# Chapter 1.4
# One-parameter semigroups

Let's now look at iteration theory from another point of view. Let $X$ be a Riemann surface; then $\mathrm{Hol}(X, X)$, endowed, as usual, with the compact-open topology, is a *topological semigroup with identity,* that is the composition product $(f, g) \mapsto f \circ g$ is continuous, associative and has an identity. Consider now a semigroup homomorphism $\Phi\colon \mathbf{N} \to \mathrm{Hol}(X, X)$. Then $\Phi$ is the same thing as the sequence of iterates of the single function $\Phi(1)$; in other words, in the previous chapter we have actually studied semigroup homomorphisms of $\mathbf{N}$ into $\mathrm{Hol}(X, X)$.

It is now evident that a natural generalization of the sequence of iterates is a *one-parameter semigroup,* i.e., a continuous semigroup homomorphism $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X, X)$. In this chapter we shall thoroughly study these objects, aiming toward a complete classification. This will be possible because on Riemann surfaces with non-abelian fundamental group every one-parameter semigroup $\Phi$ is *trivial,* i.e., $\Phi_t = \mathrm{id}_X$ for all $t \geq 0$. Furthermore, the one-parameter semigroups on other Riemann surfaces (but the disk) are easily classified (section 1.4.3); so the main problem is the description of one-parameter semigroups on $\Delta$.

We shall actually provide three different descriptions on $\Delta$, answering three different kinds of problems. We shall show how to replace $\Delta$ by another simply connected domain (in essentially a unique way) so to express a generic one-parameter semigroup in a particularly simple form. We shall show how to relate one-parameter semigroups to Cauchy problems and ordinary differential equations, proving that a semigroup is completely determined by a holomorphic function $F\colon \Delta \to \mathbf{C}$, its infinitesimal generator. Finally, we shall give both a differential characterization and a completely explicit description of infinitesimal generators.

A last word of advice: we shall freely use standard facts about ordinary differential equations (collected for easy reference in Theorem 1.4.9). Proofs can be found, e.g., in Narasimhan [1968] or Hörmander [1973].

## 1.4.1 *The infinitesimal generator*

In this section we collect some general facts regarding one-parameter semigroups: univalence, fixed points and the like. Our principal aim will be the construction of the infinitesimal generator of a one-parameter semigroup, thus relating our theory to ordinary differential equations.

We begin recalling some (well known) general lemmas about homomorphic images of $\mathbf{R}^+$:

**Lemma 1.4.1:** *Let $G$ be a group. Then:*
(i) *every semigroup homomorphism $\Phi\colon \mathbf{R}^+ \to G$ can be extended in a unique way to a group homomorphism of $\mathbf{R}$ into $G$;*
(ii) *if $G$ is finite, every semigroup homomorphism $\Phi\colon \mathbf{R}^+ \to G$ is trivial.*

*Proof:* (i) The (unique) extension is obviously given by $\Phi(t) = [\Phi(-t)]^{-1}$ for $t < 0$.

(ii) Extend $\Phi$ to a group homomorphism $\Phi\colon \mathbf{R} \to G$, and let $K$ be the kernel of $\Phi$. If $n$ is the order of $G$, we have $nt \in K$ for all $t \in \mathbf{R}$. Hence $\mathbf{R} = n\mathbf{R} \subset K \subset \mathbf{R}$, that is $\Phi$ is trivial, **q.e.d.**

**Lemma 1.4.2:** *Let $\Phi\colon \mathbf{R}^+ \to (\mathbf{C}, +)$ be a continuous semigroup homomorphism. Then $\Phi(t) = at$ for some $a \in \mathbf{C}$.*

*Proof:* Fix $t_0 > 0$. Then $\Phi(nt_0) = n\Phi(t_0)$ for all $n \in \mathbf{N}$. Hence for any $p \in \mathbf{N}^*$ we have

$$\Phi(1) = \Phi(p/p) = p\Phi(1/p),$$

and $\Phi(1/p) = \Phi(1)/p$. Therefore $\Phi(r) = r\Phi(1)$ for all $r \in \mathbf{Q}^+$ and, by continuity, for all $r \in \mathbf{R}^+$. The assertion follows setting $a = \Phi(1)$, **q.e.d.**

**Lemma 1.4.3:** *Let $\Phi\colon \mathbf{R}^+ \to (\mathbf{R}^+, \cdot)$ be a continuous semigroup homomorphism. Then $\Phi(t) = e^{at}$ for some $a \in \mathbf{R}$.*

*Proof:* Since $\Phi$ takes values in $\mathbf{R}^+$, $\log\Phi$ is a continuous semigroup homomorphism from $\mathbf{R}^+$ to $(\mathbf{R}, +)$. By Lemma 1.4.2, $\log\Phi(t) = at$ for some $a \in \mathbf{R}$, and the assertion follows, **q.e.d.**

**Lemma 1.4.4:** *Let $\Phi\colon \mathbf{R}^+ \to (\mathbf{S}^1, \cdot)$ be a continuous semigroup homomorphism. Then $\Phi(t) = e^{ibt}$ for some $b \in \mathbf{R}$.*

*Proof:* Let $\pi\colon \mathbf{R} \to \mathbf{S}^1$ be the covering map $\pi(x) = e^{ix}$; note that $\pi$ is a group homomorphism from $(\mathbf{R}, +)$ to $(\mathbf{S}^1, \cdot)$. Then $\Phi$ lifts to a continuous semigroup homomorphism $\tilde{\Phi}\colon \mathbf{R}^+ \to (\mathbf{R}, +)$ such that $\Phi = \pi \circ \tilde{\Phi}$. Then the assertion follows from Lemma 1.4.2, **q.e.d.**

**Lemma 1.4.5:** *Let $\Phi\colon \mathbf{R}^+ \to (\mathbf{C}^*, \cdot)$ be a continuous semigroup homomorphism. Then $\Phi(t) = e^{\lambda t}$ for some $\lambda \in \mathbf{C}$.*

*Proof:* Starting from $\Phi$ we can construct two new homomorphisms: $|\Phi|\colon \mathbf{R}^+ \to (\mathbf{R}^+, \cdot)$ and $\Phi/|\Phi|\colon \mathbf{R}^+ \to (\mathbf{S}^1, \cdot)$. Then Lemmas 1.4.3 and 1.4.4 imply $\Phi(t) = e^{(a+ib)t}$ for suitable $a$, $b \in \mathbf{R}$, **q.e.d.**

Now we may begin a systematic study of one-parameter semigroups on Riemann surfaces. The first result shows that not every function can be imbedded in a one-parameter semigroup:

**Proposition 1.4.6:** *Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a Riemann surface $X$. Then $\Phi_t$ is injective for all $t \geq 0$.*

*Proof:* First of all note that, since $\Phi'_t \to 1$ as $t \to 0$, for $t$ small enough every $\Phi_t$ is locally injective.

Assume, by contradiction, that $\Phi_{t_0}(z_1) = \Phi_{t_0}(z_2)$ for some $t_0 > 0$ and $z_1$, $z_2 \in X$, with $z_1 \neq z_2$. Then if $t > t_0$ we have $\Phi_t(z_1) = \Phi_{t-t_0}\big(\Phi_{t_0}(z_1)\big) = \Phi_{t-t_0}\big(\Phi_{t_0}(z_2)\big) = \Phi_t(z_2)$. In other words, the two curves $t \mapsto \Phi_t(z_1)$ and $t \mapsto \Phi_t(z_2)$ start at distinct points, meet at $t = t_0$ and coincide thereafter. Let $t_0 > 0$ be the least $t > 0$ such that $\Phi_t(z_1) = \Phi_t(z_2)$, and set $z_0 = \Phi_{t_0}(z_1) = \Phi_{t_0}(z_2)$. Then no $\Phi_t$ can be injective in a neighbourhood of $z_0$, and this is a contradiction, **q.e.d.**

In particular, it may happen that $\Phi_t \in \mathrm{Aut}(X)$ for all $t \geq 0$. In this case $\Phi$ extends to a group homomorphism $\widetilde{\Phi}: \mathbf{R} \to \mathrm{Aut}(X)$, by Lemma 1.4.1, and we say that $\Phi$ is a *one-parameter group.* Actually, $\Phi$ is a one-parameter group iff $\Phi_{t_0}$ is an automorphism for some $t_0 > 0$:

