_2 - \psi_s(\eta)}{\eta}\zeta.$$

Then $\phi_s(0) = z_1$, $\phi_s(\eta) = z_2$ and we can choose s_0 so close to 1 that $\phi_{s_0}(\Delta_{1/s_0}) \subset\subset D_{r_1}$. Finally, define $\phi \in \mathrm{Hol}(\Delta, D_{r_1})$ by

$$\phi(\zeta) = \phi_{s_0}(\zeta/s_0).$$

Then $\phi(0) = z_1$, $\phi(s_0\eta) = z_2$, and so

$$k_{D_{r_1}}(z_1, z_2) \leq \omega(0, s_0\eta) < k_{D_{r_2}}(z_1, z_2),$$

as claimed.

Analogously, take $\psi \in \mathrm{Hol}(\Delta, D_{r_2})$ and $\xi \in \mathbf{C}$ such that $\psi(0) = z_1$, $d\psi_0(\xi) = v$ and $|\xi| = \kappa_{D_{r_2}}(z_1; v)$. Clearly, $\psi(\Delta)$ is again relatively compact in D_{r_1}. For every $s < 1$ define as before $\psi_s\colon \Delta_{1/s} \to D_{r_1}$ by $\psi_s(\zeta) = \psi(s\zeta)$. We have $\psi_s(0) = z_1$ and $d(\psi_s)_0(\xi) = sv$. Let $\phi_s\colon \Delta_{1/s} \to \mathbf{C}^n$ be given by

$$\phi_s(\zeta) = \psi_s(\zeta) + \zeta\frac{1 - s}{\xi}v.$$

Then $\phi_s(0) = z_1$, $d(\phi_s)_0(\xi) = v$ and we can choose s_0 so close to 1 that $\phi_{s_0}(\Delta_{1/s_0}) \subset\subset D_{r_1}$. Finally, define $\phi \in \mathrm{Hol}(\Delta, D)$ by

$$\phi(\zeta) = \phi_{s_0}(\zeta/s_0).$$

Then $\phi(0) = z_1$, $d\phi_0(s_0\xi) = v$ and so

$$\kappa_{D_{r_1}}(z_1; v) \leq s_0|\xi| < \kappa_{D_{r_2}}(z_1; v),$$

and (i) is proved.

(ii) We can assume, without loss of generality, $\zeta_1 = 0$ and $\zeta_2 = \eta > 0$. Take $f \in L(\mathcal{D}_1, \mathbf{D}_1)$, and set $r_f = P_D(f)$. Clearly, $f(\Delta) \subset D_{r_f+\varepsilon}$ for all $\varepsilon > 0$; in particular, $k_{D_{r_f+\varepsilon}}(z_1, z_2) \le \omega(0, \eta)$. Since this holds for every $\varepsilon > 0$ and $f \in L(\mathcal{D}_1, \mathbf{D}_1)$, it follows that $k_{D_{r_0}}(z_1, z_2) \le \omega(0, \eta)$, where $r_0 = m_D(\mathcal{D}_1, \mathbf{D}_1)$.

Assume, by contradiction, $k_{D_{r_0}}(z_1, z_2) < \omega(0, \eta)$; then we can find $\varphi \in \mathrm{Hol}(\Delta, D_{r_0})$ such that $\varphi(0) = z_1$ and $\varphi(\tilde{\eta}) = z_2$ for some $0 < \tilde{\eta} < \eta$. In particular, the map $\psi\colon \Delta \to D_{r_0}$ given by $\psi(\zeta) = \varphi(\tilde{\eta}\zeta/\eta)$ belongs to $L(\mathcal{D}_1, \mathbf{D}_1)$. But $\psi(\Delta) \subset\subset D_{r_0}$, and so $P_D(\psi) < r_0 = m_D(\mathcal{D}_1, \mathbf{D}_1)$, contradiction.

Thus $k_{D_{r_0}}(z_1, z_2) = \omega(0, \eta)$, and the uniqueness follows from part (i).

(iii) Exactly as in part (ii), **q.e.d.**

Corollary 2.6.16: *Let D be a bounded convex domain in $\mathbf{C}^n$ containing the origin, and choose two distinct points ζ_1, $\zeta_2 \in \Delta$ and $\xi \in \mathbf{C}^*$. Let $\varphi \in \mathrm{Hol}(\Delta, D)$, and set $z_1 = \varphi(\zeta_1)$, $z_2 = \varphi(\zeta_2)$ and $v = d\varphi_{\zeta_1}(\xi) \in \mathbf{C}^n$. Then:*
(i) *φ is K-extremal with respect to z_1 and z_2 iff $P_D(\varphi) = 1$ and φ is an extremal map of $L(\mathcal{D}, \mathbf{D})$, where $\mathcal{D} = [\zeta_1][\zeta_2]$ and $\mathbf{D} = \{z_1, z_2\}$;*
(ii) *φ is infinitesimal K-extremal with respect to z_1 and v iff $P_D(\varphi) = 1$ and φ is an extremal map of $L(\mathcal{D}, \mathbf{D})$, where $\mathcal{D} = [\zeta_1]^2$ and $\mathbf{D} = \{z_1, v/\xi\}$.*

Proof: (i) Clearly $\varphi \in L(\mathcal{D}, \mathbf{D})$ and $m_D(\mathcal{D}, \mathbf{D}) \le P_D(\varphi) \le 1$. Furthermore, φ is K-extremal with respect to z_1 and z_2 iff $k_D(z_1, z_2) = \omega(\zeta_1, \zeta_2)$ iff $m_D(\mathcal{D}, \mathbf{D}) = 1$ (by Theorem 2.6.15), and the assertion follows.

(ii) Exactly as in part (i), **q.e.d.**

In the previous section we saw that an extremal problem is strictly correlated to its dual extremal problem, and we proved a duality principle. To apply those general facts, we need still another notation. If $\mathcal{D}$ is a (positive, as usual) divisor on Δ the space

$$Y_{\mathcal{D}} = \gamma_D\, A_n^0(\Delta) \tag{2.6.15}$$

is a closed subspace of $C_n^0(\partial\Delta)$. By Theorem 2.6.12.(iv), the annihilator of $Y_{\mathcal{D}}$ is

$$Y_{\mathcal{D}}^\circ = \gamma_{[0]}(\gamma_{\mathcal{D}})^{-1}\, H_n^1(\Delta), \tag{2.6.16}$$

where $(\gamma_{\mathcal{D}})^{-1}$ should be considered as an element of $C^0(\partial\Delta) \subset L^1(\partial\Delta) \subset \mathcal{M}(\partial\Delta)$.

We are now able to prove a very useful characterization of extremal maps:

Theorem 2.6.17: *Let D be a bounded convex domain in $\mathbf{C}^n$ containing the origin, $\mathcal{D}$ a divisor on Δ and $\mathbf{D}$ a set of data associated to $\mathcal{D}$. Then for any $f \in L(\mathcal{D}, \mathbf{D})$ the following facts are equivalent:*
(i) *f is an extremal map of $L(\mathcal{D}, \mathbf{D})$;*
(ii) *there exists $h \in Y_{\mathcal{D}}^\circ$ such that for almost all $\theta \in \mathbf{R}$ we have $p_D\big(f(e^{i\theta})\big) = m_D(\mathcal{D}, \mathbf{D})$ and*

$$\mathrm{Re}\big\langle h(e^{i\theta}), f(e^{i\theta})\big\rangle = p_D\big(f(e^{i\theta})\big)p_D^*\big(h(e^{i\theta})\big); \tag{2.6.17}$$

(iii) *there exists* $h \in Y_{\mathcal{D}}^{\circ}$ *such that*

$$\operatorname{Re} \frac{1}{2\pi} \int_0^{2\pi} \langle h(e^{i\theta}), f(e^{i\theta}) \rangle \, d\theta = P_D(f) P_D^*(h). \tag{2.6.18}$$

Proof: We shall apply the duality principle to the Banach space $X = C_n^0(\partial\Delta)$, equipped with the Minkowski functional (2.6.10). Fix $f_0 \in L(\mathcal{D}, \mathbf{D}) \cap A_n^0(\Delta)$; then

$$L(\mathcal{D}, \mathbf{D}) \cap A_n^0(\Delta) = f_0 + Y_{\mathcal{D}}.$$

Set $m_D = m_D(\mathcal{D}, \mathbf{D})$, $\widetilde{m}_D = \inf\{P_D(f_0 + f) \mid f \in Y_{\mathcal{D}}\}$ and

$$\widetilde{M}_D = \inf\{P_D^*(h) \mid h \in Y_{\mathcal{D}}^{\circ} \text{ and } \operatorname{Re} h(f_0) = 1\};$$

clearly, $m_D \leq \widetilde{m}_D$, by Lemma 2.6.14, and $\widetilde{M}_D = \widetilde{m}_D^{-1}$, by Theorem 2.6.11.(i).

Now, the duality principle provides us with $h \in Y_{\mathcal{D}}^{\circ}$ such that $\operatorname{Re} h(f_0) = 1$ and $P_D^*(h) = \widetilde{M}_D$. Take $f \in L(\mathcal{D}, \mathbf{D}) \subset H_n^\infty(\Delta) \subset H_n^1(\Delta)$. Since for any $\varphi \in H^\infty(\Delta)$ and $\psi \in H^1(\Delta)$ the product $\varphi\psi$ is contained in $H^1(\Delta)$, and since moreover we have $f_0 - f \in \gamma_{\mathcal{D}} H_n^\infty(\Delta)$, (2.6.16) implies that $\langle h, f_0 - f \rangle \in \gamma_{[0]} H^1(\Delta)$. Therefore, by Theorem 2.6.12.(ii), we have $h(f_0 - f) = 0$; in particular, $\operatorname{Re} h(f) = 1$. But then (2.6.12) and (2.6.13) yield

$$\begin{aligned} 1 = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\langle h(e^{i\theta}), f(e^{i\theta}) \rangle \, d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} p_D\big(f(e^{i\theta})\big) p_D^*\big(h(e^{i\theta})\big) \, d\theta \\ &\leq P_D(f) P_D^*(h) \leq P_D(f) \, m_D^{-1}. \end{aligned} \tag{2.6.19}$$

If f is an extremal map, then $P_D(f) = m_D$ and all the inequalities in (2.6.19) are actually equalities; in particular (i) implies (ii).

It is clear that (ii) implies (iii), by (2.6.13); finally, assume (iii) holds. Then, up to replacing h by $\big(P_D(f) P_D^*(h)\big)^{-1} h$, we have $\operatorname{Re} h(f) = P_D(f) P_D^*(h) = 1$; furthermore, $h \in L(\mathcal{D}, \mathbf{D})^\circ$, thanks to (2.6.16), (2.6.14) and Theorem 2.6.12.(ii), and thus (i) follows from Theorem 2.6.11.(iii), **q.e.d.**

Using this characterization, we can show that the property of being an extremal map is preserved passing to larger divisors:

Corollary 2.6.18: *Let* D *be a bounded convex domain in* $\mathbf{C}^n$ *containing the origin,* $\mathcal{D}$, $\mathcal{D}'$ *two divisors on* Δ *with* $\deg \mathcal{D}' \geq \deg \mathcal{D}$, $\mathbf{D}$ *a set of data associated to* $\mathcal{D}$, $\mathbf{D}'$ *a set of data associated to* $\mathcal{D}'$, *and take* $f \in L(\mathcal{D}, \mathbf{D}) \cap L(\mathcal{D}', \mathbf{D}')$. *Then if* f *is an extremal map of* $L(\mathcal{D}, \mathbf{D})$, *it is an extremal map of* $L(\mathcal{D}', \mathbf{D}')$ *too.*

Proof: We claim that, since $\deg \mathcal{D}' \geq \deg \mathcal{D}$, there is a meromorphic function χ on $\mathbf{C}$ with the same zeroes and poles as $\gamma_{\mathcal{D}}(\gamma_{\mathcal{D}'})^{-1}$ in Δ, and positive on $\partial\Delta$. Indeed, using $\operatorname{Aut}(\Delta)$, it is enough to prove the claim when $\gamma_{\mathcal{D}}(\gamma_{\mathcal{D}'})^{-1}$ has just a simple pole at the origin, and when $\gamma_{\mathcal{D}}(\gamma_{\mathcal{D}'})^{-1}$ has a simple pole at 0 and a simple zero at $t > 0$. In the first case $\chi(\zeta) = 3 + \zeta + \zeta^{-1}$ will do; in the second case $\chi(\zeta) = t + t^{-1} - \zeta - \zeta^{-1}$ will do.

