# Chapter 1.1
# Schwarz's lemma and Riemann surfaces

A characteristic feature of the theory of holomorphic functions is the very strong relationship between analytical properties of functions and geometrical properties of domains. One of the most striking examples of this phenomenon is the path connecting Schwarz's lemma to Montel's theorem passing through the Poincaré distance on hyperbolic Riemann surfaces. One of the main goals of this chapter is to unwind this thread starting from the very beginning, both for its own interest and because it will provide us with a number of tools we shall need later on.

The main connection between analytical and geometrical aspects of the theory is the invariant version of Schwarz's lemma proved by Pick, stating that any holomorphic function of the unit disk $\Delta$ of $\mathbf{C}$ into itself is a contraction for the Poincaré metric. In other words, the geometry — i.e., the Poincaré metric — imposes a strong analytic constraint on the space $\mathrm{Hol}(\Delta, \Delta)$: the equicontinuity. Now, using the universal covering map, we can carry over the construction of the Poincaré metric to any hyperbolic Riemann surface and, by means of the Ascoli-Arzelà theorem, this will eventually lead to a geometrical, and slightly unusual, proof of Montel's theorem.

Thus it turns out that the normality of the family of holomorphic functions into a hyperbolic Riemann surface (and related facts, like Vitali's theorem) depends essentially on the existence of a complete distance contracted by holomorphic functions. As we shall see in the second part of this book, this approach still works for complex manifolds of dimension greater than one, and this is the main reason we choosed it here. Actually, we shall push it a little farther, to recover Picard's theorems; in Kobayashi [1970] and Lang [1987] it is used to prove even Schottky's and Bloch's theorems.

The second main goal of this chapter, besides the setting up of all the preliminary material, is the description of the boundary behavior of the universal covering map of hyperbolic domains. This was an argument in great favour in the first thirty years of this century, but it is almost completely neglected today. We shall need it to study iteration theory on hyperbolic domains, and so the last section of this chapter is devoted to the proof of the main statements.

One word about our approach to Riemann surfaces. Our starting point is the quotation of Riemann's uniformization theorem; from there the exposition is almost self-contained, relying heavily on the theory of covering spaces. The inexperienced reader is urged to look at the first few sections of Forster [1981]; the experienced reader — well, you know what you have to do, don't you?

## 1.1.1 *The Poincaré metric*

In this section we shall discuss the connections between Schwarz's lemma and the Poincaré metric on the open unit disk $\Delta$ of $\mathbf{C}$. We shall also describe the basic geometry of the Poincaré metric.

We begin with *Schwarz's lemma*:

**Theorem 1.1.1:** *Let $f\colon\Delta\to\Delta$ be a holomorphic function such that $f(0)=0$. Then*

$$\forall z\in\Delta \qquad |f(z)|\le|z| \tag{1.1.1}$$

*and*

$$|f'(0)|\le 1. \tag{1.1.2}$$

*Moreover, equality in (1.1.1) for some non-zero $z$ or in (1.1.2) occurs iff there is $\theta\in\mathbf{R}$ such that $f(z)=e^{i\theta}z$ for all $z\in\Delta$.*

*Proof:* Since $f(0)=0$, we can define a function $g\colon\Delta\to\mathbf{C}$ by setting $g(z)=f(z)/z$. If $z\in\Delta$ and we pick $|z|<r<1$, then by the maximum principle

$$|g(z)|\le\sup_{|w|=r}|g(w)|=\sup_{|w|=r}\frac{|f(w)|}{r}\le\frac{1}{r}.$$

Letting $r\to 1$ we get $|g(z)|\le 1$, that is (1.1.2) and (1.1.1) (for $g(0)=f'(0)$). If equality holds in (1.1.2) or in (1.1.1) for some non-zero $z$, then, again by the maximum principle, $g$ is constant, and the last assertion follows, **q.e.d.**

The first application of Schwarz's lemma is the computation of the automorphism group of $\Delta$:

**Proposition 1.1.2:** *Every automorphism $\gamma\colon\Delta\to\Delta$ of $\Delta$ is of the form*

$$\gamma(z)=e^{i\theta}\frac{z-a}{1-\overline{a}z} \tag{1.1.3}$$

*for some $\theta\in\mathbf{R}$, where $a=\gamma^{-1}(0)\in\Delta$. In particular, every $\gamma\in\mathrm{Aut}(\Delta)$ extends continuously to a homeomorphism of $\overline{\Delta}$ onto itself.*