**Proposition 1.4.7:** *Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a hyperbolic Riemann surface $X$. Assume $\Phi_{t_0} \in \mathrm{Aut}(X)$ for some $t_0 > 0$; then $\Phi$ is a one-parameter group.*

*Proof:* Since $(\Phi_{t_0/n})^n = \Phi_{t_0} \in \mathrm{Aut}(X)$ for all $n \in \mathbf{N}^*$, we clearly have $\Phi_{rt_0} \in \mathrm{Aut}(X)$ for all $r \in \mathbf{Q}^+$. By continuity, $\Phi_{rt_0} \in \mathrm{Aut}(X)$ for all $r \in \mathbf{R}^+$ (for $\mathrm{Aut}(X)$ is closed in $\mathrm{Hol}(X,X)$, by Corollary 1.1.47), **q.e.d.**

As usual, a main role in our theory will be played by the fixed points. The definition is completely natural: a point $z_0 \in X$ is a *fixed point* of the semigroup $\Phi$ if $\Phi_t(z_0) = z_0$ for all $t \geq 0$. When we studied iteration theory, we saw that not necessarily a fixed point of an iterate of a function $f$ is fixed by $f$ itself. In the present context, the situation is much simpler:

**Proposition 1.4.8:** *Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a Riemann surface $X$. Assume that $\Phi_{t_0} \neq \mathrm{id}_X$ has a fixed point $z_0 \in X$ for some $t_0 > 0$. Then $z_0$ is a fixed point of $\Phi$.*

*Proof:* For any $t > 0$ we have

$$\Phi_{t_0}\big(\Phi_t(z_0)\big) = \Phi_t\big(\Phi_{t_0}(z_0)\big) = \Phi_t(z_0).$$

Therefore $t \mapsto \Phi_t(z_0)$ is a curve issuing from $z_0$ contained in the fixed point set of $\Phi_{t_0}$. Since $\Phi_{t_0} \neq \mathrm{id}_X$, its fixed point set is discrete, and we infer $\Phi_t(z_0) = z_0$ for all $t \geq 0$, **q.e.d.**

Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a Riemann surface $X$ with a fixed point $z_0 \in X$. Then we can define a continuous semigroup homomorphism $\mu: \mathbf{R}^+ \to \mathbf{C}^*$ setting $\mu(t) = \Phi'_t(z_0)$ (note that $\mu(t) \neq 0$ by Proposition 1.4.6).

By Lemma 1.4.5, $\mu(t) = e^{\lambda t}$ for some $\lambda \in \mathbf{C}$; $\lambda$ will be called the *spectral value* at $z_0$ of the semigroup $\Phi$. Note that, by Theorem 1.3.4.(i), if $X$ is hyperbolic then $\operatorname{Re}\lambda \le 0$, and $\operatorname{Re}\lambda = 0$ iff $\Phi$ is a one-parameter group.

We shall see later on (Proposition 1.4.24) that the theory of one-parameter semigroups on Riemann surfaces is interesting only for simply and doubly connected surfaces. For this reason, in the rest of this section we shall limit ourselves to domains in $\mathbf{C}$.

Our main goal is to relate one-parameter semigroups and ordinary differential equations, thus making our work easier calling in an already well-established (and quite powerful) theory. We shall use the following basic existence theorem:

**Theorem 1.4.9:** *Let $\Omega$ be an open subset of $\mathbf{R}^n$, and $F\colon\Omega \to \mathbf{R}^n$ a real analytic map. Then for any compact subset $K$ of $\Omega$ there are $\delta > 0$, a neighbourhood $U \subset \Omega$ of $K$ and a real analytic map $u\colon(-\delta,\delta)\times U \to \Omega$ such that*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = F\big(u(t,x)\big), \\ u(0,x) = x. \end{cases} \tag{1.4.1}$$

*Furthermore, the solution of* (1.4.1) *is unique in the sense that if there are $\delta' > 0$, another neighbourhood $U' \subset \Omega$ of $K$ and another map $u'\colon(-\delta',\delta')\times U' \to \Omega$ satisfying* (1.4.1), *then $u \equiv u'$ on $\big((-\delta,\delta)\times U\big)\cap\big((-\delta',\delta')\times U'\big)$. Finally, if $\Omega$ is actually a domain in $\mathbf{C}^n$ and $F\colon\Omega \to \mathbf{C}^n$ is holomorphic, then for every $t \in (-\delta,\delta)$ the map $u(t,\cdot)\colon U \to \Omega$ is holomorphic.*

As already mentioned, a proof can be found in Narasimhan [1968] or Hörmander [1973].

Using the uniqueness statement of the latter theorem, we can now introduce the link between one-parameter semigroups and ordinary differential equations:

**Corollary 1.4.10:** *Let $\Omega$ be an open subset of $\mathbf{R}^n$, $F\colon\Omega \to \mathbf{R}^n$ a real analytic map, and $K$ a compact subset of $\Omega$. Choose $\delta > 0$ and a neighbourhood $U \subset \Omega$ of $K$ such that there is a real analytic solution $u\colon(-\delta,\delta)\times U \to \Omega$ of the Cauchy problem* (1.4.1). *Then for every $s$, $t \in (-\delta,\delta)$ and $x \in K$ such that $s + t \in (-\delta,\delta)$ and $u(t,x) \in U$ we have*

$$u\big(s, u(t,x)\big) = u(s+t,x). \tag{1.4.2}$$

*Proof:* Fix $t_0 \in (-\delta,\delta)$ and $x_0 \in K$ such that $u(t_0,x_0) \in U$, and take $\delta' \le \delta - |t_0|$. Now define $v_1$, $v_2\colon(-\delta',\delta') \to \Omega$ setting $v_1(s) = u\big(s, u(t_0,x_0)\big)$ and $v_2(s) = u(s+t_0,x_0)$. Then $v_1$ and $v_2$ are two real-analytic solutions of

$$\begin{cases} \dfrac{dv}{ds} = F \circ v, \\ v(0) = u(t_0,x_0). \end{cases}$$

Then, by uniqueness, $v_1 = v_2$, and (1.4.2) is proved, **q.e.d.**

In other words, the solution of the Cauchy problem (1.4.1) is locally a one-parameter group. In particular, if $D$ is a domain in $\mathbf{C}$ and $F \in \mathrm{Hol}(D, \mathbf{C})$ is such that the Cauchy problem

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = F \circ \Phi \\ \Phi(0, z) = z \end{cases} \tag{1.4.3}$$

has a global solution $\Phi: \mathbf{R}^+ \times D \to D$, then $\Phi$ is automatically a one-parameter semigroup, holomorphic in $z$ and real analytic in $t$. In this case, $F$ is called the *infinitesimal generator* of $\Phi$. Note that $\Phi$ is completely determined by its infinitesimal generator.

The main result of this section is that every one-parameter semigroup is obtained in this way:

**Theorem 1.4.11:** *Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ be a one-parameter semigroup on a domain $D \subset \mathbf{C}$. Then there is a holomorphic function $F: D \to \mathbf{C}$ such that*

$$\frac{\partial \Phi}{\partial t} = F \circ \Phi. \tag{1.4.4}$$

*Proof:* Let $K$ be a compact convex subset of $D$. We can choose $\alpha \in (0, 1)$ such that the convex hull $\widehat{K}$ of $\Phi\bigl([0, \alpha] \times K\bigr)$ is still contained in $D$. Take $\delta \in (0, \alpha]$ such that

$$\sup_{z \in \widehat{K}} |\Phi_t'(z) - 1| \le 1/10,$$

for all $t \le \delta$ — the number $1/10$ is cabalistic; what we really need is the last step in (1.4.6). Hence for all $t \in [0, \delta]$ and $z \in K$

$$|\Phi_{2t}(z) - 2\Phi_t(z) + z| = \left| \int_z^{\Phi_t(z)} \frac{d}{d\zeta}[\Phi_t(\zeta) - \zeta]\, d\zeta \right| \le \tfrac{1}{10} |\Phi_t(z) - z|, \tag{1.4.5}$$

where the integration path is the segment from $z$ to $\Phi_t(z)$, and thus is contained in $\widehat{K}$.