Now let $h \in Y^\circ_{\mathcal{D}}$ be given by Theorem 2.6.17, and set $h_1 = \chi h$. Then $h_1 \in Y^\circ_{\mathcal{D}'}$ and for almost all $\theta \in \mathbf{R}$ we have

$$\begin{aligned}\operatorname{Re}\langle h_1(e^{i\theta}), f(e^{i\theta})\rangle &= \chi(e^{i\theta}) \operatorname{Re}\langle h(e^{i\theta}), f(e^{i\theta})\rangle = \chi(e^{i\theta}) p_D\big(f(e^{i\theta})\big) p^*_D\big(h(e^{i\theta})\big)\\ &= p_D\big(f(e^{i\theta})\big) p^*_D\big(h_1(e^{i\theta})\big).\end{aligned}$$

Since $p_D\big(f(e^{i\theta})\big) = m_D(\mathcal{D}, \mathbf{D}) = P_D(f)$ almost everywhere, this yields

$$\operatorname{Re} \frac{1}{2\pi} \int_0^{2\pi} \langle h_1(e^{i\theta}), f(e^{i\theta})\rangle \, d\theta = P_D(f) P^*_D(h_1),$$

and the assertion follows from Theorem 2.6.17, **q.e.d.**

And finally we are able to prove the existence of complex geodesics in convex domains:

Theorem 2.6.19: *Let D be a bounded convex domain of $\mathbf{C}^n$, and choose two distinct points z_1, $z_2 \in D$ and a non-zero vector $v \in \mathbf{C}^n$. Let $\varphi \in \operatorname{Hol}(\Delta, D)$ be a K-extremal map with respect to z_1 and z_2 (an infinitesimal K-extremal map with respect to z_1 and v). Then φ is a complex geodesic. In particular, there always exists a geodesic disk containing z_1 and z_2, as well as a geodesic disk containing z_1 and tangent to v.*

Proof: Without loss of generality we can assume $0 \in D$. Let ζ_1, $\zeta_2 \in \Delta$ be such that $\varphi(\zeta_j) = z_j$ for $j = 1, 2$. Choose two distinct points η_1, $\eta_2 \in \Delta$, and set $\mathcal{D} = [\zeta_1][\zeta_2]$, $\mathcal{D}' = [\eta_1][\eta_2]$, $\mathbf{D} = \{z_1, z_2\}$ and $\mathbf{D}' = \{\varphi(\eta_1), \varphi(\eta_2)\}$. Then clearly $\deg \mathcal{D}' \geq \deg \mathcal{D}$ and $\varphi \in L(\mathcal{D}, \mathbf{D}) \cap L(\mathcal{D}', \mathbf{D}')$. By Corollary 2.6.16, $P_D(\varphi) = 1$ and φ is an extremal map of $L(\mathcal{D}, \mathbf{D})$. Hence, by Corollary 2.6.18, φ is an extremal map of $L(\mathcal{D}', \mathbf{D}')$ and then, again by Corollary 2.6.16, φ is K-extremal with respect to $\varphi(\eta_1)$ and $\varphi(\eta_2)$. Since η_1 and η_2 are arbitrary, φ is a complex geodesic. The same argument works for infinitesimal K-extremal maps, and the latter assertion follows from Lemma 2.6.13, **q.e.d.**

In particular,

Corollary 2.6.20: *Let D be a bounded convex domain of $\mathbf{C}^n$; then every infinitesimal complex geodesic is a complex geodesic, and vice versa.*

Proof: By Corollary 2.6.18 and Theorem 2.6.19, **q.e.d.**

Now we wish to discuss a bit the geometric significance of Theorem 2.6.17. Let D be a bounded convex domain of $\mathbf{C}^n$ containing the origin, and $\varphi \in \operatorname{Hol}(\Delta, D)$ a complex geodesic. By Corollaries 2.6.16 and 2.6.20, if we set $\mathcal{D} = [0]^2$ and $\mathbf{D} = \{\varphi(0), \varphi'(0)\}$, then

φ is an extremal element of $L(\mathcal{D}, \mathbf{D})$, and $P_D(\varphi) = 1$. Let $h \in (\gamma_{[0]})^{-1} H^1_n(\Delta)$ be a map provided by Theorem 2.6.17. Then (2.6.17) shows that

$$\mathrm{Re}\langle h(\tau), \varphi(\tau)\rangle = p^*_D\big(h(\tau)\big)$$

for almost all $\tau \in \partial\Delta$; in particular,

$$\forall z \in D \qquad \mathrm{Re}\langle h(\tau), \varphi(\tau) - z\rangle > 0 \tag{2.6.20}$$

for almost all $\tau \in \partial\Delta$. In other words, $h(\tau)$ defines a supporting hyperplane to D at $\varphi(\tau)$ for almost all $\tau \in \partial\Delta$.

We can push this argument even farther. Let $\varphi^* = \gamma_{[0]} h \in H^1_n(\Delta)$; φ^* is called a *dual map* of the complex geodesic φ. We need

Lemma 2.6.21: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic and $\varphi^* \in H^1_n(\Delta)$ a dual map of φ. Then φ^* is never vanishing in Δ and*

$$\mathrm{Re}\langle \varphi'(0), \varphi^*(0)\rangle > 0. \tag{2.6.21}$$

Proof: Since $\varphi(0) \in D$, (2.6.20) yields

$$\mathrm{Re}\Big\langle \big(\varphi(\tau) - \varphi(0)\big)/\tau, \varphi^*(\tau)\Big\rangle > 0$$

for almost all $\tau \in \partial\Delta$. The left-hand side is the real part of a function in $H^1(\Delta)$; hence Theorem 2.6.12.(ii) yields (2.6.21), and, in particular, $\varphi^*(0) \neq 0$. Finally, since it is easy to check that $\varphi^* \circ \gamma$ is a dual map of $\varphi \circ \gamma$ for every $\gamma \in \mathrm{Aut}(\Delta)$, it follows that φ^* is never vanishing in Δ, **q.e.d.**

Now, consider the complex supporting hyperplane H_ζ to D at $\varphi(\zeta)$ defined by

$$H_\zeta = \{z \in \mathbf{C}^n \mid \langle z - \varphi(\zeta), h(\zeta)\rangle = 0\}$$

for almost all $\zeta \in \partial\Delta$. Clearly, H_ζ is also defined by the equation

$$\langle z - \varphi(\zeta), \varphi^*(\zeta)\rangle = 0 \tag{2.6.22}$$

for almost all $\zeta \in \partial\Delta$. But now (2.6.22) makes sense for all $\zeta \in \Delta$; thus the family of hyperplanes H_ζ, defined initially only for almost all $\zeta \in \partial\Delta$, extends to a holomorphic family of complex hyperplanes H_ζ defined for all $\zeta \in \Delta$ (because φ^* is never zero on Δ). In this way, we have obtained a nice fibration of D:

Proposition 2.6.22: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and let $\varphi \in \mathrm{Hol}(\Delta, D)$ be a complex geodesic. Then there is $\tilde{p} \in \mathrm{Hol}(D, \Delta)$ such that $\tilde{p} \circ \varphi = \mathrm{id}_\Delta$.*

Proof: Fix $z \in D$ and consider (2.6.22) as an equation for the unknown $\zeta \in \Delta$. We claim that (2.6.22) has exactly one solution in Δ for every $z \in D$. Indeed, the number of solutions is given by $\operatorname{wind} g$, the winding number of the function $g(\zeta) = \langle z - \varphi(\zeta), \varphi^*(\zeta)\rangle$. Now

$$\mathrm{Re}\langle z - \varphi(\zeta), h(\zeta)\rangle < 0$$

for almost every $\zeta \in \partial\Delta$, by (2.6.20); hence

$$\operatorname{wind} g = \operatorname{wind} \zeta + \operatorname{wind} \langle z - \varphi(\zeta), h(\zeta)\rangle = 1.$$

So for every $z \in D$ the equation (2.6.22) has exactly one solution $\tilde{p}(z) \in \Delta$. By the implicit function theorem $\tilde{p}$ is holomorphic, and clearly $\tilde{p} \circ \varphi = \mathrm{id}_\Delta$, **q.e.d.**

The function $\tilde{p}$ is called a *left inverse* of the complex geodesic φ. Note that $p = \varphi\circ\tilde{p}$ is a holomorphic retraction of D onto $\varphi(\Delta)$, called the *holomorphic retraction associated* to φ. In particular, then, every geodesic disk is a one-dimensional holomorphic retract of D. Conversely, every one-dimensional holomorphic retract is a geodesic disk; more precisely, we have

Corollary 2.6.23: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain; take $f \in \mathrm{Hol}(D, D)$ and let z_1, z_2 be two distinct fixed points of f. Then there exists a geodesic disk passing through z_1 and z_2 contained in* $\mathrm{Fix}(f)$.

Proof: Let $\rho\colon D \to \mathrm{Fix}(f)$ be the holomorphic retraction provided by Theorem 2.5.12, and take a complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ passing through z_1 and z_2. Then, by Theorem 2.6.19, $\rho\circ\varphi$ is a complex geodesic passing through z_1 and z_2 whose image is contained in $\mathrm{Fix}(f)$, **q.e.d.**

Corollary 2.6.24: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and M a one-dimensional submanifold of D. Then M is the fixed point set of a map $f \in \mathrm{Hol}(D, D)$ iff M is a geodesic disk.*

Proof: If $M = \mathrm{Fix}(f)$, then M is a geodesic disk by Corollary 2.6.23. The converse follows from Proposition 2.6.22, **q.e.d.**

Another important consequence of Proposition 2.6.22 is

Proposition 2.6.25: *Let $D \subset\subset \mathbf{C}^n$ be a bounded convex domain. Then we have $c_D = k_D$ and $\gamma_D = \kappa_D$.*

Proof: Choose two distinct points z_1, $z_2 \in D$, and let $\varphi \in \mathrm{Hol}(\Delta, D)$ be a complex geodesic passing through z_1 and z_2. Let $\tilde{p} \in \mathrm{Hol}(D, \Delta)$ be a left inverse of φ; in particular, if ζ_1, $\zeta_2 \in \Delta$ are such that $\varphi(\zeta_j) = z_j$, we have $\tilde{p}(z_j) = \zeta_j$ (for $j = 1, 2$). Then

$$k_D(z_1, z_2) = \omega(\zeta_1, \zeta_2) = \omega\big(\tilde{p}(z_1), \tilde{p}(z_2)\big) \le c_D(z_1, z_2) \le k_D(z_1, z_2),$$

and the assertion is proved for the invariant distances; a completely analogous argument works for the invariant metrics, **q.e.d.**

2.6.4 *Strongly convex domains*

Now we focus on strongly convex domains, where we shall prove that every complex geodesic extends continuously to $\partial\Delta$, and that every pair of distinct points is contained in a unique geodesic disk. We shall also study in detail the dual map of a complex geodesic.

Our first goal is to prove that every complex geodesic in a strongly convex domain belongs to the Hölder space $C^{0,1/2}(\overline{\Delta})$. As a matter of notations, we shall write $A^{0,\alpha}(\Delta) = A^0(\Delta) \cap C^{0,\alpha}(\overline{\Delta})$ for every $\alpha \in (0, 1)$. We need the following elegant criterion, due to Hardy and Littlewood:

Theorem 2.6.26: *Let $f\colon \Delta \to \mathbf{C}$ be holomorphic, and fix $\alpha \in (0,1)$. Then the following statements are equivalent:*
(i) *$f \in A^{0,\alpha}(\Delta)$;*
(ii) *$f \in H^1(\Delta)$ and there is $c_1 > 0$ such that*

$$|f(e^{i\theta_2}) - f(e^{i\theta_1})| \le c_1|\theta_2 - \theta_1|^\alpha,$$

for almost all θ_1, $\theta_2 \in \mathbf{R}$;
(iii) *$f \in H^1(\Delta)$ and there is $c_2 > 0$ such that*

$$|\operatorname{Re} f(e^{i\theta_2}) - \operatorname{Re} f(e^{i\theta_1})| \le c_2|\theta_2 - \theta_1|^\alpha,$$

for almost all θ_1, $\theta_2 \in \mathbf{R}$;
(iv) *there is $c_3 > 0$ such that*

$$\forall \zeta \in \Delta \qquad |f'(\zeta)| \le c_3(1 - |\zeta|)^{\alpha - 1}. \tag{2.6.23}$$

Proof: (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii): it suffices to notice that for every θ_1, $\theta_2 \in \mathbf{R}$ we have

$$|e^{i\theta_2} - e^{i\theta_1}| \le |\theta_2 - \theta_1|.$$

(iii)$\Longrightarrow$(iv): Let $u = \operatorname{Re} f$; then f is given by

$$f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta}\, u(e^{i\theta})\, d\theta.$$

In particular, for every $\zeta_0 = r_0 e^{i\theta_0} \in \Delta$ we have

$$f'(\zeta_0) = \frac{1}{\pi}\int_0^{2\pi} \frac{u(e^{i\theta}) - u(e^{i\theta_0})}{(e^{i\theta} - \zeta_0)^2}\, e^{i\theta}\, d\theta,$$

and thus, integrating over $[\theta_0 - \pi, \theta_0 + \pi]$ instead of $[0, 2\pi]$,

$$|f'(\zeta_0)| \le \frac{c_2}{\pi}\int_{-\pi}^{\pi} \frac{|\theta|^\alpha}{1 - 2r_0\cos\theta + r_0^2}\, d\theta.$$

Now,

$$1 - 2r_0\cos\theta + r_0^2 = (1 - r_0)^2 + 4r_0\sin^2\frac{\theta}{2} \ge (1 - r_0)^2 + \frac{4r_0\theta^2}{\pi^2}$$

for all $\theta \in [-\pi, \pi]$. Hence

$$|f'(\zeta_0)| \leq \frac{c_2}{\pi} \int_{-\pi}^{\pi} \frac{|\theta|^\alpha}{(1 - r_0)^2 + 4r_0(\theta/\pi)^2} \, d\theta \leq \frac{c_3}{(1 - |\zeta_0|)^{1-\alpha}},$$

because the integral

$$\int_0^\infty \frac{t^\alpha}{1 + t^2} \, dt,$$

obtained after the substitution $t = \theta/(1 - r_0)$, is convergent.