*Proof:* First of all, every $\gamma$ given by (1.1.3) is an automorphism of $\Delta$. Indeed,

$$\forall z\in\Delta \qquad 1-|\gamma(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}, \tag{1.1.4}$$

and so $\gamma(\Delta)\subset\Delta$; furthermore,

$$\gamma^{-1}(z)=e^{-i\theta}\,\frac{z+ae^{i\theta}}{1+\overline{a}e^{-i\theta}z}$$

is the inverse of $\gamma$.

Let $\Gamma$ be the set composed by the automorphisms of type (1.1.3); it is easy to check that $\Gamma$ is a group, acting transitively on $\Delta$. Therefore if $\gamma$ is another automorphism of $\Delta$, there exists $\gamma_1\in\Gamma$ such that $\gamma_1\circ\gamma(0)=0$; hence it suffices to show that every automorphism $\gamma$ of $\Delta$ leaving 0 fixed is of the form $\gamma(z)=e^{i\theta}z$ for some $\theta\in\mathbf{R}$, and thus belongs to $\Gamma$. But if we apply (1.1.2) to $\gamma$ and $\gamma^{-1}$, we see that $|\gamma'(0)|=1$, and the assertion follows from Schwarz's lemma, **q.e.d.**

Thus every automorphism of $\Delta$ extends continuously to a neighbourhood of $\overline{\Delta}$; actually, they are restrictions of automorphisms of $\widehat{\mathbf{C}}$ sending $\Delta$ into itself (cf. Proposition 1.1.22). In particular, every automorphism of $\Delta$ sends $\partial\Delta$ into itself. The corresponding action of $\mathrm{Aut}(\Delta)$ on $\partial\Delta$ is fairly good. We say that a group $\Gamma$ acting on a set $X$ is *transitive* if for every $x_1$, $x_2 \in X$ there is $\gamma \in \Gamma$ such that $\gamma(x_1) = x_2$; *simply transitive* if the previous $\gamma$ is unique; *doubly transitive* if for every $x_1$, $x_2$, $y_1$, $y_2 \in X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ there is $\gamma \in \Gamma$ such that $\gamma(x_1) = y_1$ and $\gamma(x_2) = y_2$. Then

**Corollary 1.1.3:** $\mathrm{Aut}(\Delta)$ *acts transitively on* $\Delta$, *and doubly transitively on* $\partial\Delta$.

*Proof:* The transitivity on $\Delta$ and $\partial\Delta$ follows immediately from Proposition 1.1.2. So choose $\sigma_1$, $\sigma_2$, $\tau_1$, $\tau_2 \in \partial\Delta$ with $\sigma_1 \neq \sigma_2$ and $\tau_1 \neq \tau_2$; we seek $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(\sigma_1) = \tau_1$ and $\gamma(\sigma_2) = \tau_2$. Obviously, it is enough to show that such a $\gamma$ exists when $\sigma_1 = 1$ and $\sigma_2 = -1$. Moreover, we can also assume $\tau_1 = 1$, for $\mathrm{Aut}(\Delta)$ contains the rotations. In conclusion, given $\tau \in \partial\Delta$ we want $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(1) = 1$ and $\gamma(-1) = \tau$. Let $\sigma \in \partial\Delta$ be the square root of $-\tau$ with positive real part; set $a = (\sigma - 1)/(\sigma + 1)$ and $\alpha = (\sigma + 1)/(\overline{\sigma} + 1)$. Then $|a| < 1 = |\alpha|$, and

$$\gamma(z) = \alpha\,\frac{z - a}{1 - \overline{a}z}$$

behaves as required, **q.e.d.**

In particular, then, if we have to prove something (invariant by automorphisms, of course) about two points $\sigma_1$, $\sigma_2 \in \partial\Delta$ we may assume, without loss of generality, that $\sigma_1 = 1$ and $\sigma_2 = -1$.