Therefore for all $t \in [0, \delta)$ and $z \in K$ we have

$$|\Phi_t(z) - z| \le \tfrac{10}{19} |\Phi_{2t}(z) - z| \le 2^{-2/3} |\Phi_{2t}(z) - z|. \tag{1.4.6}$$

Let $k \in \mathbf{N}$ be such that $2^k \delta \ge 1$, and put

$$M = 2^{2k/3} \sup\bigl\{ |\Phi_t(z) - z| \bigm| z \in K,\ t \in [2^{-k}, 1] \bigr\}.$$

Then (1.4.6) implies

$$\forall t \in [0, 1] \quad \forall z \in K \qquad |\Phi_t(z) - z| \le M t^{2/3}. \tag{1.4.7}$$

Now repeat the same argument on a compact convex subset $K_1$ of $D$ containing properly $\widehat{K}$, coming up with a constant $M_1 > 0$ such that

$$\forall t \in [0,1] \quad \forall z \in K_1 \qquad |\Phi_t(z) - z| \le M_1 t^{2/3}.$$

Then the Cauchy inequalities produce a constant $\widetilde{M} > 0$ such that

$$\forall t \in [0,1] \quad \forall z \in \widehat{K} \qquad |\Phi_t'(z) - 1| \le \widetilde{M} t^{2/3}. \tag{1.4.8}$$

If we plug (1.4.7) and (1.4.8) in (1.4.5), we find that for all $t \in [0, \alpha]$ and $z \in K$

$$|\Phi_{2t}(z) - 2\Phi_t(z) + z| \le \widetilde{M} t^{2/3} |\Phi_t(z) - z| \le M\widetilde{M} t^{4/3}.$$

Thus

$$\left| \frac{\Phi_{2t}(z) - z}{2t} - \frac{\Phi_t(z) - z}{t} \right| \le \frac{M\widetilde{M}}{2} t^{1/3},$$

for $z \in K$ and $t \in (0, \alpha]$. Hence

$$\lim_{n \to \infty} \frac{\Phi(2^{-n}, z) - z}{2^{-n}} = F(z)$$

exists uniformly on compact subsets of $D$, defining a holomorphic function $F\colon D \to \mathbf{C}$.

For $z_0 \in D$ and $t_0 > 0$, $\Phi\bigl([0, t_0] \times \{z_0\}\bigr)$ is a compact subset of $D$. Hence, as $n \to +\infty$, the function $2^n[\Phi(t + 2^{-n}, z_0) - \Phi(t, z_0)]$ tends uniformly to $F\bigl(\Phi(t, z_0)\bigr)$ for $t \in [0, t_0]$. This implies

$$\Phi_t(z) = z + \int_0^t F\bigl(\Phi_s(z)\bigr)\, ds$$

for all $z \in D$ and $t \in \mathbf{R}^+$, and (1.4.4) is proved, **q.e.d.**

So every one-parameter semigroup is the solution of a Cauchy problem, and there is a one-to-one correspondance between infinitesimal generators and one-parameter semigroups. In particular, the classification of one-parameter semigroups is reduced to the classification of their infinitesimal generators, which will be attained using the following observation:

**Corollary 1.4.12:** *Let $D$ be a domain in $\mathbf{C}$, and take $F \in \mathrm{Hol}(D, \mathbf{C})$. Then $F$ is an infinitesimal generator iff* (1.4.3) *has a global solution on $\mathbf{R}^+ \times D$.*

*Proof:* This follows from Corollary 1.4.10 and Theorem 1.4.11, **q.e.d.**

As we shall see, this approach will eventually provide us with a complete classification of one-parameter semigroups. For the moment, we end this section showing how to recover fixed points and spectral values using infinitesimal generators:

**Proposition 1.4.13:** *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ be a one-parameter semigroup on a domain $D \subset \mathbf{C}$, and let $F\colon D \to \mathbf{C}$ be its infinitesimal generator. Then:*
(i) *$z_0 \in D$ is a fixed point of $\Phi$ iff $F(z_0) = 0$;*
(ii) *if $\Phi$ has a fixed point $z_0$, then its spectral value at $z_0$ is $F'(z_0)$.*

*Proof:* If $z_0 \in D$ is a fixed point of $\Phi$, then (1.4.4) immediately yields $F(z_0) = 0$. Conversely, assume $F(z_0) = 0$, and set $\varphi(t) = \Phi(t, z_0)$. Then $\varphi$ solves the Cauchy problem

$$\begin{cases} \dfrac{d\psi}{dt} = F \circ \psi, \\ \psi(0) = z_0. \end{cases}$$

Since $F(z_0) = 0$, $\psi \equiv z_0$ is a solution and thus, by Theorem 1.4.9, it is the only solution. Hence $\varphi \equiv z_0$, and $z_0$ is a fixed point of $\Phi$.

Finally, assume $z_0 \in D$ is a fixed point of $\Phi$. Hence, if $\lambda$ is the spectral value of $\Phi$ at $z_0$, we have

$$F'(z_0)e^{\lambda t} = F'(z_0)\Phi_t'(z_0) = \frac{\partial}{\partial z}(F \circ \Phi)(t, z_0) = \frac{\partial^2 \Phi}{\partial z \partial t}(t, z_0) = \frac{\partial}{\partial t}\Phi_t'(z_0) = \lambda e^{\lambda t},$$

and $F'(z_0) = \lambda$, **q.e.d.**

## 1.4.2 *One-parameter semigroups on the unit disk*

This section is devoted to the description of one-parameter semigroups on $\Delta$. We shall first give a differential characterization of infinitesimal generators; next, after the description of the asymptotic behavior modelled on the Wolff-Denjoy theorem, we shall present an explicit classification of infinitesimal generators. Finally, we shall describe a concrete realization of one-parameter semigroups by means of accurately chosen biholomorphisms of $\Delta$ with specific domains in $\mathbf{C}$.

Without losing any more time, we immediately begin with the first characterization:

**Theorem 1.4.14:** *A holomorphic function $F\colon \Delta \to \mathbf{C}$ is the infinitesimal generator of a one-parameter semigroup on $\Delta$ iff*

$$\forall z \in \Delta \qquad \mathrm{Re}\big[2\,\overline{z}\,F(z) + (1 - |z|^2)F'(z)\big] \le 0. \tag{1.4.9}$$

*Proof:* Assume first $F$ is the infinitesimal generator of a one-parameter semigroup $\Phi$ on $\Delta$. Fix $z_0 \in \Delta$; then, by the Schwarz-Pick lemma, for every $t_2 > t_1 > 0$ we have

$$\frac{|\Phi_{t_1}'(z_0)|}{1 - |\Phi_{t_1}(z_0)|^2} \ge \frac{|\Phi_{t_1}'(z_0)|\,|\Phi_{t_2 - t_1}'\big(\Phi_{t_1}(z_0)\big)|}{1 - |\Phi_{t_2}(z_0)|^2} = \frac{|\Phi_{t_2}'(z_0)|}{1 - |\Phi_{t_2}(z_0)|^2}.$$

In other words, for every $z_0 \in \Delta$ the function $t \mapsto |\Phi_t'(z_0)|/(1 - |\Phi_t(z_0)|^2)$ is not increasing. Therefore

$$0 \ge \frac{\partial}{\partial t}\left.\frac{|\Phi_t'(z_0)|}{1 - |\Phi_t(z_0)|^2}\right|_{t=0} = \frac{1}{(1 - |z_0|^2)^2}\ \mathrm{Re}\big[2\,\overline{z_0}\,F(z_0) + (1 - |z_0|^2)F'(z_0)\big],$$

and (1.4.9) is proved.