(iv)$\Longrightarrow$(i): By (2.6.23), the radial limit

$$f(e^{i\theta}) = f(0) + \lim_{r_0 \to 1} \int_0^{r_0} f'(re^{i\theta}) \, dr$$

exists bounded for every $\theta \in \mathbf{R}$; in particular, $f \in H^\infty(\Delta)$, by Theorem 2.6.12.(v). To show that $f \in A^{0,\alpha}(\Delta)$ it suffices to find a constant $c_0 > 0$ such that

$$|f(e^{i\theta_2}) - f(e^{i\theta_1})| \leq c_0 |\theta_2 - \theta_1|^\alpha, \tag{2.6.24}$$

$$|f(re^{i\theta_2}) - f(re^{i\theta_1})| \leq c_0 r |\theta_2 - \theta_1|^\alpha, \tag{2.6.25}$$

for all θ_1, $\theta_2 \in \mathbf{R}$ with $0 < \theta_2 - \theta_1 < 1$ and $r \in (0, 1)$, and

$$|f(r_2 e^{i\theta_0}) - f(r_1 e^{i\theta_0})| \leq c_0 |r_2 - r_1|^\alpha, \tag{2.6.26}$$

for all $0 \leq r_1 \leq r_2 < 1$ and $\theta_0 \in \mathbf{R}$.

We start with (2.6.24). Let $\rho = 1 - (\theta_2 - \theta_1) > 0$, and denote by Σ the path consisting of the radial segment from $e^{i\theta_1}$ to $\rho e^{i\theta_1}$, the arc of circle $|\zeta| = \rho$ from $\rho e^{i\theta_1}$ to $\rho e^{i\theta_2}$, and the radial segment from $\rho e^{i\theta_2}$ to $e^{i\theta_2}$ (see Figure 2.3). Then

$$f(e^{i\theta_2}) - f(e^{i\theta_1}) = \int_\Sigma f'(\zeta) \, d\zeta.$$

Breaking up the integral, we find

$$\begin{aligned}
|f(e^{i\theta_2}) - f(e^{i\theta_1})| &\leq \int_\rho^1 |f'(re^{i\theta_1})| \, dr + \int_\rho^1 |f'(re^{i\theta_2})| \, dr + \int_{\theta_1}^{\theta_2} |f'(\rho e^{i\theta})| \, d\theta \\
&\leq 2c_3 \int_\rho^1 \frac{dr}{(1 - r)^{1-\alpha}} + c_3 \frac{\theta_2 - \theta_1}{(1 - \rho)^{1-\alpha}} \\
&\leq c_3 \left(1 + \frac{2}{\alpha}\right) |\theta_2 - \theta_1|^\alpha = c_0 |\theta_2 - \theta_1|^\alpha,
\end{aligned}$$

Figure 2.3

and (2.6.24) is proved. In particular, $f \in A^0(\Delta)$, by the Cauchy formula.

For (2.6.25), consider the function $\zeta \mapsto \bigl(f(\zeta e^{i\theta_2}) - f(\zeta e^{i\theta_1})\bigr)/\zeta$, which is holomorphic in Δ. Then the maximum principle yields

$$\frac{|f(re^{i\theta_2}) - f(re^{i\theta_1})|}{r} \le \sup_{|\tau|=1} |f(\tau e^{i\theta_2}) - f(\tau e^{i\theta_1})| \le c_0|\theta_2 - \theta_1|^\alpha,$$

and (2.6.25) is proved. Finally,

$$|f(r_2e^{i\theta_0}) - f(r_1e^{i\theta_0})| \le \int_{r_1}^{r_2} |f'(re^{i\theta_0})|\,dr \le c_3 \int_{r_1}^{r_2} \frac{dr}{(1-r)^{1-\alpha}}.$$

If $r_2 - r_1 \le 1 - r_2$ we have

$$|f(r_2e^{i\theta_0}) - f(r_1e^{i\theta_0})| \le c_3 \frac{r_2 - r_1}{(1-r_2)^{1-\alpha}} \le c_3|r_2 - r_1|^\alpha \le c_0|r_2 - r_1|^\alpha.$$

On the other hand, if $r_2 - r_1 > 1 - r_2$ we get

$$|f(r_2e^{i\theta_0}) - f(r_1e^{i\theta_0})| \le c_3 \frac{(1-r)^\alpha}{\alpha}\bigg|_{r_1}^{r_2} \le \frac{c_3}{\alpha}|r_2 - r_1|^\alpha \le c_0|r_2 - r_1|^\alpha,$$

and we are done, **q.e.d.**

There is a corollary concerning harmonic functions. A *conjugate function* of a harmonic function $u\colon \Delta \to \mathbf{R}$ is a harmonic function $v\colon \Delta \to \mathbf{R}$ such that $u+iv$ is holomorphic; it is well known that v is unique up to an additive constant. Then

Corollary 2.6.27: *Take $r \in \mathbf{N}$ and $\alpha \in (0,1)$; let $u \in C^{r,\alpha}(\overline{\Delta})$ be harmonic in Δ, and choose a conjugate function $v\colon \Delta \to \mathbf{R}$. Then $v \in C^{r,\alpha}(\overline{\Delta})$.*

Proof: Indeed, by Theorem 2.6.26, $f = u + iv \in C^{r,\alpha}(\overline{\Delta})$, and the assertion follows, **q.e.d.**

We are almost ready to deal with complex geodesics. We need a last estimate:

Lemma 2.6.28: *Take $R > 0$, $z_0 \in \mathbf{C}^n$ and $f \in \mathrm{Hol}\big(\Delta, B(z_0, R)\big)$. Then*

$$\|f'(0)\| \le \big(R^2 - \|f(0) - z_0\|^2\big)^{1/2}. \tag{2.6.27}$$

Proof: We can assume $z_0 = 0$. Let $g = R^{-1} f \in \mathrm{Hol}(\Delta, B^n)$. Then g is a contraction for the Poincaré metric on Δ and the Bergmann metric on B^n; hence

$$\frac{1}{(1 - \|g(0)\|^2)^2} \left[(1 - \|g(0)\|^2)\|g'(0)\|^2 + \big|\big(g(0), g'(0)\big)\big|^2\right] \le 1,$$

that is

$$\|g'(0)\|^2 \le 1 - \|g(0)\|^2 - \frac{\big|\big(g(0), g'(0)\big)\big|^2}{1 - \|g(0)\|^2} \le 1 - \|g(0)\|^2,$$

and (2.6.27) follows, **q.e.d.**

Then

Theorem 2.6.29: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, and $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic. Then $\varphi \in A_n^{0,1/2}(\Delta)$.*

Proof: By Theorem 2.3.51 there is a constant $c_1 \in \mathbf{R}$ such that for all $\zeta \in \Delta$

$$\tfrac{1}{2} \log \frac{1}{1 - |\zeta|} \le \omega(0, \zeta) = k_D\big(\varphi(0), \varphi(\zeta)\big) \le c_1 - \tfrac{1}{2} \log d\big(\varphi(\zeta), \partial D\big),$$

and so

$$d\big(\varphi(\zeta), \partial D) \le e^{2c_1}(1 - |\zeta|). \tag{2.6.28}$$

Let $R > 0$ be so large that for every $x \in \partial D$ the euclidean ball B_x of radius R tangent to ∂D in x contains D. Fix $\zeta_0 \in \Delta$, and let $x \in \partial D$ be such that $\|x - \varphi(\zeta_0)\| = d\big(\varphi(\zeta_0), \partial D\big)$. Let w_0 be the center of B_x. If we set

$$\psi(\zeta) = \varphi\left(\frac{\zeta + \zeta_0}{1 + \overline{\zeta_0}\zeta}\right),$$

then $\psi \in \mathrm{Hol}(\Delta, B_x)$ and $\psi(0) = \varphi(\zeta_0)$. By (2.6.27), then

$$\|\psi'(0)\| \le (R^2 - \|\varphi(\zeta_0) - w_0\|^2)^{1/2} \le c_2\, d\big(\varphi(\zeta_0), \partial D\big)^{1/2},$$

where $c_2 = \sqrt{2R} > 0$. Therefore (2.6.28) yields

$$\|\varphi'(\zeta_0)\| \le \frac{1}{1 - |\zeta_0|^2} \|\psi'(0)\| \le c_2 e^{c_1} (1 - |\zeta_0|)^{-1/2},$$

and the assertion follows from Theorem 2.6.26, **q.e.d.**

A first consequence of this theorem is the uniqueness of the geodesic disk containing two given points:

Corollary 2.6.30: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain. Take two distinct points z_0, $z_1 \in D$ and a non-zero vector $v \in \mathbf{C}^n$. Then:*
(i) *there exists a unique geodesic disk containing z_0 and z_1;*
(ii) *there exists a unique geodesic disk containing z_0 and tangent to v.*

Proof: Since the proof of (ii) is almost identical to the proof of (i), we describe in detail the latter only. Assume that $\varphi_0, \varphi_1 \in \mathrm{Hol}(\Delta, D)$ are two complex geodesics passing through z_0 and z_1; up to parametrization we can assume $\varphi_0(0) = \varphi_1(0) = z_0$ and $\varphi_0(\zeta_0) = \varphi_1(\zeta_0) = z_1$ for some $\zeta_0 \in \Delta$. Set

$$\forall \lambda \in [0,1] \qquad\qquad \varphi_\lambda = (1-\lambda)\varphi_0 + \lambda\varphi_1.$$

Clearly, every φ_λ is a holomorphic map of Δ into D; moreover, $\varphi_\lambda(0) = z_0$ and $\varphi_\lambda(\zeta_0) = z_1$ for all $\lambda \in [0,1]$. By Theorem 2.6.19, this implies that every φ_λ is a complex geodesic; in particular, every φ_λ extends continuously to ∂D and $\varphi_\lambda(\partial\Delta) \subset \partial D$ for every $\lambda \in [0,1]$. But D is strongly convex; hence $\varphi_\lambda|_{\partial\Delta}$ does not depend on λ. In particular, $\varphi_0|_{\partial\Delta} = \varphi_1|_{\partial\Delta}$, and then $\varphi_0 = \varphi_1$, **q.e.d.**

Now we shall examine in detail the dual map of a complex geodesic. First of all, in strongly convex domains it is (almost) uniquely determined:

Lemma 2.6.31: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, and $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic. Then the dual map of φ is unique up to a positive multiple.*

Proof: By definition, it suffices to prove that there is a unique $h \in (\gamma_{[0]})^{-1} H^1_n(\Delta)$ such that $P^*_D(h) = 1$ and (2.6.17) holds. Assume, by contradiction, that $h_1 \in (\gamma_{[0]})^{-1} H^1_n(\Delta)$ is another map satisfying the same requisites. In particular, both $h(\tau)$ and $h_1(\tau)$ define a supporting hyperplane to D at $\varphi(\tau)$ for almost all $\tau \in \partial\Delta$; being D strongly convex, this implies that $h_1 = \mu h$ for some measurable function $\mu\colon \partial\Delta \to \mathbf{R}^+$.

Now $P^*_D(h) = P^*_D(h_1)$ implies

$$\int_0^{2\pi} [1 - \mu(e^{i\theta})] \sup_{w \neq 0} \frac{\mathrm{Re}\langle h(e^{i\theta}), w\rangle}{p_D(w)}\, d\theta = 0. \tag{2.6.29}$$

Since $\gamma_{[0]} h \in H^1_n(\Delta)$, $h \neq 0$ almost everywhere on $\partial\Delta$, by Theorem 2.6.12.(iii). Assume, by contradiction, that $\mu > 1$ on a set $A \subset \partial\Delta$ of positive measure, and define a measurable map $w\colon \partial\Delta \to \mathbf{C}^n$ by setting $w(\tau) = -\overline{h(\tau)}$ if $\tau \in A$, and choosing $w(\tau)$ tangent to ∂D in $\varphi(\tau)$ if $\tau \notin A$. Then (2.6.29) yields

$$0 \geq \int_A [\mu(\tau) - 1] \frac{\|h(\tau)\|^2}{p_D\big(h(\tau)\big)}\, d\tau > 0,$$

impossible. For the same reason, μ cannot be less than 1 on a subset of $\partial\Delta$ of positive measure; so $\mu = 1$ almost everywhere, and $h = h_1$, **q.e.d.**

So if $\varphi \in \mathrm{Hol}(\Delta, D)$ is a complex geodesic, its dual map restricted to $\partial\Delta$ is of the form

$$\varphi^*(\tau) = \tau\mu(\tau)\overline{\mathbf{n}_{\varphi(\tau)}},$$

where $\mathbf{n}_x$ is, as usual, the outer unit normal vector to ∂D in x, and $\mu \in L^1(\partial\Delta)$ is unique up to a positive multiple.