On the other hand, the action of $\mathrm{Aut}(\Delta)$ on $\Delta$ is *not* doubly transitive. Indeed, if $z_0$ and $z_1$ are in $\Delta$, we can surely find $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(z_0) = 0$, but then $|\gamma(z_1)|$ depends only on the two points, and not on the particular $\gamma$ chosen. This is the first clue to the existence of an underlying geometrical structure that must be preserved by $\mathrm{Aut}(\Delta)$, and therefore strictly correlated to the holomorphic structure.

To be more specific, using the automorphism group of $\Delta$ we first express Theorem 1.1.1 in a more invariant form:

**Corollary 1.1.4:** *Let* $f\colon\Delta \to \Delta$ *be holomorphic. Then*

$$\forall z, w \in \Delta \qquad \left|\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}\right| \leq \left|\frac{z - w}{1 - \overline{w}z}\right|, \tag{1.1.5}$$

*and*

$$\forall z \in \Delta \qquad \frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{1.1.6}$$

*Moreover, equality in* (1.1.5) *for some* $z$, $w \in \Delta$ *or in* (1.1.6) *for some* $z \in \Delta$ *occurs iff* $f \in \mathrm{Aut}(\Delta)$.

*Proof:* Fix $w \in \Delta$, and let $\gamma_1$, $\gamma_2 \in \mathrm{Aut}(\Delta)$ be given by

$$\gamma_1(z) = \frac{z + w}{1 + \overline{w}z} \qquad \text{and} \qquad \gamma_2(z) = \frac{z - f(w)}{1 - \overline{f(w)}z}.$$

Then the assertion follows applying Schwarz's lemma to $\gamma_2 \circ f \circ \gamma_1$, **q.e.d.**

Note that then every $f \in \mathrm{Hol}(\Delta, \Delta)$ different from the identity has at most one fixed point: if $f(z_0) = z_0$ and $f(z_1) = z_1$ for $z_0 \neq z_1$, then (assuming, as we can, $z_0 = 0$) $f(z) = e^{i\theta} z$ for a suitable $\theta \in \mathbf{R}$; but $e^{i\theta} z_1 = z_1$ implies $e^{i\theta} = 1$, and $f = \mathrm{id}_\Delta$.

Led by (1.1.6), we introduce on $\Delta$ the *Poincaré metric*

$$d\kappa_z^2 = \frac{dz\, d\bar{z}}{(1 - |z|^2)^2}. \tag{1.1.7}$$

The differential geometry lovers can verify that $d\kappa^2$ is a Kähler metric of constant Gaussian curvature $-4$. The corresponding distance $\omega$ on $\Delta$ will be called the *Poincaré distance.*

**Lemma 1.1.5:** *For any $z \in \Delta$ we have*

$$\omega(0, z) = \tfrac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$

*Proof:* The expression of $d\kappa^2$ shows that the complex conjugation and the rotations about the origin are isometries for the Poincaré metric; hence the reflection with respect to any straight line passing through the origin must be an isometry too.

Let $\sigma\colon [0, \varepsilon) \to \Delta$ be a geodesic issuing from 0, and let $\dot{\sigma}(0) = v$. The reflection of $\sigma$ with respect to the line determined by $v$ is another geodesic with the same initial tangent vector; therefore $\sigma$ should be radial.

Hence we may compute $\omega(0, z)$ for any $z \in \Delta$ integrating $d\kappa$ along the segment from 0 to $z$, and we obtain

$$\omega(0, z) = \int_0^{|z|} \frac{dt}{1 - t^2} = \tfrac{1}{2} \log \frac{1 + |z|}{1 - |z|},$$

**q.e.d.**

So the Poincaré distance from 0 to $z$ is the inverse hyperbolic tangent of $|z|$.

We may now rephrase Corollary 1.1.4, obtaining the *Schwarz-Pick lemma*:

**Theorem 1.1.6:** *Let $f\colon \Delta \to \Delta$ be holomorphic. Then*

$$\forall z, w \in \Delta \qquad \omega\big(f(z), f(w)\big) \leq \omega(z, w), \tag{1.1.8}$$

*and*

$$\forall z \in \Delta \qquad f^*(d\kappa_z^2) \leq d\kappa_z^2. \tag{1.1.9}$$

*Moreover, equality in* (1.1.8) *for some $z \neq w \in \Delta$ or in* (1.1.9) *for some $z \in \Delta$ occurs iff $f \in \mathrm{Aut}(\Delta)$.*