Conversely, assume $F$ satisfies (1.4.9); we should show that (1.4.3) has a global solution on $\mathbf{R}^+ \times D$ (by Corollary 1.4.12). Fix $z_0 \in \Delta$, and let $\phi\colon [0,\delta) \to \Delta$ be a maximal solution of the Cauchy problem

$$\begin{cases} \dfrac{d\phi}{dt} = F \circ \phi, \\ \phi(0) = z_0; \end{cases} \tag{1.4.10}$$

we have to show that $\delta = +\infty$. Assume, by contradiction, $\delta < +\infty$; we claim that $\phi\bigl([0,\delta)\bigr)$ is contained in a compact subset of $\Delta$. Indeed, we have

$$\frac{d}{dt}\frac{|\dot\phi(t)|^2}{(1-|\phi(t)|^2)^2} = \frac{2\bigl|F\bigl(\phi(t)\bigr)\bigr|^2}{(1-|\phi(t)|^2)^3}\,\mathrm{Re}\bigl[2\overline{\phi(t)}F\bigl(\phi(t)\bigr) + (1-|\phi(t)|^2)F'\bigl(\phi(t)\bigr)\bigr] \le 0,$$

by (1.4.9); thus the function $t \mapsto |\dot\phi(t)|/(1-|\phi(t)|^2)$ is not increasing. Fix $t_0 < \delta$, and let $\sigma\colon [0,t_0] \to \Delta$ be the curve $\sigma(t) = \phi(t)$; then

$$\omega\bigl(z_0, \phi(t_0)\bigr) \le \int_\sigma d\kappa = \int_0^{t_0} \frac{|\dot\phi(t)|}{1-|\phi(t)|^2}\,dt \le t_0\,\frac{|F(z_0)|}{1-|z_0|^2} \le \delta\frac{|F(z_0)|}{1-|z_0|^2},$$

and so $\phi\bigl([0,\delta)\bigr)$ is contained in a closed Poincaré disk $K$, as claimed. Let $\delta_1 > 0$ and $u\colon (-\delta_1,\delta_1) \times K \to \Delta$ be given by Theorem 1.4.9 applied to $K$, and choose $t_0 \in [0,\delta)$ such that $\delta - t_0 < \delta_1$. Then the uniqueness statement of Theorem 1.4.9 shows that the function $\psi\colon [0, \delta_1 + t_0) \to \Delta$ given by

$$\psi(t) = \begin{cases} \phi(t) & \text{if } t < \delta, \\ u\bigl(t - t_0, \phi(t_0)\bigr) & \text{if } t \ge t_0, \end{cases}$$

is still a solution of (1.4.10), against the maximality of $\delta$, **q.e.d.**

So the infinitesimal generators are characterized by a sort of ordinary differential inequality. A first consequence is:

**Corollary 1.4.15:** *The set of all infinitesimal generators of one-parameter semigroups on $\Delta$ is a real convex cone in* $\mathrm{Hol}(\Delta, \mathbf{C})$ *with vertex at* $0$.

At this point, the knowledgeable reader may wonder whether the infinitesimal generators of one-parameter groups are maybe characterized by a differential equation; after all, $\mathrm{Aut}(\Delta)$ is a Lie group, and so we can apply the third Lie theorem, and... Indeed, the knowledgeable reader is right, and the differential equation is easily find:

**Corollary 1.4.16:** *A holomorphic function $F\colon \Delta \to \mathbf{C}$ is the infinitesimal generator of a one-parameter group on $\Delta$ iff*

$$\forall z \in \Delta \qquad \mathrm{Re}\bigl[2\,\overline{z}\,F(z) + (1-|z|^2)F'(z)\bigr] = 0. \tag{1.4.11}$$

*Proof:* Assume first $F$ is the infinitesimal generator of a one-parameter group $\Phi$ on $\Delta$. For every $t \geq 0$ set $\Phi_t^+ = \Phi_t$ and $\Phi_t^- = \Phi_{-t}$. Then $\Phi^+$ and $\Phi^-$ are two one-parameter semigroups, of infinitesimal generators $F$ and $-F$ respectively, and so Theorem 1.4.14 yields (1.4.11).

Conversely, assume (1.4.11) holds; then, by Theorem 1.4.14 and Corollary 1.4.12, (1.4.3) has a global solution $\Phi$ on $\mathbf{R} \times D$. Hence we can quote Corollary 1.4.10, showing that $\Phi$ is a one-parameter group of infinitesimal generator $F$, **q.e.d.**

Our next aim is an explicit description of the infinitesimal generators; in other words, we want to solve (1.4.9). We begin describing the asymptotic behavior of one-parameter semigroups, generalizing the Wolff-Denjoy theorem.

Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ be a non-trivial one-parameter semigroup. By Proposition 1.4.8, either $\Phi$ has a fixed point, or $\mathrm{Fix}(\Phi_t) = \emptyset$ for all $t > 0$; in this latter case, we shall say that $\Phi$ is *fixed point free.*

If $\Phi$ has a fixed point $z_0 \in \Delta$, then the spectral value $\lambda$ of $\Phi$ at $z_0$ satisfies $\mathrm{Re}\,\lambda \leq 0$, and $\mathrm{Re}\,\lambda = 0$ iff $\Phi$ is a one-parameter group of elliptic automorphisms (Theorem 1.3.4 and Proposition 1.4.7). Furthermore, in the latter case $\Phi_t$ has no limit in $\mathrm{Hol}(\Delta, \mathbf{C})$ as $t \to +\infty$ (by Lemma 1.4.4). Looking at the Wolff-Denjoy theorem it is now clear what is going on:

**Theorem 1.4.17:** *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ be a non-trivial one-parameter semigroup on $\Delta$. Assume $\Phi$ is not a one-parameter group of elliptic automorphisms. Then $\Phi_t$ converges as $t \to +\infty$ to a constant $\tau \in \overline{\Delta}$, the Wolff point of $\Phi_1$.*

*Proof:* By assumption, $\Phi$ either is fixed point free or has a fixed point with spectral value having nonzero real part. By the Wolff-Denjoy theorem, the sequence $\{\Phi_k\}$ converges as $k \to +\infty$ to a point $\tau \in \overline{\Delta}$, the Wolff point of $\Phi_1$.

Fix $z_0 \in \Delta$, and let $K = \Phi\big([0,1] \times \{z_0\}\big)$. $K$ is a compact subset of $\Delta$; hence for every $\varepsilon > 0$ there is $k_0 \in \mathbf{N}$ such that for any $k \geq k_0$ and $z \in K$ we have $|\Phi_k(z) - \tau| < \varepsilon$. Hence

$$\forall r \in [0,1] \quad \forall k \geq k_0 \qquad |\Phi_{k+r}(z_0) - \tau| < \varepsilon.$$

This means that $\lim\limits_{t \to +\infty} \Phi_t(z_0) = \tau$. But $z_0$ was arbitrary; hence, by Corollary 1.1.41, $\Phi_t \to \tau$ as $t \to +\infty$, **q.e.d.**

In particular we have (cf. Theorems 1.3.22 and 1.3.24):

**Corollary 1.4.18:** *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ be a non-trivial one-parameter semigroup on $\Delta$. Then the Wolff point of $\Phi_t$ is independent from $t$.*

*Proof:* If $\Phi$ has a fixed point $z_0 \in \Delta$, $\tau(\Phi_t) = z_0$ for all $t > 0$. If $\Phi$ is fixed point free,

$$\lim_{t \to +\infty} \Phi_t = \lim_{k \to +\infty} \Phi_{kt_0} = \tau(\Phi_{t_0})$$

for any $t_0 > 0$, and again $\tau(\Phi_{t_0})$ does not depend on $t_0$, **q.e.d.**

Thus the *Wolff point* of a non-trivial one-parameter semigroup $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta,\Delta)$ is well-defined.