Our next goal is to prove that even the dual map belongs to $A^{0,1/2}(\Delta)$. We need the following facts, complementing Lemma 2.6.21:

Lemma 2.6.32: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic, and set $v(\tau) = \left|\left(\varphi'(\tau), \mathbf{n}_{\varphi(\tau)}\right)\right|$ for almost all $\tau \in \partial\Delta$. Then v, $1/v \in L^\infty(\partial\Delta)$.*

Proof: Theorem 2.3.70 yields

$$v(\tau) = 2\lim_{t\to1} \kappa_D\big(\varphi(t\tau); \varphi'(t\tau)\big) d\big(\varphi(t\tau), \partial D\big) = 2\lim_{t\to1} \frac{d\big(\varphi(t\tau), \partial D\big)}{1-t^2},$$

for almost all $\tau \in \partial\Delta$, where the last step follows from Corollary 2.6.20. But Theorems 2.3.51 and 2.3.52 provide us with constants K_1, $K_2 > 0$ such that

$$\forall\zeta \in \Delta \qquad\qquad K_1 \le \frac{d\big(\varphi(\zeta), \partial D\big)}{1-|\zeta|^2} \le K_2,$$

because $k_D\big(\varphi(0), \varphi(\zeta)\big) = \omega(0, \zeta)$, and the assertion follows, **q.e.d.**

Lemma 2.6.33: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic, and $\varphi^* \in H^1_n(\Delta)$ a dual map of φ. Then the function $\langle\varphi', \varphi^*\rangle\colon \Delta \to \mathbf{C}$ is a positive real constant.*

Proof: The curve $\theta \mapsto \varphi(e^{i\theta})$ is almost everywhere differentiable, by Theorem 2.6.29, and lies entirely in ∂D; hence its tangent in a point $\varphi(e^{i\theta})$ is orthogonal to $\mathbf{n}_{\varphi(e^{i\theta})}$. It follows that

$$0 = \mathrm{Re}\langle i\varphi'(e^{i\theta}), \varphi^*(e^{i\theta})\rangle = -\,\mathrm{Im}\langle\varphi'(e^{i\theta}), \varphi^*(e^{i\theta})\rangle$$

for almost all $\theta \in \mathbf{R}$. Now, $\langle\varphi', \varphi^*\rangle \in H^1(\Delta)$, by Lemma 2.6.32; therefore $\mathrm{Im}\langle\varphi', \varphi^*\rangle \equiv 0$ on Δ, and then $\langle\varphi', \varphi^*\rangle$ is a (positive by Lemma 2.6.21) real constant, **q.e.d.**

In particular, we can normalize the dual map of a complex geodesic φ requiring

$$\langle\varphi', \varphi^*\rangle \equiv 1. \tag{2.6.30}$$

Therefore from now on we shall talk of *the* dual map of φ, and consequently of *the* left inverse and holomorphic retraction associated to φ.

And now

Theorem 2.6.34: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic, and $\varphi^* \in H^1_n(\Delta)$ its dual map. Then $\varphi^* \in A^{0,1/2}_n(\Delta)$.*

Proof: Define $\mu \in L^1(\partial\Delta)$ by $\mu(\tau) = \|\varphi^*(\tau)\|$; we first of all claim that $\mu \in L^\infty(\partial\Delta)$. Let $v \in L^\infty(\partial\Delta)$ be given by $v(\tau) = |(\varphi'(\tau), \mathbf{n}_{\varphi(\tau)})|$; by Lemma 2.6.33 and (2.6.30), $\mu(\tau)v(\tau) = 1$ almost everywhere on $\partial\Delta$. But Lemma 2.6.32 yields $1/v \in L^\infty(\partial\Delta)$; thus $\mu \in L^\infty(\partial\Delta)$, as claimed. In particular, then, $\varphi^* \in H^\infty_n(\Delta)$.

Now to prove that $\varphi^* \in A^{0,1/2}(\Delta)$, by Theorem 2.6.26, it suffices to show that there exist δ, $c > 0$ such that

$$|\mu(\tau_1) - \mu(\tau_2)| \le c|\tau_1 - \tau_2|^{1/2}$$

for almost all τ_1, $\tau_2 \in \partial\Delta$ with $|\tau_1 - \tau_2| \le \delta$, because $\tau \mapsto \mathbf{n}_{\varphi(\tau)}$ is already in $C^{0,1/2}(\partial\Delta)$.

Fix $\tau_1 \in \partial\Delta$; we can assume $(\mathbf{n}_{\varphi(\tau_1)})_1 = 1$, where $(\mathbf{n}_x)_1$ denotes the first component of $\mathbf{n}_x$. Let $\delta \in (0, 1/4)$ be such that $|(\mathbf{n}_{\varphi(\tau)})_1 - 1| < 1/2$ if $|\tau - \tau_1| \le 2\delta$, and choose $\chi \in C^{0,1/2}(\partial\Delta)$ such that

(a) $\chi(\tau) = \overline{(\mathbf{n}_{\varphi(\tau)})_1}$ if $|\tau - \tau_1| \le 2\delta$;

(b) $|\chi(\tau) - 1| < 1/2$ for all $\tau \in \partial\Delta$;

(c) $\|\chi\|_{1/2} = \|\mathbf{n} \circ \varphi\|_{1/2}$.

We can extend χ to a complex-valued function (still denoted by χ) continuous in $\overline{\Delta}$ and harmonic in Δ such that $|\chi(\zeta) - 1| < 1/2$ for all $\zeta \in \overline{\Delta}$; furthermore, by Theorem 2.6.26, $\chi \in C^{0,1/2}(\overline{\Delta})$. Let $\xi = \mathrm{Im} \log \chi \in C^{0,1/2}(\overline{\Delta})$. ξ is a (real-valued) harmonic function; then, by Corollary 2.6.27, the conjugate function $\eta\colon \overline{\Delta} \to \mathbf{R}$ of ξ belongs to $C^{0,1/2}(\overline{\Delta})$ too. In conclusion, $\rho = -\eta - \mathrm{Re} \log \chi \in C^{0,1/2}(\overline{\Delta})$ and $h = \rho + \log \chi \in A^{0,1/2}(\Delta)$.

Set $g = (\varphi^*)_1 \exp(-h)$ and $G(\zeta) = g(\zeta)/\zeta$. Clearly, g is holomorphic in Δ and G is holomorphic in Δ^*; furthermore, $g \in H^\infty(\Delta)$, for $\varphi^* \in H^\infty_n(\Delta)$, and so G is bounded on the set $\Delta \cap D(\tau_1, 2\delta)$. Now,

$$G(\tau) = \mu(\tau) \exp\bigl(-\rho(\tau)\bigr) \in \mathbf{R}$$

for almost all $\tau \in \partial\Delta \cap D(\tau_1, 2\delta)$. Therefore, by the Schwarz reflection principle, G extends holomorphically to $D(\tau_1, 2\delta) \backslash \partial\Delta$, and it is bounded there; hence it extends holomorphically to all $D(\tau_1, 2\delta)$. In particular, G is Lipschitz in $\overline{D(\tau_1, \delta)}$; being $\mu = G(\exp h)/\overline{\mathbf{n} \circ \varphi}$, it follows that μ is Hölder of exponent $1/2$ in $\overline{D(\tau_1, \delta)}$, and we are done, **q.e.d.**

Corollary 2.6.35: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic, $\tilde{p}$ its left inverse and $p = \varphi \circ \tilde{p}$. Then $\tilde{p} \in C^1(\overline{D})$ and $p \in C^{0,1/2}(\overline{D})$.*

Proof: Looking at the definition of $\tilde{p}$ it is clear that $\tilde{p}$ and p extend continuously to ∂D; furthermore

$$\tilde{p}\bigl(\,\overline{D} \setminus \varphi(\partial\Delta)\bigr) \subset \Delta. \tag{2.6.31}$$

Moreover, the regularity of $d\tilde{p}$ is the same as the regularity of the family (2.6.22) of hyperplanes; therefore Theorems 2.6.29 and 2.6.34 yields $\tilde{p} \in C^1(\overline{\Delta})$ and $p = \varphi \circ \tilde{p} \in C^{0,1/2}(\overline{\Delta})$, **q.e.d.**

In particular, we have the following characterization of complex geodesics in strongly convex domains:

Corollary 2.6.36: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain. A map $\varphi \in \mathrm{Hol}(\Delta, D)$ is a complex geodesics iff $\varphi \in A_n^{0,1/2}(\Delta)$, $\varphi(\partial\Delta) \subset \partial D$ and there is a positive function $\mu \in C^{0,1/2}(\partial\Delta)$ such that $\tau \mapsto \mu(\tau)\tau\overline{\mathbf{n}_{\varphi(\tau)}}$ extends to a map $\varphi^* \in A_n^{0,1/2}(\Delta)$.*

Proof: We already saw that a complex geodesic satisfies the given conditions. Conversely, take $\varphi \in A_n^{0,1/2}(\Delta)$ and $\mu \in C^{0,1/2}(\partial\Delta)$ such that $\varphi(\Delta) \subset D$, $\varphi(\partial\Delta) \subset \partial D$, $\mu > 0$ and the map $\tau \mapsto \mu(\tau)\tau\overline{\mathbf{n}_{\varphi(\tau)}}$ extends to a map $\varphi^* \in A_n^{0,1/2}(\Delta)$. Choose $\zeta_0 \in (0,1)$ and set $z_0 = \varphi(0)$ and $z_1 = \varphi(\zeta_0)$; by Theorem 2.6.19, it suffices to show that $k_D(z_0, z_1) = \omega(0, \zeta_0)$. If this is not the case, there would exist $\psi \in \mathrm{Hol}(\Delta, D)$ with $\psi(0) = z_0$ and $\psi(\zeta_1) = z_1$ for some $0 \leq \zeta_1 < \zeta_0$. Define $\phi\colon \overline{\Delta} \to D$ by $\phi(\zeta) = \psi(\zeta_1\zeta/\zeta_0)$; then $\varphi(0) = z_0$, $\varphi(\zeta_0) = z_1$ and $\varphi(\overline{\Delta}) \subset\subset D$. Then

$$\forall \tau \in \partial\Delta \qquad \mathrm{Re}\langle \varphi(\tau) - \phi(\tau), \overline{\mathbf{n}_{\varphi(\tau)}} \rangle > 0$$

and hence

$$\forall \tau \in \partial\Delta \qquad \mathrm{Re}\big\langle \big(\varphi(\tau) - \phi(\tau)\big)\tau^{-1}, \varphi^*(\tau) \big\rangle > 0,$$

but, by the minimum principle for harmonic functions, this is impossible, because the harmonic function on the left vanishes at ζ_0, **q.e.d.**

The usefulness of this characterization is that to see if a given map φ is a complex geodesic, it suffices to check certain properties of the map φ alone, instead of comparing it with all the other maps from Δ into the domain.

2.6.5 *Boundary smoothness*

In chapter 2.7 we shall need more precise information regarding the boundary smoothness of complex geodesics; so this section is devoted to prove that in a strongly convex domains with C^r boundary ($r \geq 3$) the complex geodesics belongs to $C^{r-2}(\overline{\Delta})$. As an application, we shall study the existence and uniqueness of geodesic disks passing through boundary points.

We begin with (the statement of) Whitney's extension theorem. We shall call *multi-index* an element of $\mathbf{N}^N$. If $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}^N$ is a multi-index, we define, as customary, the *order* $|\alpha|$ of α by $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and we set $\alpha! = (\alpha_1!)\cdots(\alpha_N!)$, $x^\alpha = x_1^{\alpha_1}\cdots x_N^{\alpha_N}$ for any $x \in \mathbf{R}^N$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}.$$

Let K be a compact subset of $\mathbf{R}^N$, and $r \in \mathbf{N}$. A *r-jet* F on K is a subset $F = \{F_\alpha\}_{|\alpha|\leq r}$ of $C^0(K)$. We associate to every function $g \in C^r(\mathbf{R}^N)$ a jet $J^r(g)$ setting $J^r(g)_\alpha = D^\alpha g|_K$ for every multi-index α with $|\alpha| \leq r$.

If F is a r-jet on K and $a \in K$, let

$$\forall x \in \mathbf{R}^N \qquad T_a^r F(x) = \sum_{|\alpha| \le r} \frac{(x-a)^\alpha}{\alpha!} F_\alpha(a).$$

Clearly, $T_a^r F \in C^\infty(\mathbf{R}^N)$; put $\widetilde{T}_a^r F = J^r(T_a^r F)$.