*Proof:* (1.1.9) is exactly (1.1.6) in the new language. In particular, then, the automorphisms of $\Delta$ are isometries for the Poincaré metric, and Lemma 1.1.5 together with Proposition 1.1.2 show that the Poincaré distance is given by

$$\omega(z_1, z_2) = \tfrac{1}{2} \log \frac{1 + \left| \dfrac{z_1 - z_2}{1 - \overline{z_2} z_1} \right|}{1 - \left| \dfrac{z_1 - z_2}{1 - \overline{z_2} z_1} \right|}. \tag{1.1.10}$$

Since $\frac{1}{2}\log[(1+t)/(1-t)]$ is strictly increasing in $t$, (1.1.8) is exactly (1.1.5), and the assertion follows, **q.e.d.**

So we have constructed a distance on $\Delta$ which is contracted by holomorphic functions. This is exactly the geometrical structure whose existence we suggested before:

**Corollary 1.1.7:** *Let $z_1$, $z_2$, $w_1$, $w_2$ be four points of $\Delta$. Then there exists an automorphism $\gamma$ of $\Delta$ such that $\gamma(z_1) = w_1$ and $\gamma(z_2) = w_2$ iff $\omega(z_1, z_2) = \omega(w_1, w_2)$.*

*Proof:* It is clear that, by Theorem 1.1.6, the existence of such an automorphism implies the equality of the Poincaré distances. Conversely, assume $\omega(z_1, z_2) = \omega(w_1, w_2)$. Clearly, we can suppose $w_1 = 0$. Let $\gamma_1 \in \mathrm{Aut}(\Delta)$ be such that $\gamma_1(z_1) = 0 = w_1$. Then

$$\omega\big(0, \gamma_1(z_2)\big) = \omega(z_1, z_2) = \omega(0, w_2),$$

that is $|\gamma(z_2)| = |w_2|$. Hence there exists a rotation $\gamma_2$ about the origin such that $\gamma_2\big(\gamma_1(z_2)\big) = w_2$, and $\gamma = \gamma_2 \circ \gamma_1$ is the automorphism we were seeking, **q.e.d.**

The Poincaré metric is really intimately linked to the holomorphic structure of $\Delta$. It can be easily checked that it is the unique (up to a multiplicative constant) Riemannian metric invariant under $\mathrm{Aut}(\Delta)$; furthermore, practically the only isometries of $d\kappa^2$ are the automorphisms of $\Delta$:

**Proposition 1.1.8:** *The group of all isometries for the Poincaré metric consists of all holomorphic and antiholomorphic automorphisms of $\Delta$.*

*Proof:* Let $f = u+iv$ be an isometry for the Poincaré metric, where $u$ and $v$ are real-valued functions, and write $z = x + iy$, with $x$, $y \in \mathbf{R}$. Then

$$\frac{1}{\big(1-(u^2+v^2)\big)^2}\big((du)^2+(dv)^2\big) = \frac{1}{\big(1-(x^2+y^2)\big)^2}\big((dx)^2+(dy)^2\big),$$

and this implies

$$\begin{cases} \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 = \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2, \\ \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} = 0, \end{cases}$$

that is

$$\begin{cases} \dfrac{\partial u}{\partial x} = (-1)^\varepsilon \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -(-1)^\varepsilon \dfrac{\partial v}{\partial x} \end{cases}$$

with $\varepsilon = 0$ or $1$, and thus $f$ is either holomorphic or antiholomorphic, **q.e.d.**

Now we would like to describe a bit more the geometry of the Poincaré metric, beginning with the aspect of an *open Poincaré disk*

$$B_\omega(z,R)=\{w\in\Delta\mid \omega(z,w)<R\}$$

of center $z\in\Delta$ and radius $R>0$. Since the function $t\mapsto\frac{1}{2}\log[(1+t)/(1-t)]$ is strictly increasing, $B_\omega(z,R)$ is given by

$$B_\omega(z,R)=\left\{w\in\Delta\;\middle|\;\left|\frac{w-z}{1-\overline{z}w}\right|<\tanh R\right\}.$$

It is then easy to check that $B_\omega(z,R)$ is the open euclidean disk of center

$$z_0=\frac{1-(\tanh R)^2}{1-(\tanh R)^2|z|^2}\,z,$$

and radius

$$\rho=\frac{(\tanh R)(1-|z|^2)}{1-(\tanh R)^2|z|^2}. \tag{1.1.11}$$

In particular, every $B_\omega(z,R)$ is relatively compact in $\Delta$, and so the Poincaré metric is complete.