We are now able to solve explicitely (1.4.9). Denote by $\mathcal{P}$ the set of holomorphic functions $f\colon \Delta \to \mathbf{C}$ such that $f \not\equiv 0$ and $\operatorname{Re} f \geq 0$. Note that, by the minimum principle for harmonic functions, either $\operatorname{Re} f > 0$ on $\Delta$ or $f \equiv ib$ for some $b \in \mathbf{R}^*$. Then

**Theorem 1.4.19:** *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta,\Delta)$ be a non-trivial one-parameter semigroup on $\Delta$, and $\tau \in \overline{\Delta}$ its Wolff point. Then the infinitesimal generator of $\Phi$ is of the form*

$$F(z) = (\overline{\tau} z - 1)(z - \tau) f(z), \tag{1.4.12}$$

*for a suitable $f \in \mathcal{P}$. In particular, $\Phi$ has a fixed point iff $\tau \in \Delta$, the fixed point is exactly $\tau$ and the spectral value is $(|\tau|^2 - 1) f(\tau)$. Conversely, given $f \in \mathcal{P}$ and $\tau \in \overline{\Delta}$, the function $F\colon \Delta \to \mathbf{C}$ given by (1.4.12) is the infinitesimal generator of a one-parameter semigroup on $\Delta$ with Wolff point $\tau$.*

*Proof:* Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta,\Delta)$ be a non-trivial one-parameter semigroup on $\Delta$, with Wolff point $\tau$ and infinitesimal generator $F$. Using Corollary 1.2.15, we see that the function

$$t \mapsto \frac{|1 - \overline{\tau}\Phi_t(z)|^2}{1 - |\Phi_t(z)|^2}$$

is non-increasing for any $z \in \Delta$. Hence its derivative in 0 is non-positive, and we get

$$\operatorname{Re}\bigl((1 - \tau\overline{z})(\overline{z} - \overline{\tau})F(z)\bigr) \leq 0. \tag{1.4.13}$$

Let $f(z) = (\overline{\tau} z - 1)^{-1}(z - \tau)^{-1} F(z)$; $f$ is well-defined, for if $\tau \in \Delta$ then $F(\tau) = 0$ by Proposition 1.4.13. Then (1.4.13) implies $f \in \mathcal{P}$, and $F$ is given by (1.4.12).

Conversely, let $F \in \mathrm{Hol}(\Delta, \mathbf{C})$ be given by (1.4.12); we should prove that $F$ satisfies (1.4.9). If $f \equiv ib$ for some $b \in \mathbf{R}^*$, then (1.4.9) is easily verified. Assume then $\operatorname{Re} f(z) > 0$ for all $z \in \Delta$. Then Theorem 1.1.40 applied to $(f-1)/(f+1)$ yields

$$\forall z \in \Delta \qquad \frac{|f'(z)|}{2 \operatorname{Re} f(z)} \leq \frac{1}{1 - |z|^2}. \tag{1.4.14}$$

Now

$$2\,\overline{z}\,F + (1 - |z|^2)F' = (1 - |z|^2)(\overline{\tau} z - 1)(z - \tau) f' - \bigl[|\overline{\tau} z - 1|^2 + |z - \tau|^2\bigr] f;$$

hence (1.4.14) yields

$$\begin{aligned} \operatorname{Re}[2\,\overline{z}\,F + (1 - |z|^2)F'] &= (1 - |z|^2) \operatorname{Re}\bigl[(\overline{\tau} z - 1)(z - \tau) f'\bigr] - \bigl[|\overline{\tau} z - 1|^2 + |z - \tau|^2\bigr] \operatorname{Re} f \\ &\leq (1 - |z|^2)|\overline{\tau} z - 1|\,|z - \tau|\,|f'| - \bigl[|\overline{\tau} z - 1|^2 + |z - \tau|^2\bigr] \operatorname{Re} f \\ &\leq -(|\overline{\tau} z - 1| - |z - \tau|)^2 \operatorname{Re} f \leq 0, \end{aligned} \tag{1.4.15}$$

and we are done, **q.e.d.**

We now spend a couple of words about one-parameter groups on $\Delta$. A one-parameter group $\Phi\colon \mathbf{R} \to \mathrm{Aut}(\Delta)$ clearly acts on $\overline{\Delta}$; therefore, by Proposition 1.1.13, the fixed point set of $\Phi_t$ is independent from $t > 0$. In particular, every $\Phi_t$ is either elliptic (and $\Phi$ is an *elliptic one-parameter group*), or parabolic (and $\Phi$ is a *parabolic one-parameter group*), or hyperbolic (and $\Phi$ is a *hyperbolic one-parameter group*). Furthermore, since the group of all elliptic — parabolic, hyperbolic — automorphisms with given fixed points is isomorphic to $(\mathbf{S}^1, \cdot)$ — $(\mathbf{R}, +)$, $(\mathbf{R}^+, \cdot)$ —, the one-parameter groups on $\Delta$ are essentially determined by Lemmas 1.4.3, 1.4.4 and 1.4.5. For sake of completeness, in the next corollary we explicitely describe the infinitesimal generators of one-parameter groups:

**Corollary 1.4.20:** *Let $\Phi\colon \mathbf{R} \to \mathrm{Aut}(\Delta)$ be a non-trivial one-parameter group on $\Delta$, and $\tau \in \overline{\Delta}$ its Wolff point. Then the infinitesimal generator of $\Phi$ is of the form:*
(i) *if $\Phi$ is an elliptic one-parameter group*

$$F(z) = (\overline{\tau} z - 1)(z - \tau) i a, \tag{1.4.16}$$

*for some $a \in \mathbf{R}^*$, and the spectral value of $\Phi$ at $\tau$ is $(|\tau|^2 - 1) i a$;*
(ii) *if $\Phi$ is a parabolic one-parameter group*

$$F(z) = (z - \tau)^2 \overline{\tau} i a, \tag{1.4.17}$$

*for some $a \in \mathbf{R}^*$;*
(iii) *if $\Phi$ is a hyperbolic one-parameter group*

$$F(z) = (z^2 - \tau^2)\overline{\tau} a + i b \overline{\tau} (z - \tau)^2, \tag{1.4.18}$$

*for some $a \in \mathbf{R}^*$ and $b \in \mathbf{R}$, and the other fixed point is $\sigma = \tau(-a + ib)/(a + ib) \in \partial\Delta$. Conversely, every function of the form* (1.4.16) — (1.4.17), (1.4.18) — *is the infinitesimal generator of an elliptic — parabolic, hyperbolic — one-parameter group of Wolff point $\tau$.*

*Proof:* Let $F \in \mathrm{Hol}(\Delta, \mathbf{C})$ be an infinitesimal generator of a non-trivial one-parameter group $\Phi\colon \mathbf{R} \to \mathrm{Aut}(\Delta)$; therefore $F$ satisfies (1.4.11), and it is of the form (1.4.12) for suitable $f \in \mathcal{P}$ and $\tau \in \overline{\Delta}$.

If $f \equiv ia$ for some $a \in \mathbf{R}^*$, then $F$ is given either by (1.4.16) — if $\tau \in \Delta$ — or by (1.4.17) — if $\tau \in \partial\Delta$. Assume then $\mathrm{Re}\, f(z) > 0$ for all $z \in \Delta$. Now (1.4.11) implies that all the inequalities in (1.4.15) are actually equalities. The last one yields $\tau \in \partial\Delta$, the second one implies that (1.4.14) is an equality too, and the first one yields $f'(z) = 2a\tau(z - \tau)^{-2}$ for a suitable $a \in \mathbf{R}$. It follows that $f(z) = 2a\tau(\tau - z)^{-1} + c$ for some $c \in \mathbf{C}$; moreover, the equality in (1.4.14) forces $\mathrm{Re}\, c = -a$, and so

$$f(z) = a\,\frac{\tau + z}{\tau - z} + ib,$$

for some $b \in \mathbf{R}$. In particular, since we assumed $\mathrm{Re}\, f > 0$ everywhere, $a \neq 0$, and then, by (1.4.12), $F$ is of the form (1.4.18), as desired.

Finally, it is easy to check that every $F$ of the form (1.4.16), (1.4.17) or (1.4.18) satisfies (1.4.11); furthermore, since (1.4.16) has only one zero which is in $\Delta$, (1.4.17) has a unique zero which is in $\partial\Delta$, and (1.4.18) has two distinct zeroes on $\partial\Delta$, it is evident that (1.4.16) characterizes elliptic, (1.4.17) parabolic and (1.4.18) hyperbolic one-parameter groups, **q.e.d.**

We end this section giving another, quite concrete, description of the action of one-parameter semigroups on $\Delta$, relying on two particular kind of domains.

We start with fixed point free one-parameter semigroups. A domain $D \subset \mathbf{C}$ such that $z + it \in D$ for every $z \in D$ and $t \geq 0$ is called *vertically invariant.* Let $D$ be a simply connected vertically invariant domain (different from $\mathbf{C}$), and take a biholomorphism $g\colon \Delta \to D$. Then define $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ by

$$\Phi_t(z) = g^{-1}\bigl(g(z) + it\bigr).$$

$\Phi$ is clearly a one-parameter semigroup without fixed points, of infinitesimal generator $F(z) = i/g'(z)$. Then the idea is that every fixed point free one-parameter semigroup on $\Delta$ is obtained in this way.