The *Whitney extension theorem* then says:

Theorem 2.6.37: *Let K be a compact subset of $\mathbf{R}^N$, and F a r-jet on K. Then there is $g \in C^r(\mathbf{R}^N)$ such that $J^r(g) = F$ iff for every multi-index α with $|\alpha| \le r$ we have*

$$\|F_\alpha(y) - (\widetilde{T}_x^r F)_\alpha(y)\| = o(\|x-y\|^{r-|\alpha|}) \tag{2.6.32}$$

as $\|x-y\| \to 0$ with x, $y \in K$.

For a proof see Malgrange [1966].

We shall apply this theorem in the proof of the next lemma. A real submanifold M of a complex manifold X is *totally real* in $z \in M$ if $T_z M \cap i T_z M = \{0\}$; it is *totally real* if it is so at each point. Clearly, the real dimension of M is at most equal to the complex dimension of X.

Lemma 2.6.38: *Let M be a closed C^r $(1 \le r \le \infty)$ totally real submanifold of a domain $U \subset\subset \mathbf{C}^n$, and take a complex-valued function $f \in C^r(M)$. Then for every open set $V \subset\subset U$ there exists a function $F \in C^k(\mathbf{C}^n)$ such that $F|_{M\cap\overline{V}} = f|_{M\cap\overline{V}}$ and*

$$D^\alpha \frac{\partial F}{\partial \bar z}(z) = 0 \tag{2.6.33}$$

for every $z \in M \cap V$ and every multi-index α with $|\alpha| \le r-1$.

Proof: Fix $z_0 \in M$, choose a basis $\{u_1, \ldots, u_k\}$ of $T_{z_0}M$, and let $J: T_{z_0}\mathbf{C}^n \to T_{z_0}\mathbf{C}^n$ be the almost complex structure on $T_{z_0}\mathbf{C}^n$ induced by the complex structure of $\mathbf{C}^n$. Since M is totally real, the set $\{u_1, \ldots, u_k, Ju_1, \ldots, Ju_k\}$ is linearly independent over $\mathbf{R}$; choose $v_1, \ldots, v_{2(n-k)} \in T_{z_0}\mathbf{C}^n$ so that $\{u_1, \ldots, u_k, Ju_1, \ldots, Ju_k, v_1, \ldots, v_{2(n-k)}\}$ is a $\mathbf{R}$-basis of $T_{z_0}\mathbf{C}^n$.

Clearly, $u_j(f)$ is defined for $j = 1, \ldots, k$; set $(Ju_j)(f) = iu_j(f)$ for $j = 1, \ldots, k$ and $v_j(f) = 0$ for $j = 1, \ldots, 2(n-k)$. Denote by $z_j = x_j + iy_j$ the natural coordinates of $\mathbf{C}^n$; we have

$$\frac{\partial}{\partial x_j}(z_0) = \sum_{\mu=1}^{k} \big[\alpha_{j\mu} u_\mu + \beta_{j\mu} J u_\mu\big] + \sum_{\mu=1}^{2(n-k)} \gamma_{j\mu}^1 v_\mu$$

$$\frac{\partial}{\partial y_j}(z_0) = J\frac{\partial}{\partial x_j}(z_0) = \sum_{\mu=1}^{k} \big[-\beta_{j\mu} u_\mu + \alpha_{j\mu} J u_\mu\big] + \sum_{\mu=1}^{2(n-k)} \gamma_{j\mu}^2 v_\mu$$

for suitable $\alpha_{j\mu}$, $\beta_{j\mu}$, $\gamma^1_{j\mu}$, $\gamma^2_{j\mu} \in \mathbf{R}$. In this way we can define

$$f_{x_j}(z_0) = \sum_{\mu=1}^k (\alpha_{j\mu} + i\beta_{j\mu}) u_\mu(f)$$

$$f_{y_j}(z_0) = \sum_{\mu=1}^k (-\beta_{j\mu} + i\alpha_{j\mu}) u_\mu(f).$$

Clearly, f_{x_j}, $f_{y_j} \in C^{r-1}(M)$, and $f_{\bar z_j} = \frac{1}{2}(f_{x_j} + i f_{y_j}) = 0$ for every $j = 1, \ldots, n$.

Iterating this argument, we can define the formal derivatives of f on M up to order r. It is easy to see that this set of continuous functions is a r-jet on the compact set $M \cap \overline{V}$ satisfying (2.6.32) — after all, f is a C^r function on M —; then by Whitney's extension theorem $f|_{M\cap V}$ is the restriction of a function $F \in C^k(\mathbf{C}^n)$. Moreover, it follows from the previous construction that F satisfies (2.6.33), and we are done, **q.e.d.**

Next we recall some facts about the $\overline{\partial}$-equation in $\mathbf{C}$. Let D be an open subset of $\mathbf{C}$, and take f, $g \in L^1_{\mathrm{loc}}(D)$. We shall say that $\partial f/\partial \bar z = g$ *in the weak sense* in D if for every $\varphi \in C^\infty(\mathbf{C})$ with compact support contained in D we have

$$\int f(\zeta) \frac{\partial \varphi}{\partial \bar\zeta}(\zeta)\, d\zeta \wedge d\bar\zeta = -\int g(\zeta)\varphi(\zeta)\, d\zeta \wedge d\bar\zeta.$$

By the way, we shall denote by $C^\infty_c(D)$ the space of smooth functions with compact support contained in D.

The first fact we shall need is

Lemma 2.6.39: *Let D be an open subset of D, and $f \in L^1_{\mathrm{loc}}(D)$. Then f is holomorphic in D iff $\partial f/\partial \bar z = 0$ in the weak sense in D.*

Proof: If f is holomorphic, integrating by parts we immediately get $\partial f/\partial \bar z = 0$ in the weak sense.

Conversely, assume $\partial f/\partial \bar z = 0$ in the weak sense in D; it suffices to show that f is holomorphic in a neighbourhood of a point of D, which we can assume to be the origin. Choose $\delta > 0$ such that $\Delta_{2\delta} \subset\subset D$, take $\rho \in C^\infty_c(\Delta_{2\delta})$ such that $\rho \equiv 1$ in a neighbourhood U of $\overline{\Delta}_\delta$, and set $g = \rho f$.

Choose a sequence of non-negative functions $\{\chi_\nu\} \subset C^\infty_c(\mathbf{C})$ such that $\chi_\nu \in C^\infty_c(\Delta_{1/\nu})$ and $\int \chi_\nu(\zeta)\, d\zeta \wedge d\bar\zeta = 1$, and set

$$g_\nu(z) = \chi_\nu \star g(z) = \int \chi_\nu(\zeta) g(z-\zeta)\, d\zeta \wedge d\bar\zeta;$$

clearly, $g_\nu \in C^\infty_c(\Delta_{2\delta})$ for all ν large enough. Now, if $\varphi \in C^\infty_c(\Delta_\delta)$ we have

$$\begin{aligned} \int g_\nu(z) \frac{\partial \varphi}{\partial \bar z}(z)\, dz \wedge d\bar z &= \int \chi_\nu(\zeta) \left[\int g(z-\zeta) \frac{\partial \varphi}{\partial \bar z}(z)\, dz \wedge d\bar z \right] d\zeta \wedge d\bar\zeta \\ &= \int_{\Delta_{1/\nu}} \chi_\nu(\zeta) \left[\int g(z) \frac{\partial \varphi}{\partial \bar z}(z+\zeta)\, dz \wedge d\bar z \right] d\zeta \wedge d\bar\zeta; \end{aligned} \tag{2.6.34}$$

hence if ν is so large that $\Delta_{\delta+1/\nu} \subset U$, then the latter integral in (2.6.34) is zero, and so $\partial g_\nu/\partial\bar{z} = 0$ in the weak sense in Δ_δ. Since $g_\nu \in C_c^\infty(\Delta_{2\delta})$, it follows, integrating by parts, that g_ν is holomorphic in Δ_δ for ν large enough.

Now, it is easy to check that $g_\nu \to g$ in $L^1(\Delta_\delta)$; since, by the Cauchy formula, $\mathrm{Hol}(\Delta_\delta, \mathbf{C}) \cap L^1(\Delta_\delta)$ is closed in $L^1(\Delta_\delta)$, it follows that g is holomorphic in Δ_δ, and we are done, **q.e.d.**

The second fact is an accurate description of the regularity of solutions of the $\overline{\partial}$-equation in $\mathbf{C}$:

Proposition 2.6.40: *Take $g \in L^\infty(\mathbf{C})$ with compact support K and define*

$$\forall z \in \mathbf{C} \qquad f(z) = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{g(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

Then:
(i) $\partial f/\partial\bar{z} = g$ *in the weak sense in* $\mathbf{C}$*;*
(ii) $f \in C^\infty(\mathbf{C} \setminus K)$*;*
(iii) $f \in C^{0,\alpha}(D)$ *for every* $\alpha \in (0,1)$ *and every bounded domain* $D \subset\subset \mathbf{C}$*;*
(iv) *if* $g \in C^{r,\alpha}(\mathbf{C})$ *for some* $r \in \mathbf{N}$ *and* $\alpha \in (0,1)$ *then* $f \in C^{r+1,\alpha}(\mathbf{C})$.

Proof: (i) Take $\varphi \in C_c^\infty(\mathbf{C})$; then

$$\begin{aligned}\int f(\zeta)\frac{\partial\varphi}{\partial\bar{\zeta}}(\zeta)\, d\zeta \wedge d\bar{\zeta} &= \frac{-1}{2\pi i}\int g(z)\left[\int \frac{\partial\varphi/\partial\bar{\zeta}(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\right] dz \wedge d\bar{z} \\ &= -\int g(z)\varphi(z)\, dz \wedge d\bar{z},\end{aligned}$$

by the Cauchy formula for C^1 functions.

(ii) Obvious.

(iii) Choose z_1, $z_2 \in D$, and set $\delta = |z_2 - z_1|$ and $z_0 = (z_1 + z_2)/2$. Then

$$f(z_2) - f(z_1) = \frac{1}{2\pi i}(I_1 + I_2 + I_3)$$

where

$$\begin{aligned} I_1 &= \int_{D(z_0,\delta)} \frac{g(\zeta)}{\zeta - z_2}\, d\zeta \wedge d\bar{\zeta}, \\ I_2 &= \int_{D(z_0,\delta)} \frac{g(\zeta)}{z_1 - \zeta}\, d\zeta \wedge d\bar{\zeta}, \\ I_3 &= \int_{\mathbf{C}\setminus D(z_0,\delta)} g(\zeta)\left[\frac{1}{\zeta - z_2} - \frac{1}{\zeta - z_1}\right] d\zeta \wedge d\bar{\zeta}. \end{aligned}$$

Now, $D(z_0, \delta) \subset D(z_j, 3\delta/2)$ for $j = 1$, 2, and so

$$|I_1| + |I_2| \leq 6\pi \|g\|_\infty |z_2 - z_1|.$$

Furthermore, for every $\zeta \in \mathbf{C} \setminus D(z_0, \delta)$ we have $|\zeta - z_j| \geq |\zeta - z_0|/2$ for $j = 1$, 2; hence

$$|I_3| \leq 8\pi \|g\|_\infty |z_2 - z_1| \log \frac{M}{|z_2 - z_1|},$$

where $M = \sup\bigl\{|\zeta - z_0| \bigm| \zeta \in K,\, z_1, z_2 \in D\bigr\}$, and (iii) follows.

(iv) After a change of variable we can write

$$f(z) = \frac{-1}{2\pi i} \int \frac{g(z-\zeta)}{\zeta}\, d\zeta \wedge d\bar{\zeta};$$

so if $g \in C^r(\mathbf{C})$ we have

$$D^\alpha f(z) = \frac{-1}{2\pi i} \int \frac{D^\alpha g(z-\zeta)}{\zeta}\, d\zeta \wedge d\bar{\zeta}$$

for every multi-index α of order at most r, and thus $f \in C^r(\mathbf{C})$. Hence it remains to show that $g \in C^{0,\alpha}(\mathbf{C})$ implies $f \in C^{1,\alpha}(C)$.

Choose a function $\eta \in C^\infty(\mathbf{R})$ such that $0 \leq \eta \leq 1$, $0 \leq \eta' \leq 2$, $\eta(t) = 0$ for all $t \leq 1$ and $\eta(t) = 1$ for all $t \geq 2$. For every $\varepsilon > 0$ set $\eta_\varepsilon(z, \zeta) = \eta(|\zeta - z|/\varepsilon)$ and

$$f_\varepsilon(z) = \frac{1}{2\pi i} \int \frac{g(\zeta)}{\zeta - z} \eta_\varepsilon(z, \zeta)\, d\zeta \wedge d\bar{\zeta}.$$

Clearly, $f_\varepsilon \in C^\infty(\mathbf{C})$ for every $\varepsilon > 0$; furthermore,

$$|f(z) - f_\varepsilon(z)| \leq \|g\|_\infty \int_0^{2\varepsilon} [1 - \eta(r/\varepsilon)]\, dr \leq 2\varepsilon \|g\|_\infty,$$

and $f_\varepsilon \to f$ uniformly on $\mathbf{C}$.