The other thing we shall need to know is the aspect of the geodesics for the Poincaré metric. We have already seen in the proof of Lemma 1.1.5 that the geodesics issuing from the origin are the diameters of the circle $\partial\Delta$. Now, $\mathrm{Aut}(\Delta)$ acts simply transitively on the so-called *line elements,* i.e., on the pairs composed by a point and a tangent direction at that point (in a fancier language, on the unit sphere bundle of $\Delta$); this is exactly the content of Schwarz's lemma. In particular, then, $\mathrm{Aut}(\Delta)$ acts transitively on the set of all geodesics; so it suffices to find the images of the diameters of $\partial\Delta$ through the elements of $\mathrm{Aut}(\Delta)$. Up to compose by rotations, this amounts to finding $\gamma(C)$, where $C=\{z\in\Delta\mid \mathrm{Re}\,z=0\}$ and $\gamma(z)=(z-a)/(1-\overline{a}z)$ for some $a\in\Delta$.

A point $w$ belongs to $\gamma(C)$ iff $\gamma^{-1}(w)\in C$, i.e., iff

$$0=\mathrm{Re}\,\frac{w+a}{1+\overline{a}w}=\frac{1}{|1+\overline{a}w|^2}\big\{(1+|w|^2)\,\mathrm{Re}\,a+\mathrm{Re}[(1+\overline{a}^2)w]\big\}.$$

If $\mathrm{Re}\,a=0$, $1+\overline{a}^2$ is real and positive, and so $\gamma(C)=C$. If $\mathrm{Re}\,a\neq 0$, $w\in\Delta$ belongs to $\gamma(C)$ iff

$$|w|^2+2\,\mathrm{Re}\left[\frac{1+\overline{a}^2}{\mathrm{Re}\,a}\,w\right]+1=0,$$

that is iff

$$|w-b|^2=|b|^2-1, \tag{1.1.12}$$

where $b=-(1+a^2)/\,\mathrm{Re}\,a$. Since $|b|>1$, (1.1.12) is the equation of an euclidean circle orthogonal to $\partial\Delta$.

In conclusion, we have proved that the geodesics of the Poincaré metric are the diameters of the circle $\partial\Delta$ and the intersections of $\Delta$ with the circles orthogonal to $\partial\Delta$. In particular, given two points of $\overline{\Delta}$ there exists a unique geodesic connecting them.

We end this section introducing a different model of the unit disk, the *upper half-plane* $H^+ = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$. The *Cayley transform* is the function $\Psi\colon \Delta \to H^+$ given by

$$\Psi(z) = i\,\frac{1+z}{1-z}. \tag{1.1.13}$$

It is easily verified that $\Psi$ is a biholomorphism between $\Delta$ and $H^+$, with inverse

$$\Psi^{-1}(w) = \frac{w-i}{w+i}.$$

If we imbed $H^+$ in the extended complex plane, then $\Psi$ extends to a homeomorphism of $\overline{\Delta}$ with $\overline{H^+}$, sending 1 in $\infty$, 0 in $i$ and $-1$ in 0.

Using the Cayley transform, we can transfer the Poincaré metric and distance from $\Delta$ to $H^+$. The *Poincaré metric* on $H^+$ is

$$d\kappa^2 = \frac{dw\, d\bar{w}}{4(\operatorname{Im} w)^2},$$

while the *Poincaré distance* on $H^+$ is given by

$$\omega(w_1, w_2) = \tfrac{1}{2} \log \frac{1 + \left|\dfrac{w_1 - w_2}{w_1 - \overline{w_2}}\right|}{1 - \left|\dfrac{w_1 - w_2}{w_1 - \overline{w_2}}\right|}. \tag{1.1.14}$$

Again, a holomorphic map $f\colon H^+ \to H^+$ contracts both Poincaré metric and distance, that is

$$\forall w \in H^+ \qquad \frac{|f'(w)|}{\operatorname{Im} f(w)} \le \frac{1}{\operatorname{Im} w}, \tag{1.1.15}$$

and

$$\forall w_1, w_2 \in H^+ \qquad \left|\frac{f(w_1) - f(w_2)}{f(w_1) - \overline{f(w_2)}}\right| \le \left|\frac{w_1 - w_2}{w_1 - \overline{w_2}}\right|, \tag{1.1.16}$$

with equality in (1.1.15) for some $w \in H^+$ or in (1.1.16) for some $w_1 \ne w_2 \in H^+$ iff $f \in \operatorname{Aut}(H^+)$.