To prove this assertion we need a preliminary lemma:

**Lemma 1.4.21:** *Let $f\colon H^+ \to \mathbf{C}$ be a holomorphic function such that $\operatorname{Re} f' \geq 0$. Then $f$ is either constant or injective.*

*Proof:* If there is $z_0 \in H^+$ so that $\operatorname{Re} f'(z_0) = 0$, then $f(z) = a + ibz$ for some $a$, $b \in \mathbf{R}$, and $f$ is either constant or injective. We henceforth suppose $\operatorname{Re} f' > 0$.

Assume, by contradiction, that $f(z_1) = f(z_2)$ for two distinct points $z_1$, $z_2 \in H^+$. Integrating along the segment $\sigma$ from $z_1$ to $z_2$ we obtain

$$0 = f(z_2) - f(z_1) = \int_\sigma f'(\zeta)\, d\zeta = (z_2 - z_1) \int_0^1 f'\bigl(z_1 + t(z_2 - z_1)\bigr)\, dt.$$

Hence $z_1 \neq z_2$ implies

$$0 = \operatorname{Re} \int_0^1 f'\bigl(z_1 + t(z_2 - z_1)\bigr)\, dt = \int_0^1 \operatorname{Re} f'\bigl(z_1 + t(z_2 - z_1)\bigr)\, dt > 0,$$

contradiction, **q.e.d.**

Then we have

**Theorem 1.4.22:** *Every fixed point free one-parameter semigroup $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ is of the form*

$$\Phi_t(z) = g^{-1}\bigl(g(z) + it\bigr), \tag{1.4.19}$$

*where $g$ is a biholomorphism between $\Delta$ and a domain $D \subset \mathbf{C}$ vertically invariant; $g$ is uniquely determined up to an additive constant. Furthermore, $\Phi$ is a one-parameter group iff $D$ is either a vertical strip or a vertical half-plane.*

*Proof:* Let $\Phi$ be a fixed point free one-parameter semigroup with infinitesimal generator $F$. We know that $F(z) = \overline{\tau}(z - \tau)^2 f(z)$ for a suitable $\tau \in \partial\Delta$ and $f \in \mathcal{P}$. Since we are assuming $\Phi$ without fixed points, $f(z) \neq 0$ for all $z \in \Delta$.

Let $\mu: \Delta \to H^+$ be given by

$$\mu(z) = \frac{i}{2} \frac{\tau + z}{\tau - z}.$$

Then $\mu$ is a biholomorphism between $\Delta$ and $H^+$ such that $\mu'(z) = i\tau(z - \tau)^{-2}$. Let $h \in \mathrm{Hol}(H^+, \mathbf{C})$ be such that $h'(z) = 1/f\bigl(\mu^{-1}(z)\bigr)$; by Lemma 1.4.21, $h$ is injective. Therefore $g = h \circ \mu: \Delta \to \mathbf{C}$ is injective too, and furthermore

$$g'(z) = \frac{i}{F(z)}.$$

Now fix $z_0 \in \Delta$. Then

$$\frac{\partial}{\partial t} g \circ \Phi_t(z_0) = g'\bigl(\Phi_t(z_0)\bigr) \cdot F\bigl(\Phi_t(z_0)\bigr) \equiv i;$$

therefore

$$g\bigl(\Phi_t(z)\bigr) = g(z) + it.$$

This means that $D = g(\Delta)$ is vertically invariant, and $\Phi$ is given by (1.4.19).

For the uniqueness, let $g_1: \Delta \to D_1$ be another biholomorphism between $\Delta$ and a vertically invariant domain $D_1$ such that (1.4.19) holds. Differentiating

$$g^{-1}\bigl(g(z) + it\bigr) = g_1^{-1}\bigl(g_1(z) + it\bigr)$$

with respect to $t$ at $t = 0$ we get $g'(z) = g_1'(z)$ for all $z \in \Delta$, and then $g - g_1$ is constant. Conversely, it is easy to check that $g + c$ satisfies (1.4.19) for every $c \in \mathbf{C}$.

Finally, $\Phi$ is a one-parameter group iff for every $t \in \mathbf{R}$ the function $z \mapsto z + it$ is an automorphism of $D$, that is iff $D$ is either a vertical strip or a vertical half-plane, **q.e.d.**

An analogous theorem holds for one-parameter groups with a fixed point. Again, let's first describe the example. Fix $\lambda \in \mathbf{C}^*$; a domain $D \subset \mathbf{C}$ is *$\lambda$-invariant* if $e^{-\lambda t} z \in D$ for all $t \geq 0$ and $z \in D$. Let $D \subset \mathbf{C}$ be a simply connected $\lambda$-invariant domain (different from $\mathbf{C}$), with $\mathrm{Re}\,\lambda \geq 0$ and $0 \in D$. Let $g: \Delta \to D$ be a biholomorphism such that $g(0) = 0$. Then $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ given by

$$\Phi_t(z) = g^{-1}\bigl(e^{-\lambda t} g(z)\bigr)$$

is a one-parameter semigroup on $\Delta$ with fixed point $0$ and spectral value $-\lambda$ at $0$. That's all:

**Theorem 1.4.23:** *Every non-trivial one-parameter semigroup* $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ *with fixed point* $\tau \in \Delta$ *is of the form*

$$\Phi_t(z) = g^{-1}\bigl(e^{-\lambda t} g(z)\bigr) \tag{1.4.20}$$

*where* $\lambda \in \mathbf{C}^*$ *with* $\operatorname{Re}\lambda \geq 0$*, and* $g$ *is a biholomorphism between* $\Delta$ *and a* $\lambda$*-invariant domain* $D \subset \mathbf{C}$ *so that* $g(\tau) = 0$*;* $\lambda$ *is uniquely determined, and* $g$ *is uniquely determined up to a multiplicative constant. Furthermore,* $\Phi$ *is a one-parameter group iff* $\operatorname{Re}\lambda = 0$ *and* $D$ *is a disk.*

*Proof:* By (1.4.12), the infinitesimal generator of $\Phi$ is given by $F(z) = -(1-\overline{\tau}z)(z-\tau)f(z)$ for a suitable $f \in \mathcal{P}$; note that $0 \notin f(\Delta)$ for $\Phi$ is non-trivial. Let $\gamma_0 \in \operatorname{Aut}(\Delta)$ be given by $\gamma_0(z) = (z-\tau)/(1-\overline{\tau}z)$, and set $f_0 = (1-|\tau|^2)f \circ \gamma_0^{-1}$ and $\tilde{f} = f_0 \circ \pi \colon H^+ \to \mathbf{C}$, where $\pi\colon H^+ \to \Delta^*$ is the universal covering map $\pi(w) = e^{2\pi i w}$. Choose a holomorphic function $\tilde{h}\colon H^+ \to \mathbf{C}$ such that $\tilde{h}' = 1/\tilde{f}$. Now, $\tilde{f}(w+1) = \tilde{f}(w)$ for all $w \in H^+$; hence there is $\mu \in \mathbf{C}$ such that $\tilde{h}(w+1) = \tilde{h}(w) + \mu$ for all $w \in H^+$. Since $\tilde{h}$ is injective (Lemma 1.4.21), $\mu \neq 0$; set $h = \mu^{-1}\tilde{h}$. $h$ is a biholomorphism between $H^+$ and $\widetilde{D} = h(H^+)$; moreover, $h' = \mu^{-1}/\tilde{f}$ and $h(w+1) = h(w) + 1$ for all $w \in H^+$. This implies that $h$ factorizes through $\pi$, defining a biholomorphism $\hat{g}\colon \Delta^* \to D^* = \pi(\widetilde{D})$ such that $\hat{g} \circ \pi = \pi \circ h$. In particular, $\hat{g}$ satisfies

$$\forall z \in \Delta^* \qquad z\hat{g}'(z) = \frac{\lambda\,\hat{g}(z)}{f_0(z)},$$

where $\lambda = \mu^{-1}$.