Now we have

$$\frac{\partial f_\varepsilon}{\partial \bar{z}}(z) = \frac{1}{2\pi i} \int \frac{g(\zeta) - g(z)}{\zeta - z} \frac{\partial \eta_\varepsilon}{\partial \bar{z}}(z, \zeta)\, d\zeta \wedge d\bar{\zeta} + g(z),$$

because

$$\int \frac{1}{\zeta - z} \frac{\partial \eta_\varepsilon}{\partial \bar{z}}(z, \zeta)\, d\zeta \wedge d\bar{\zeta} = -\int \frac{1}{\zeta - z} \frac{\partial \eta_\varepsilon}{\partial \bar{\zeta}}(z, \zeta)\, d\zeta \wedge d\bar{\zeta} = \int_{\partial D(z, 3\varepsilon)} \frac{d\zeta}{\zeta - z} = 2\pi i.$$

Furthermore,

$$\left| \int \frac{g(\zeta) - g(z)}{\zeta - z} \frac{\partial \eta_\varepsilon}{\partial \bar{z}}(z, \zeta)\, d\zeta \wedge d\bar{\zeta} \right| \leq \frac{\pi \|g\|_\alpha}{\varepsilon} \int_0^{3\varepsilon} r^\alpha \eta'(r/\varepsilon)\, dr \leq 2\pi \frac{3^{\alpha+1}}{\alpha+1} \|g\|_\alpha \, \varepsilon^\alpha; \quad (2.6.35)$$

hence $\partial f_\varepsilon / \partial \bar{z} \to g$ uniformly on $\mathbf{C}$.

Next,

$$\frac{\partial f_\varepsilon}{\partial z}(z) = \frac{1}{2\pi i} \int \frac{g(\zeta) - g(z)}{\zeta - z} \frac{\partial \eta_\varepsilon}{\partial z}(z, \zeta)\, d\zeta \wedge d\bar{\zeta} + \frac{1}{2\pi i} \int \frac{g(\zeta) - g(z)}{(\zeta - z)^2} \eta_\varepsilon(z, \zeta)\, d\zeta \wedge d\bar{\zeta}, \quad (2.6.36)$$

because, by Stokes' formula

$$\int \frac{\partial}{\partial z} \left(\frac{\eta_\varepsilon(z, \zeta)}{\zeta - z} \right) d\zeta \wedge d\bar{\zeta} = - \int_{\partial D(z, 3\varepsilon)} \frac{d\bar{\zeta}}{\zeta - z} = 0.$$

We already know, by (2.6.35), that the first summand in (2.6.36) goes uniformly to 0 as $\varepsilon \to 0$. For the second summand, we have

$$\left| \int \big[1 - \eta_\varepsilon(z, \zeta)\big] \frac{g(\zeta) - g(z)}{(\zeta - z)^2}\, d\zeta \wedge d\bar{\zeta} \right| \leq 2\pi \|g\|_\alpha \int_0^{2\varepsilon} \frac{dr}{r^{1-\alpha}} = \frac{2^{\alpha+1}\pi}{\alpha} \|g\|_\alpha \, \varepsilon^\alpha;$$

hence

$$\frac{\partial f_\varepsilon}{\partial z}(z) \longrightarrow h(z) = \frac{1}{2\pi i} \int \frac{g(\zeta) - g(z)}{(\zeta - z)^2}\, d\zeta \wedge d\bar{\zeta}$$

as $\varepsilon \to 0$, uniformly on $\mathbf{C}$, and, together with $f_\varepsilon \to f$ and $\partial f_\varepsilon / \partial \bar{z} \to g$ uniformly on $\mathbf{C}$, this yields $f \in C^1(\mathbf{C})$, $\partial f / \partial \bar{z} = g$ and $\partial f / \partial z = h$. Therefore it remains to show that $h \in C^{0,\alpha}(\mathbf{C})$.

Choose z_1, $z_2 \in D$, and set again $\delta = |z_2 - z_1|$ and $z_0 = (z_1 + z_2)/2$. Then

$$h(z_2) - h(z_1) = \frac{1}{2\pi i} \big[I_1 + I_2 + \big(g(z_1) - g(z_2)\big) I_3 + I_4 \big],$$

where

$$I_1 = \int_{D(z_0, \delta)} \frac{g(\zeta) - g(z_2)}{(\zeta - z_2)^2}\, d\zeta \wedge d\bar{\zeta};$$

$$I_2 = \int_{D(z_0, \delta)} \frac{g(z_1) - g(\zeta)}{(\zeta - z_1)^2}\, d\zeta \wedge d\bar{\zeta};$$

$$I_3 = \int_{\mathbf{C} \setminus D(z_0, \delta)} \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z_1)^2};$$

$$I_4 = \int_{\mathbf{C} \setminus D(z_0, \delta)} \big(g(z_2) - g(\zeta)\big) \left[\frac{1}{(\zeta - z_1)^2} - \frac{1}{(\zeta - z_2)^2} \right] d\zeta \wedge d\bar{\zeta}.$$

Now, $D(z_0,\delta)\subset D(z_j,3\delta/2)$ for $j=1$, 2, and so

$$|I_1|+|I_2|\le \frac{4\pi}{\alpha}\|g\|_\alpha\left(\frac{3}{2}\right)^\alpha|z_2-z_1|^\alpha.$$

Furthermore, using Stokes' formula we find

$$I_3=-\int\limits_{\partial D(z_0,\delta)}\frac{d\bar\zeta}{\zeta-z_1}=0,$$

because $|\zeta-z_1|\ge\delta/2$ for every $\zeta\in\partial D(z_0,\delta)$.

Finally, for every $\zeta\in\mathbf{C}\setminus D(z_0,\delta)$ we have

$$\frac{3|\zeta-z_0|}{2}\ge|\zeta-z_j|\ge\frac{|\zeta-z_0|}{2}$$

for $j=1$, 2; so

$$|I_4|\le\frac{24\pi}{1-\alpha}\|g\|_\alpha|z_2-z_1|^\alpha,$$

and we are done, **q.e.d.**

We shall use these results to cook up a sort of reflection principle with respect to a totally real manifold, as shown in the following proposition, which is the core of our argument:

Proposition 2.6.41: *Let $U\subset\mathbf{C}$ be an open neighborhood of a point $\tau_0\in\partial\Delta$ such that $U\cap\Delta$ is connected, X a complex manifold, and M a totally real closed C^r submanifold ($r=2,3,\ldots,\infty,\omega$) of X with $\dim_{\mathbf{R}}M=\dim_{\mathbf{C}}X$. Take $g\in\mathrm{Hol}(U\cap\Delta,X)\cap C^{0,1/2}(U\cap\overline{\Delta})$ such that $g(U\cap\partial\Delta)\subset M$; then $g\in C^{r-1,\alpha}(U\cap\overline{\Delta})$ for every $\alpha\in(0,1)$.*

Proof: Since the statement is local, we can made some reduction on the hypotheses. First of all, we can replace Δ by H^+, assume that U is convex, symmetric with repect to the real axis, and that g satisfies the hypotheses in a slightly larger neighbourhood of τ_0 (so that $g(U\cap\mathbf{R})\subset\subset M$). Furthermore, we can assume $X=\mathbf{C}^n$ and that there is a C^r-diffeomorphism Φ of M with an open subset V of $\mathbf{R}^n$. Finally, set $A=U\cap\mathbf{R}$.

If $r=\omega$, Φ is the restriction of a biholomorphic map $\widetilde\Phi$ of a neighbourhood of M with a neighbourhood of V in $\mathbf{C}^n\supset\mathbf{R}^n$. Then $\widetilde\Phi\circ g$ is holomorphic in $U\cap H^+$, continuous on A and $\widetilde\Phi\circ g(A)\subset\mathbf{R}^n$; by the Schwarz reflection principle, $\widetilde\Phi\circ g$ extends holomorphically across A, and so $g\in C^\omega(U\cap A)$.

Assume now $r<\omega$. By Lemma 2.6.38, slightly shrinking M if necessary, we can find $\widetilde\Phi\in C^r(\mathbf{C}^n)$ such that $\widetilde\Phi|_M=\Phi$ and

$$D^\alpha\frac{\partial\widetilde\Phi}{\partial\bar z}\bigg|_M\equiv 0$$

for all multi-index α with $|\alpha| \le r-1$.

Set $h = \widetilde{\Phi} \circ g \colon \overline{U \cap H^+} \to \mathbf{C}^n$ and define $H \colon U \to \mathbf{C}^n$ by

$$H(\zeta) = \begin{cases} h(\zeta) & \text{if } \operatorname{Im}\zeta \ge 0; \\ \overline{h(\bar\zeta)} & \text{if } \operatorname{Im}\zeta \le 0. \end{cases}$$

Clearly, $H \in C^0(U) \cap C^r(U \setminus \mathbf{R})$. Now set

$$\phi(\zeta) = \begin{cases} \dfrac{\partial H}{\partial \bar\zeta}(\zeta) & \text{if } \zeta \in U \setminus A; \\ 0 & \text{if } \zeta \in A; \end{cases}$$

we claim that ϕ is continuous in U. We have

$$\forall \zeta \in U \cap H^+ \qquad \frac{\partial H}{\partial \bar\zeta}(\zeta) = \left\langle \frac{\partial \widetilde{\Phi}}{\partial \bar z}\big(g(\zeta)\big), \overline{g'(\zeta)} \right\rangle. \tag{2.6.37}$$

Take $z \in \mathbf{C}^n$, and choose $z_0 \in M$ such that $\|z - z_0\| = d(z, M)$. Then

$$\begin{aligned} \left\| \frac{\partial \widetilde{\Phi}}{\partial \bar z}(z) \right\| &= \left\| \frac{\partial \widetilde{\Phi}}{\partial \bar z}(z) - \frac{\partial \widetilde{\Phi}}{\partial \bar z}(z_0) \right\| \\ &\le \|z - z_0\| \sum_{j=1}^n \int_0^1 \left[\left\| \frac{\partial^2 \widetilde{\Phi}}{\partial z_j \partial \bar z}\big(z_0 + t(z - z_0)\big) \right\| + \left\| \frac{\partial^2 \widetilde{\Phi}}{\partial \bar z_j \partial \bar z}\big(z_0 + t(z - z_0)\big) \right\| \right] dt \\ &\le o(1)\, d(z, M) \end{aligned}$$

as $z \to M$. Thus, since if $\zeta \in U \setminus A$ then $\operatorname{Re}\zeta \in A$ (for U is convex and symmetric with respect to the real axis), we have

$$\left\| \frac{\partial \widetilde{\Phi}}{\partial \bar z}\big(g(\zeta)\big) \right\| \le o(1)\, d\big(g(\zeta), M\big) \le o(1) \|g(\zeta) - g(\operatorname{Re}\zeta)\| \le o(1) |\operatorname{Im}\zeta|^{1/2}$$

as $\zeta \to A$, for $g \in C^{0,1/2}(\overline{U \cap H^+})$. Finally, $\|g'(\zeta)\| \le O(1) |\operatorname{Im}\zeta|^{-1/2}$, by Theorem 2.6.26, and hence ϕ is continuous in U, as claimed.

Now fix $U' \subset\subset U$, and choose a non-negative function $\chi \in C^\infty_c(U)$ such that $\chi \equiv 1$ in a neighbourhood of $\overline{U'}$. Then applying Proposition 2.6.40 to $\chi\phi$ we find a function $\psi \in C^0(\mathbf{C})$ such that $\partial\psi/\partial\bar\zeta = \phi$ in the weak sense on U', and moreover $\psi \in C^{0,\alpha}(U')$ for all $\alpha \in (0,1)$. Then (by Lemma 2.6.39) $H - \psi$ is holomorphic in $U' \setminus A$, and continuous in U'; therefore $H - \psi$ is holomorphic in U', and then $H \in C^{0,\alpha}(U')$. Being U' arbitrary, it follows that $h, g \in C^{0,\alpha}(\overline{U \cap H^+})$.

Now, arguing as before, we find that

$$\left\| \frac{\partial \widetilde{\Phi}}{\partial \bar z}\big(g(\zeta)\big) \right\| \le o(1) |\operatorname{Im}\zeta|^\alpha$$
$$\|g'(\zeta)\| \le O(1) |\operatorname{Im}\zeta|^{\alpha - 1}$$

as $\zeta \to A$; therefore $\phi \in C^{0,2\alpha-1}(U')$, $\psi \in C^{1,2\alpha-1}(U')$ and finally $h, g \in C^{1,2\alpha-1}(\overline{U \cap H^+})$.

If $r = 2$ we have finished. If $r > 2$, we repeat the argument: we know that $g' \in C^{0,2\alpha-1}(\overline{U \cap H^+})$, and thus $\|g''(\zeta)\| \le O(1) |\operatorname{Im}\zeta|^{2\alpha-2}$ as $\zeta \to A$. Differentiating (2.6.37) we find $\phi \in C^{1,3\alpha-2}(U)$; thus $\psi \in C^{2,3\alpha-2}(U')$ and $h, g \in C^{2,3\alpha-2}(\overline{U \cap H^+})$, and so on, **q.e.d.**

To apply this bootstrap lemma, we need examples of totally real submanifolds. Denote by $\mathbf{P}_{n-1}$ the complex projective space of complex hyperplanes in $\mathbf{C}^n$. Then:

Lemma 2.6.42: *Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain, and define $\Psi\colon \partial D \to \mathbf{C}^n \times \mathbf{P}_{n-1}$ by*

$$\forall x \in \partial D \qquad \Psi(x) = \big(x, T_x^{\mathbf{C}} \partial D\big), \tag{2.6.38}$$

and set $M = \Psi(\partial D)$. Then M is a totally real submanifold of $\mathbf{C}^n \times \mathbf{P}_{n-1}$ iff the Levi form of D is nondegenerate.

Proof: Let $\rho \in C^2(\mathbf{C}^n)$ be a defining function for D. Fix $x \in \partial D$; we can assume

$$\frac{\partial \rho}{\partial z}(x) = (0, \dots, 0, 1). \tag{2.6.39}$$

Let $z_1, \dots, z_n$, $p_1, \dots, p_{n-1}$ be coordinates of $\mathbf{C}^n \times \mathbf{P}_{n-1}$ in a neighbourhood of $\Psi(x)$; clearly, M is defined by

$$\begin{cases} \rho(z) = 0 \\ \rho_\alpha(z) - \rho_n(z) p_\alpha = 0 \quad \text{for } \alpha = 1, \dots, n-1, \end{cases}$$

where the subscript α in ρ_α denotes partial differentiation with respect to z_α. Therefore $(v, a) \in \mathbf{C}^n \times \mathbf{C}^{n-1}$ belongs to $T_{\Psi(x)}M$ iff

$$\begin{cases} \operatorname{Re} \sum\limits_{\mu=1}^{n} \rho_\mu(x) v = 0, \\ \operatorname{Re} \sum\limits_{\mu=1}^{n} \rho_{\bar\mu\alpha}(x) \overline{v}_\mu - p_\alpha \operatorname{Re} \sum\limits_{\mu=1}^{n} \rho_{\bar\mu n}(x) \overline{v}_\mu = a_\alpha \quad \text{for } \alpha = 1, \dots, n-1. \end{cases}$$

Thus M is not totally real in x iff there is $(v, a) \in \mathbf{C}^n \times \mathbf{C}^{n-1}$ such that we have (v, a), $(iv, ia) \in T_{\Psi(x)}M$, and then iff $A_\rho(x)\overline{v} = 0$, where A_ρ is the $n \times n$ matrix

$$A_\rho = \begin{pmatrix} \rho_{\bar\mu} & 1 \\ \rho_{\bar\mu\alpha} - p_\alpha \rho_{\bar\mu n} & \rho_{\bar n\alpha} - p_\alpha \rho_{\bar n n} \end{pmatrix}.$$

Recalling (2.6.39), we see that $A_\rho(x)\overline{v} = 0$ iff $v \in T_x^{\mathbf{C}} \partial D$ is in the kernel of the Levi form of D at x, and we are done, **q.e.d.**

And so we are finally ready to prove

Theorem 2.6.43: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^r domain ($r = 3, \dots, \infty, \omega$), and $\varphi \in \operatorname{Hol}(\Delta, D)$ a complex geodesic. Then we have φ, $\varphi^* \in C^{r-2,\alpha}(\overline{\Delta})$, $\tilde p \in C^{r-1,\alpha}(\overline{D})$ and $p \in C^{r-2,\alpha}(\overline{D})$ for every $\alpha \in (0,1)$, where, as usual, $\tilde p$ is the left inverse of φ and $p = \varphi \circ \tilde p$.*

Proof: By Lemmas 2.6.21, 2.6.32 and 2.6.33, φ^* is never vanishing on $\overline{\Delta}$. Therefore the map $\tilde\varphi = \pi \circ \varphi^* \colon \overline{\Delta} \to \mathbf{P}_{n-1}$ is well defined, where $\pi\colon \mathbf{C}^n \to \mathbf{P}_{n-1}$ is the canonical projection and we are identifying $\mathbf{C}^n$ and $(\mathbf{C}^n)^*$. Now, the map $(\varphi, \tilde{\varphi}): \overline{\Delta} \to \mathbf{C}^n \times \mathbf{P}_{n-1}$ is 1/2-Hölder, holomorphic in Δ and $(\varphi, \tilde{\varphi}) \subset \Psi(\partial D)$, where $\Psi: \partial D \to \mathbf{C}^n \times \mathbf{P}_{n-1}$ is defined in (2.6.38). Then Lemma 2.6.42 and Proposition 2.6.41 yield $(\varphi, \tilde{\varphi}) \in C^{r-2,\alpha}(\overline{\Delta})$; thus, in particular, $\varphi \in C^{r-2,\alpha}(\overline{\Delta})$. Since this implies that $\mathbf{n} \circ \varphi \in C^{r-2,\alpha}(\partial\Delta)$ and the regularity of p and $\tilde{p}$ immediately follows from the regularity of φ and φ^*, it remains to show that $\mu = \|\varphi^*\| \in C^{r-2,\alpha}(\partial\Delta)$.

Fix $\tau_0 \in \partial\Delta$; we can assume $(\mathbf{n}_{\varphi(\tau_0)})_1 \neq 0$. Choose $\chi \in C^{r-2,\alpha}(\partial\Delta)$ such that

$$\exp\bigl(\chi(\tau)\bigr) = \overline{(\mathbf{n}_{\varphi(\tau)})_1}$$

in a neighbourhood $V \cap \partial\Delta$ of τ_0 in $\partial\Delta$. As usual, extend $-\operatorname{Im}\chi$ to a harmonic function in Δ, and let $\rho: \overline{\Delta} \to \mathbf{R}$ be a conjugate function; by Corollary 2.6.27, $\rho \in C^{r-2,\alpha}(\overline{\Delta})$.

Now, the functions $\mu(\tau)\overline{(\mathbf{n}_{\varphi(\tau)})_1}$ and $\overline{(\mathbf{n}_{\varphi(\tau)})_1} \exp\{\rho(\tau) - \operatorname{Re}\chi(\tau)\}$ extends to never vanishing holomorphic functions in $\Delta \cap V$; therefore $\mu(\tau)\exp\{\operatorname{Re}\chi(\tau) - \rho(\tau)\}$ extends too. This latter function is real on $\partial\Delta \cap V$; hence it extends holomorphically across $\partial\Delta$, and, in consideration of the regularity of χ and ρ, it follows that $\mu \in C^{r-2,\alpha}(V \cap \partial\Delta)$. But $\tau_0 \in \partial\Delta$ was an arbitrary point of $\partial\Delta$; therefore $\mu \in C^{r-2,\alpha}(\partial\Delta)$, and we are done, **q.e.d.**

Now we can study in detail the existence and uniqueness of geodesic disks containing (in their closure) points of the boundary of the domain. In particular, we are interested in geodesic disks containing a point $z_0 \in D$ and a point $x \in \partial D$, or containing two distinct points x_1, $x_2 \in \partial D$. As the quick reader can imagine, we have both existence and uniqueness; by the way, the proofs rely in an interesting way on the theory developed in chapter 1.2.

We need the following lemma:

Lemma 2.6.44: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain, $\varphi \in \operatorname{Hol}(\Delta, D)$ a complex geodesic and $\tilde{p} \in \operatorname{Hol}(D, \Delta)$ its left inverse. Then for every $\tau \in \partial\Delta$ we have*

$$\forall v \in \mathbf{C}^n \qquad d\tilde{p}_{\varphi(\tau)}(v) = \langle v, \varphi^*(\tau)\rangle = \frac{(v, \mathbf{n}_{\varphi(\tau)})}{\bigl(\varphi'(\tau), \mathbf{n}_{\varphi(\tau)}\bigr)}.$$

Proof: Since for every $\zeta \in \Delta$ we have $d\tilde{p}_{\varphi(\zeta)}\bigl(\varphi'(\zeta)\bigr) = 1$ and

$$\ker d\tilde{p}_{\varphi(\zeta)} = \{v \in \mathbf{C}^n \mid \langle v, \varphi^*(\zeta)\rangle = 0\},$$

(2.6.30) implies

$$\forall v \in \mathbf{C}^n \qquad d\tilde{p}_{\varphi(\zeta)}(v) = \langle v, \varphi^*(\zeta)\rangle,$$

and, letting $\zeta \to \tau$, the assertion follows, **q.e.d.**

Then

Theorem 2.6.45: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain. Then:*
(i) *for every $z_0 \in D$ and $x \in \partial D$ there exists a unique complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ such that $\varphi(0) = z_0$ and $\varphi(1) = x$;*
(ii) *for every pair of distinct points $x_1, x_2 \in \partial D$ there exists a unique (up to parametrization) complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ such that $\varphi(1) = x_1$ and $\varphi(-1) = x_2$.*

Proof: (i) We begin with the existence. Let $\{z_\nu\} \subset D$ be a sequence converging to x, and let $\varphi_\nu \in \mathrm{Hol}(\Delta, D)$ be the complex geodesic such that $\varphi_\nu(0) = z_0$ and $z_\nu \in \varphi_\nu\big((0,1)\big)$. Since D is taut, up to a subsequence we can assume that $\{\varphi_\nu\}$ converges to a map $\varphi \in \mathrm{Hol}(\Delta, D)$. Clearly, $\varphi(0) = z_0$; moreover, for all $\zeta \in \Delta$ we have

$$k_D\big(z_0, \varphi(\zeta)\big) = \lim_{\nu\to\infty} k_D\big(z_0, \varphi_\nu(\zeta)\big) = \omega(0, \zeta),$$

and φ is a complex geodesic. Then $\varphi \in C^1(\overline{\Delta})$, and clearly $\varphi(1) = x$.

Assume now $\psi \in \mathrm{Hol}(\Delta, D)$ is another complex geodesic with $\psi(0) = z_0$ and $\psi(1) = x$; denote by $\tilde{p}$ the left inverse of φ, and by $\tilde{q}$ the left inverse of ψ. We claim that

$$\tilde{p} \circ \psi = \mathrm{id}_\Delta\,. \tag{2.6.40}$$

In fact, let $f = \tilde{p} \circ \psi$. Clearly, $f \in \mathrm{Hol}(\Delta, \Delta) \cap C^1(\overline{\Delta})$, $f(0) = 0$ and $f(1) = 1$; moreover, by Lemma 2.6.44,

$$f'(1) = d\tilde{p}_x\big(\psi'(1)\big) = \frac{(\psi'(1), \mathbf{n}_x)}{(\varphi'(1), \mathbf{n}_x)}.$$

Analogously, if we set $g = \tilde{q} \circ \varphi$, we have $g(0) = 0$, $g(1) = 1$ and

$$g'(1) = \frac{(\varphi'(1), \mathbf{n}_x)}{(\psi'(1), \mathbf{n}_x)} = \frac{1}{f'(1)}.$$

Then Corollary 1.2.10 yields $f'(1) = g'(1) = 1$ and $f = \mathrm{id}_\Delta$, as claimed.

Now, (2.6.40) implies that $\tilde{p}\big(\psi(\tau)\big) = \tau$ for every $\tau \in \partial\Delta$; hence, by (2.6.31), $\psi|_{\partial\Delta} = \varphi|_{\partial\Delta}$, and thus $\psi \equiv \varphi$.

(ii) We begin with the existence again. Let $\{z_\nu\} \subset D$ be a sequence converging to x_2, and denote by $\varphi_\nu \in \mathrm{Hol}(\Delta, D)$ a complex geodesic such that $\varphi_\nu(1) = x_1$, $z_\nu \in \varphi_\nu\big((-1,1)\big)$ and

$$\|\varphi_\nu(0) - x_2\| < \frac{\|x_2 - x_1\|}{2}. \tag{2.6.41}$$

Since D is bounded, up to a subsequence we can assume that $\{\varphi_\nu\}$ converges to a holomorphic map $\varphi\colon \Delta \to \mathbf{C}^n$. Since D is strongly convex, either $\varphi(\Delta) \subset D$ or φ is a constant contained in ∂D. The latter possibility cannot occur, by (2.6.41); so $\varphi \in \mathrm{Hol}(\Delta, D)$, and it is clear that φ is as desired.