Now set $g_0 = \hat{g} \circ \gamma_0$. Then $g_0$ satisfies

$$\forall z \in \Delta \setminus \{\tau\} \qquad g_0'(z) = -\frac{\lambda\, g_0(z)}{F(z)}.$$

Therefore for every $z_0 \in \Delta \setminus \{z\}$ we have

$$\frac{\partial}{\partial t} g_0\big(\Phi_t(z_0)\big) = g_0'\big(\Phi_t(z_0)\big) \cdot F\big(\Phi_t(z_0)\big) = -\lambda\, g_0\big(\Phi_t(z_0)\big),$$

and so

$$g_0\big(\Phi_t(z_0)\big) = e^{-\lambda t} g_0(z_0). \tag{1.4.21}$$

Now, $g_0$ extends to a biholomorphism $g\colon \Delta \to D = D^* \cup \{w_0\} \subset \widehat{\mathbf{C}}$ such that $g(\tau) = w_0$ (Lemma 1.1.50 and Theorem 1.1.51). (1.4.21) then implies $w_0 = 0$ or $w_0 = \infty$; furthermore, $D$ is $\lambda$-invariant, and $g$ satisfies (1.4.20). It remains to show that we can assume $\operatorname{Re}\lambda \geq 0$ and $w_0 = 0$.

Assume first $\operatorname{Re}\lambda > 0$. In this case, (1.4.21) shows that $g_0\big(\Phi_t(z_0)\big) \to 0$ as $t \to +\infty$ for any $z_0 \in \Delta \setminus \{\tau\}$, and this implies $w_0 = 0$; so in this case we are done.

If $\operatorname{Re}\lambda = 0$ and $w_0 = 0$, there is nothing to change. If $\operatorname{Re}\lambda = 0$ and $w_0 = \infty$, then it suffices to replace $\lambda$ by $-\lambda$ and $g$ by $1/g$ (note that $0 \notin D^*$).

Finally, if $\operatorname{Re}\lambda < 0$, (1.4.21) shows that $g_0\big(\Phi_t(z_0)\big) \to \infty$ as $t \to +\infty$ for any $z_0 \in \Delta \setminus \{\tau\}$; hence $w_0 = \infty$, and replacing $\lambda$ by $-\lambda$ and $g$ by $1/g$ we are again done.

It remains to prove the last two assertions. Assume that $\mu \in \mathbf{C}^*$ and $g_1\colon \Delta \to D_1$ are such that

$$\forall z \in \Delta\ \forall t \geq 0 \qquad g^{-1}\big(e^{-\lambda t} g(z)\big) = g_1^{-1}\big(e^{-\mu t} g_1(z)\big).$$

Differentiating with respect to $z$ at $z=\tau$ we immediately get $\mu=\lambda$. Differentiating with respect to $t$ at $t=0$ we get $gg_1'-g_1g'=0$, that is $g_1=cg$ for some constant $c\in\mathbf{C}^*$. Conversely, it is easy to check that $cg$ satisfies (1.4.20) for every $c\in\mathbf{C}^*$.

Finally, assume $\Phi$ is a one-parameter group. Then $\Phi_t(z_0)$ cannot converge as $t\to+\infty$ for any $z_0\in\Delta\setminus\{\tau\}$; this implies, by (1.4.20), $\operatorname{Re}\lambda=0$. In particular, $D$ is a simply connected domain invariant under rotations, that is a disk, **q.e.d.**

## 1.4.3 *One-parameter semigroups on Riemann surfaces*

In the latter section we thoroughly investigated one-parameter semigroups on $\Delta$; in this section we shall do the same on other Riemann surfaces. Our task is made possible by

**Proposition 1.4.24:** *Let $\Phi\colon\mathbf{R}^+\to\operatorname{Hol}(X,X)$ be a one-parameter semigroup on a Riemann surface $X$ with non-abelian fundamental group. Then $\Phi$ is trivial.*

*Proof:* By Theorem 1.2.19 we should have $\Phi_t=\mathrm{id}_X$ for small $t$, and hence for all $t$, **q.e.d.**

So we are left with just a few cases to investigate.

**Proposition 1.4.25:** *Let $\Phi\colon\mathbf{R}^+\to\operatorname{Hol}(\widehat{\mathbf{C}},\widehat{\mathbf{C}})$ be a one-parameter semigroup on the extended complex plane $\widehat{\mathbf{C}}$. Then $\Phi$ is a one-parameter group, and there is $\gamma\in\operatorname{Aut}(\widehat{\mathbf{C}})$ such that either*
(i) $\gamma^{-1}\circ\Phi_t\circ\gamma(z)=z+at$ *for some* $a\in\mathbf{C}$, *or*
(ii) $\gamma^{-1}\circ\Phi_t\circ\gamma(z)=e^{bt}z$ *for some* $b\in\mathbf{C}$.

*Proof:* By Proposition 1.4.6, $\Phi$ is a one-parameter group, for $\widehat{\mathbf{C}}$ is compact. Looking at the description of the automorphisms of $\widehat{\mathbf{C}}$ given in Proposition 1.1.22, we see that $\Phi_1$ admits either two distinct fixed points or one (double) fixed point. In the latter case, there is $\gamma_0\in\operatorname{Aut}(\widehat{\mathbf{C}})$ such that we can write $\gamma^{-1}\circ\Phi_1\circ\gamma(z)=z+a$ for a suitable $a\in\mathbf{C}$. Since $\Phi_t$ for $t\neq1$ is an automorphism of $\widehat{\mathbf{C}}$ commuting with $\Phi_1$, (1.1.22) shows that we should have $\gamma^{-1}\circ\Phi_t\circ\gamma(z)=z+\alpha(t)$, where $\alpha\colon\mathbf{R}^+\to(\mathbf{C},+)$ is a continuous semigroup homomorphism with $\alpha(1)=a$. Then Lemma 1.4.2 yields $\alpha(t)=at$, and we are in case (i).

If $\Phi_1$ has two distinct fixed points, there is again $\gamma\in\operatorname{Aut}(\widehat{\mathbf{C}})$ so that we can write $\gamma^{-1}\circ\Phi_1\circ\gamma(z)=e^bz$, for a suitable $b\in\mathbf{C}$. Using again (1.1.22) we see that an automorphism $\phi$ of $\widehat{\mathbf{C}}$ commuting with $\Phi_1$ should be of the form $\gamma^{-1}\circ\phi\circ\gamma(z)=\lambda z^{\varepsilon(\phi)}$, where $\lambda\in\mathbf{C}^*$, and $\varepsilon(\phi)=\pm1$.

The map $\varepsilon\colon\mathbf{R}^+\to\mathbf{Z}_2$ given by $\varepsilon(t)=\varepsilon(\Phi_t)$ is a semigroup homomorphism; by Lemma 1.4.1, $\varepsilon\equiv1$. Hence $\gamma^{-1}\circ\Phi_t\circ\gamma(z)=\lambda(t)z$ for all $t\geq0$, where $\lambda\colon\mathbf{R}^+\to(\mathbf{C}^*,\cdot)$ is a continuous semigroup homomorphism with $\lambda(1)=e^b$. By Lemma 1.4.5, $\lambda(t)=e^{bt}$, and we are in case (ii), **q.e.d.**

We consider now the complex plane:

**Proposition 1.4.26:** *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\mathbf{C},\mathbf{C})$ be a non-trivial one-parameter semigroup on $\mathbf{C}$. Then either*
(i) $\Phi_t(z) = z + bt$ *for some* $b \in \mathbf{C}^*$, *or*
(ii) $\Phi_t(z) = e^{at}z - b(e^{at} - 1)$ *for some* $a \in \mathbf{C}^*$ *and* $b \in \mathbf{C}$.
*In case* (i), *the infinitesimal generator is* $F \equiv b$, *and* $\Phi$ *has no fixed points. In case* (ii), *the infinitesimal generator is* $F(z) = a(z - b)$, *and* $b$ *is the fixed point of* $\Phi$ *with spectral value* $a$.

*Proof:* By Proposition 1.4.6, every $\Phi_t$ is an injective entire function, that is a linear polynomial. Write $\Phi_t(z) = \alpha(t)z + \beta(t)$, where $\alpha\colon \mathbf{R}^+ \to \mathbf{C}^*$ and $\beta\colon \mathbf{R}^+ \to \mathbf{C}$ are continuous and satisfy

$$\begin{aligned} \alpha(s+t) &= \alpha(s)\alpha(t), \\ \beta(s+t) &= \alpha(s)\beta(t) + \beta(s). \end{aligned} \tag{1.4.22}$$

The first one, by Lemma 1.4.5, implies $\alpha(t) = e^{at}$ for some $a \in \mathbf{C}$. If $a = 0$, the second one implies $\beta(t) = bt$ for some $b \in \mathbf{C}^*$, by Lemma 1.4.2, and we are in case (i).

If $a \neq 0$, fix $t_0 > 0$ such that $e^{at_0} \neq 1$. Then, setting first $t = t_0$ and next $s = t_0$ in the second equation (1.4.22) and subtracting the results, we get

$$\beta(t) = \frac{\beta(t_0)}{1 - e^{at_0}}(e^{at} - 1),$$

and we are in case (ii). The last part of the assertion is just a computation, **q.e.d.**

The next case is $\mathbf{C}^*$:

**Proposition 1.4.27:** *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(\mathbf{C}^*,\mathbf{C}^*)$ be a non-trivial one-parameter semigroup on $\mathbf{C}^*$. Then $\Phi_t(z) = e^{at}z$ for some $a \in \mathbf{C}^*$. The infinitesimal generator is $F(z) = az$, and $\Phi$ has no fixed points.*

*Proof:* By Proposition 1.4.6 every $\Phi_t$ is injective, and thus the restriction of a homogeneous linear polynomial. Hence $\Phi_t(z) = \alpha(t)z^{\varepsilon(t)}$, where $\alpha\colon \mathbf{R}^+ \to \mathbf{C}^*$ and $\varepsilon\colon \mathbf{R}^+ \to \mathbf{Z}_2$ are continuous semigroup homomorphisms. Then the assertion follows from Lemmas 1.4.1 and 1.4.5, **q.e.d.**

The next example is the torus. The connected component at the identity of the automorphism group of a torus $X$ is isomorphic to $\mathbf{R}^2/\mathbf{Z}^2$, by Proposition 1.1.32. Now, since $X$ is compact, Proposition 1.4.6 implies that every one-parameter semigroup $\Phi$ on $X$ is a one-parameter group; in particular, it can be thought of as a continuous semigroup homomorphism $\Phi\colon \mathbf{R}^+ \to \mathbf{R}^2/\mathbf{Z}^2$. Then $\Phi$ lifts to a continuous semigroup homomorphism $\widetilde{\Phi}\colon \mathbf{R}^+ \to \mathbf{C}$; Lemma 1.4.2 completely determines this kind of homomorphism, and we have proved

**Proposition 1.4.28:** *Let $X$ be a torus, and $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ a one-parameter semigroup on $X$. Then $\Phi$ is a one-parameter group, and the lifting $\widetilde{\Phi}\colon \mathbf{R}^+ \to \mathrm{Hol}(\mathbf{C},\mathbf{C})$ is given by $\widetilde{\Phi}_t(z) = z + at$ for a suitable $a \in \mathbf{C}$.*

Finally we are left with the doubly connected domains, that is $\Delta^*$ and the annuli $A(r,1) = \{z \in \mathbf{C} \mid r < |z| < 1\}$ for $0 < r < 1$. We begin with $A(r,1)$:

**Proposition 1.4.29:** *Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}\big(A(r,1), A(r,1)\big)$ be a non-trivial one-parameter semigroup on an annulus $A(r,1)$ with $0 < r < 1$. Then $\Phi_t(z) = e^{iat}z$ for some $a \in \mathbf{R}^*$. The infinitesimal generator is $F(z) = iaz$ and $\Phi$ has no fixed points.*

*Proof:* By Corollary 1.2.24, $\Phi$ is a semigroup of automorphisms; hence

$$\Phi_t(z) = \alpha(t)\, r^{(1-\varepsilon(t))/2} z^{\varepsilon(t)},$$

where $a: \mathbf{R}^+ \to \mathbf{C}^*$ and $\varepsilon: \mathbf{R}^+ \to \mathbf{Z}_2$ are continuous semigroup homomorphisms. Then the assertion follows from Lemmas 1.4.1 and 1.4.5, **q.e.d.**

We end the section with $\Delta^*$:

**Proposition 1.4.30:** *Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(\Delta^*, \Delta^*)$ be a non-trivial one-parameter semigroup on $\Delta^*$. Then there is a one-parameter semigroup $\widetilde{\Phi}$ on $\Delta$ with fixed point $0$ such that $\Phi_t = \widetilde{\Phi}_t|_{\Delta^*}$ for all $t \geq 0$. In particular, $\Phi$ has no fixed points and its infinitesimal generator is of the form $F(z) = -zf(z)$, where $f \in \mathcal{P}$.*

*Proof:* Every $\Phi_t$ has a removable singularity in 0, and hence is the restriction of a function $\widetilde{\Phi}_t \in \mathrm{Hol}(\Delta, \Delta)$. Obviously, $\widetilde{\Phi}: \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ is a one-parameter semigroup on $\Delta$; it remains to show that $\widetilde{\Phi}_t(0) = 0$ for all $t > 0$, for then the last assertion follows from Theorem 1.4.19.

Assume, by contradiction, that $\widetilde{\Phi}_{t_0}(0) \neq 0$ for some $t_0 > 0$. Since $\widetilde{\Phi}_{t_0}$ sends $\Delta^*$ into itself, $\widetilde{\Phi}_{t_0}$ has no zeroes in $\Delta$. The same argument works for any $\widetilde{\Phi}_{t_0/n}$ (where $n \geq 1$), because $\widetilde{\Phi}_{t_0/n}(0) = 0$ implies $\widetilde{\Phi}_{t_0}(0) = (\widetilde{\Phi}_{t_0/n})^n(0) = 0$. But $\widetilde{\Phi}_{t_0/n} \to \mathrm{id}_\Delta$ as $n \to +\infty$, and Hurwitz's theorem provides a contradiction, **q.e.d.**

## Notes

It is difficult to date precisely the birth of the concept of one-parameter semigroups of holomorphic functions. At the beginning of this century Tricomi (see for instance Tricomi [1917]) was dealing with problems somehow regarding the asymptotic behavior of one-parameter semigroups; however, only later Wolff [1938] recognized the relevance of equations like (1.4.4). The typical approach used to be via the problem of *fractional iteration;* loosely stated, given $f \in \mathrm{Hol}(X, X)$ they looked for a reasonable way of defining the $r$-th iterate of $f$ for any $r$ positive real. One standard request was that two fractional iterates of the same function must commute; from here trying to imbed $f$ in a one-parameter semigroup is a very short step. We saw (Proposition 1.4.6) that there are non trivial obstructions to this approach, and indeed a great amount of work has been spent on this question; a recent paper in this field with a good bibliography is Cowen [1981].

Anyway, a consequence of this state of affairs is that the results described in this chapter are quite recent, coming mainly from Berkson and Porta [1978] and Heins [1981].

Theorem 1.4.11 is the main result of Berkson and Porta [1978]. It should be remarked that an infinitesimal generator actually is a holomorphic vector field on the domain; hence it could be possible, using this interpretation, to extend Theorem 1.4.11 to any Riemann surface, but Proposition 1.4.24 made such a generalization useless.

Theorem 1.4.14 has been proved for the upper half-plane by Berkson and Porta [1978], as well as Theorems 1.4.17 and 1.4.19 in the disk, using different arguments. Our proof of Theorem 1.4.17 is due to Vesentini.

Lemma 1.4.21 is due to Wolff [1934], Noshiro [1935] and Warschawski [1935]; it is also the first step in the characterization, due to Grunsky [1971], of the domains convex in the vertical direction, that is such that the intersection with any vertical line is connected (possibly empty).

Theorems 1.4.22 and 1.4.23 are in Heins [1981]. They are an indication of strong relationships between one-parameter semigroups and the theory of conformal representations; a work in this direction is Goryaĭnov [1987].

Finally, the whole section 1.4.3 comes from Heins [1981], where also the case of discontinuous one-parameter semigroups is considered.