Assume now that ψ is another complex geodesic with $\psi(1) = x_1$ and $\psi(-1) = x_2$, and denote again by $\tilde{p}$ (respectively, $\tilde{q}$) the left inverse of φ (respectively ψ). We claim that this time

$$\tilde{p} \circ \psi \in \mathrm{Aut}(\Delta).$$

Indeed, $\tilde{p} \circ \psi(1) = 1$, $\tilde{p} \circ \psi(-1) = -1$ and, by Lemma 2.6.44,

$$(\tilde{p} \circ \psi)'(1) \cdot (\tilde{p} \circ \psi)'(-1) = \frac{(\psi'(1), \mathbf{n}_{x_1})}{(\varphi'(1), \mathbf{n}_{x_1})} \cdot \frac{(\psi'(-1), \mathbf{n}_{x_2})}{(\varphi'(-1), \mathbf{n}_{x_2})}.$$

Analogously, $\tilde{q} \circ \varphi(1) = 1$, $\tilde{q} \circ \varphi(-1) = -1$ and

$$(\tilde{q} \circ \varphi)'(1) \cdot (\tilde{q} \circ \varphi)'(-1) = \frac{1}{(\tilde{p} \circ \psi)'(1) \cdot (\tilde{p} \circ \psi)'(-1)};$$

then Theorem 1.2.11 yields $(\tilde{p} \circ \psi)'(1)(\tilde{p} \circ \psi)'(-1) = 1$ and $\tilde{p} \circ \psi \in \mathrm{Aut}(\Delta)$. Hence, up to parametrization, we can assume $\tilde{p} \circ \psi = \mathrm{id}_\Delta$, and then, exactly as before, it follows that $\psi \equiv \varphi$, **q.e.d.**

Fix $z_0 \in D$, and for every $z \in \overline{D} \setminus \{z_0\}$ denote by $\varphi_z \in \mathrm{Hol}(\Delta, D)$ the unique complex geodesic such that $\varphi_z(0) = z_0$ and $z \in \varphi_z\big((0,1]\big)$. It is easy to check that, as a consequence of the uniqueness of φ_z, the map $z \mapsto \varphi_z$ from $\overline{D} \setminus \{z_0\}$ into $\mathrm{Hol}(\Delta, D)$ is continuous. Lempert [1981] has shown that even more is true: if D is a strongly convex C^r domain, with $r \geq 3$, then φ_z is a C^{r-2} function of $z \in \overline{D} \setminus \{z_0\}$. A corollary of this result is that if we set $K_{z_0}(z) = \tanh\big(k_D(z_0, z)\big)$ for $z \in D$ and $K_{z_0}(x) = 1$ for $x \in \partial D$, so that $\varphi_z\big(K_{z_0}(z)\big) = z$ for all $z \in \overline{D}$, then $K_{z_0} \in C^{r-2}(\overline{D} \setminus \{z_0\})$.

Unfortunately, the methods used in Lempert [1981] are out of the scope of this book; so we decided to omit the proof of this fact (see also the notes). However, since we shall need it once, to prove the last theorem of this chapter, we quote it officially:

Proposition 2.6.46: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and fix $z_0 \in D$. Then:*
(i) *K_{z_0} is continuous in $\overline{D}$ and $\log K_{z_0}$ is plurisubharmonic in D;*
(ii) *if D is strongly convex with C^r boundary, $r \geq 3$, then $K_{z_0} \in C^{r-2}(\overline{D} \setminus \{z_0\})$.*

Proof: We prove only part (i). The continuity is obvious. Next, by Proposition 2.6.25,

$$\forall z \in D \quad \log K_{z_0}(z) = \sup\big\{\log|f(z)| \bigm| f \in \mathrm{Hol}(D, \Delta),\ f(z_0) = 0\big\};$$

so K_{z_0} is the supremum of a family of plurisubharmonic functions, and hence it is plurisubharmonic, **q.e.d.**

Then we can prove:

Theorem 2.6.47: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain, and fix $z_0 \in D$ and $x \in \partial D$. Then*

$$\lim_{w \to x} \big[k_D(z, w) - k_D(z_0, w)\big] = \tfrac{1}{2} \log\left[\frac{\partial K_{z_0}}{\partial \mathbf{n}_x}(x) \bigg/ \frac{\partial K_z}{\partial \mathbf{n}_x}(x)\right].$$

Proof: The first observation is that

$$k_D(z, w) - k_D(z_0, w) = \tfrac{1}{2} \log\left[\frac{1 + K_z(w)}{1 - K_z(w)} \cdot \frac{1 - K_{z_0}(w)}{1 + K_{z_0}(w)}\right];$$

therefore it suffices to study the quotient $\big(1-K_{z_0}(w)\big)/\big(1-K_z(w)\big)$. Since D is strongly convex, for any $w\in D$ the real half-line issuing from w and parallel to $\mathbf{n}_x$ meets ∂D in exactly one point $\tilde w$, and $w\mapsto\tilde w$ is a C^1 function of $w\in D$. Then Lagrange's theorem applied to the real segment bounded by w and $\tilde w$ yields

$$\frac{1-K_{z_0}(w)}{1-K_z(w)}=\frac{K_{z_0}(\tilde w)-K_{z_0}(w)}{\|\tilde w-w\|}\bigg/\frac{K_z(\tilde w)-K_z(w)}{\|\tilde w-w\|}=\frac{\partial K_{z_0}}{\partial\mathbf{n}_x}(w')\bigg/\frac{\partial K_z}{\partial\mathbf{n}_x}(w''),$$

for suitable w' and w'' contained in the segment $[w,\tilde w]$. Since, by Propositions 2.3.57 and 2.6.46, $\partial K_z(x)/\partial\mathbf{n}_x$ exists and it is nonzero for all $z\in D$, we can take the limit as $w\to x$; then w', $w''\to x$ and

$$\lim_{w\to x}\frac{1-K_{z_0}(w)}{1-K_z(w)}=\frac{\partial K_{z_0}}{\partial\mathbf{n}_x}(x)\bigg/\frac{\partial K_z}{\partial\mathbf{n}_x}(x),$$

q.e.d.

It follows that in strongly convex C^3 domains big and small horospheres coincide:

Corollary 2.6.48: *Let $D\subset\subset\mathbf{C}^n$ be a strongly convex C^3 domain. Then for any $z_0\in D$, $x\in\partial D$ and $R>0$ we have*

$$E_{z_0}(x,R)=F_{z_0}(x,R).$$

Proof: Indeed Theorem 2.6.47 implies the existence of the limit in the definition of horospheres, **q.e.d.**

In particular, the horospheres are convex sets:

Corollary 2.6.49: *Let $D\subset\subset\mathbf{C}^n$ be a strongly convex C^3 domain. Then the horospheres are convex subsets of D, strongly convex near their center.*

Proof: This follows from Proposition 2.3.46 and Theorem 2.6.47, **q.e.d.**

Notes

The concept of complex geodesic was first introduced in Vesentini [1979] — to study the automorphism group of the unit ball of $L^1(M,\mu)$, where (M,μ) is a measure space — and later developed in Vesentini [1981, 1982a, b]; the general facts described in section 2.6.1 come from those papers. Corollary 2.6.7 was originally proved in a more general setting by Thorp and Whitley [1967]; our proof is due to Harris [1969].

There are just a few cases other than B^n where the complex geodesics are explicitely described: for instance, the bounded symmetric domains (Abate [1985]) and the domains

$$\{z\in\mathbf{C}^n\mid |z_1|^\alpha+\cdots+|z_n|^\alpha<1\},$$

see Poletskiĭ [1983] for $\alpha>1$, and Gentili [1988] for $\alpha=1$.

Theorem 2.6.12.(i) and (iii) goes back to Fatou [1906] and Plessner [1923]; Theorem 2.6.12.(ii), (iv) and (v) is substantially due to F. and M. Riesz [1916].

At present, the most important works on complex geodesics are Lempert [1981] and Royden and Wong [1983]; our exposition is largely inspired to these papers. The content of Royden and Wong [1983] is substantially reproduced in sections 2.6.2 and 2.6.3: Theorems 2.6.11, 2.6.17 and 2.6.19 as well as Corollaries 2.6.16, 2.6.18 and 2.6.20 come from that source. The proof of Theorem 2.6.15 has been suggested by Vigué.

Concerning Corollary 2.6.20, every complex geodesic is always an infinitesimal complex geodesic, but the converse in general is false; see Vigué [1984b] and Venturini [1988a]. On the other hand, every complex C-geodesic is an infinitesimal complex C-geodesic, and conversely; cf. Proposition 2.6.3 and Vesentini [1981].

An investigation of the non-unicity of complex geodesics in convex domains is Gentili [1985, 1986].

The discussion at the end of section 2.6.3 culminating in the definition of holomorphic retraction associated to a complex geodesic is taken from Lempert [1982]; see also Lempert [1984].

Corollaries 2.6.23 and 2.6.24 have been originally proved by Vigué [1984a, 1985]. Vesentini [1982a] proved Corollary 2.6.24 for Δ^2. In generic convex domains it is not true that every triple of points is contained in a 2-dimensional holomorphic retract; see Lempert [1982].

Proposition 2.6.25 is in Lempert [1982]. A different proof of Proposition 2.6.25 is described in Royden and Wong [1983]. It should be remarked that the equality of the Carathéodory and Kobayashi distances entails the existence of complex geodesics; see Vigué [1985]. In particular, the only hyperbolic Riemann surface X where $c_X = k_X$ is the disk. An interesting (and probably difficult) question is to characterize the hyperbolic manifolds where the Kobayashi and Carathéodory distances coincide. A first partial result follows from Barth [1983] and Vigué [1984b]: if $D \subset\subset \mathbf{C}^n$ is a balanced domain, then $c_D = k_D$ iff D is convex; see also Stanton [1980, 1983], and compare with Theorem 2.3.43.

Theorem 2.6.26 is due to Hardy and Littlewood [1932], while Corollary 2.6.27 goes back to Privalov [1918].

The theory of complex geodesics in strongly convex domains has been developed by Lempert [1981, 1984]: Theorems 2.6.29, 2.6.34 and 2.6.43 are his. Actually, Lempert used an approach slightly different from ours: he studied the maps satisfying the conditions decribed in Corollary 2.6.36, showing that they are complex geodesics, and an involved continuity argument allowed him to prove Corollary 2.6.30 directly. We tried to merge Royden's and Wong's ideas, leading to an easier proof of the existence of complex geodesics, with Lempert's techniques, quite more sensitive to boundary phenomena; unfortunately, this approach forced us to leave out the proof of Proposition 2.6.46.

In Abate [1986] there is an extension of Theorem 2.6.29 to strongly pseudoconvex domains.

Theorem 2.6.37 is due to Whitney [1934]. The proof of Lemma 2.6.38 comes from Harvey and Wells [1972]; more direct (and longer) proofs can be found in Hörmander and Wermer [1968] and in Niremberg and Wells [1969].

Proposition 2.6.41 is in Lempert [1981]. Theorem 2.6.45 is taken from Abate [1988d];

the proof is inspired by Lempert [1984].

The proof of Proposition 2.6.46.(ii) goes as follows (see Lempert [1981, 1984]): let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^r domain, with $r \geq 3$, fix a base point $z_0 \in D$, and for each point $z \in \overline{D} \setminus \{z_0\}$ denote by $\varphi_z \in \mathrm{Hol}(\Delta, D)$ the unique complex geodesic such that $\varphi_z(0) = z_0$ and $\varphi_z\big(K_{z_0}(z)\big) = z$. Then there are suitable Banach spaces X and Y, constructed starting from $A_n^{0,1/2}(\Delta)$, and a C^{r-2} map $\Psi: \overline{D} \setminus \{z_0\} \times \overline{\Delta} \times X \to Y$ such that $\Psi(z, K_{z_0}(z), \varphi_z) \equiv 0$. Then, by the implicit function theorem (the hard part of the proof is exactly to show that we can apply the implicit function theorem), the maps $z \mapsto \varphi_z$ and K_{z_0} are in $C^{r-2}(\overline{D} \setminus \{z_0\})$. The smoothness at z_0 is almost never achieved, not even for $(K_{z_0})^2$: see Patrizio [1986].

In Lempert [1981] there is an interesting application of Proposition 2.6.46.(ii). Define a map $\Phi_{z_0}: \overline{D} \to \overline{B^n}$ by setting $\Phi_{z_0}(z_0) = 0$ and

$$\Phi_{z_0}(z) = K_{z_0}(z) \, \frac{\varphi_z'(0)}{\|\varphi_z'(0)\|}.$$

It is easy to check that Φ_{z_0} is a homeomorphism of $\overline{D}$ with $\overline{B^n}$; furthermore, the regularity of φ_z and K_{z_0} implies that $\Phi_{z_0} \in C^{r-2}(\overline{D} \setminus \{z_0\})$. Then $\log \|\Phi_{z_0}\|$ is a solution of the complex Monge-Ampère equation on $D \setminus \{z_0\}$

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} \right) = 0 \tag{2.6.42}$$

with a logarithmic singularity at z_0. Later on, Lempert's approach has been used by Patrizio [1987] to study the complex geodesics in strongly pseudoconvex smooth circular domains by means of the Monge-Ampère equation. For more information on (2.6.42) consult Bedford and Taylor [1976] and Demailly [1987].

The proof of Theorem 2.6.47 is due to Venturini, and is taken from Abate [1988f]. In the terminology of Balmann, Gromov and Schroeder [1985] described in the notes to chapter 2.4, Corollary 2.6.48 says that the ideal boundary of a strongly convex C^3 domain D coincides with the topological boundary of D.