We can also compute $\operatorname{Aut}(H^+)$:

**Proposition 1.1.9:** *Every automorphism* $\gamma\colon H^+ \to H^+$ *of* $H^+$ *is of the form*

$$\gamma(w) = \frac{aw+b}{cw+d}, \tag{1.1.17}$$

*for some* $a$, $b$, $c$, $d \in \mathbf{R}$ *such that* $ad - bc = 1$. *In particular,* $\mathrm{Aut}(H^+)$ *is isomorphic to* $\mathbf{PGL}(2,\mathbf{R}) = \mathbf{SL}(2,\mathbf{R})/\{\pm I_2\}$.

*Proof:* $\gamma$ is an automorphism of $H^+$ iff $\Psi^{-1} \circ \gamma \circ \Psi$ is an automorphism of $\Delta$. Plugging the Cayley transform into (1.1.3) we find exactly (1.1.17), with $a$, $b$, $c$, $d \in \mathbf{R}$ satisfying $D = ad - bc > 0$. But now if we divide $a$, $b$, $c$ and $d$ by $\sqrt{D}$ we can express $\gamma$ in the form (1.1.17) with coefficients satisfying $ad - bc = 1$.

In other words, the group homomorphism $\mu\colon \mathbf{SL}(2,\mathbf{R}) \to \mathrm{Aut}(H^+)$ given by

$$\mu\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w) = \frac{aw+b}{cw+d}$$

is surjective. Its kernel is easily seen be equal to $\{\pm I_2\}$, and the assertion follows, **q.e.d.**

The upper half-plane model is sometimes useful to understand questions regarding the behavior of geometrical objects at a point of the boundary. For instance, the Cayley transform sends the geodesics ending at $1 \in \partial\Delta$ into the vertical lines in $H^+$, quite a simpler object. On the other hand, the study of objects linked to internal points may be formally easier in $\Delta$ than in $H^+$. For instance, the isotropy group of $i$ in $H^+$ is the composed by the automorphisms of the cumbersome form

$$\gamma(w) = \frac{w\cos\theta - \sin\theta}{w\sin\theta + \cos\theta}.$$

For these reasons, from now on in the proofs we shall often move back and forth from $\Delta$ to $H^+$, according to the current situation.

## 1.1.2 *Fixed points of automorphisms*

In this section we have collected several facts about automorphisms of $\Delta$ and $H^+$, mainly regarding their fixed points.

We recall that an automorphism $\gamma$ of $\Delta$ extends continuosly to $\overline{\Delta}$, and the extension (still denoted by $\gamma$) sends $\overline{\Delta}$ into itself; in particular, it makes sense to look for fixed points in $\overline{\Delta}$. Analogously, an automorphism of $H^+$ extends continuously to $\overline{H^+}$ (where the closure is taken in the extended complex plane, and thus contains the point at infinity), and sends $\overline{H^+}$ into itself. The leading result is:

**Proposition 1.1.10:** *Let* $\gamma \in \mathrm{Aut}(\Delta)$, $\gamma \neq \mathrm{id}_\Delta$. *Then either*
(i) $\gamma$ *has a unique fixed point in* $\Delta$, *or*
(ii) $\gamma$ *has a unique fixed point in* $\partial\Delta$, *or*

(iii) $\gamma$ *has two distinct fixed points in* $\partial\Delta$.

*Proof:* Write

$$\gamma(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z},$$

for some $\theta \in \mathbf{R}$ and $z_0 \in \Delta$. The equation satisfied by the fixed points of $\gamma$ is

$$\overline{z_0} z^2 + (e^{i\theta} - 1)z - z_0 = 0. \tag{1.1.18}$$

If $z_0 = 0$, then $\gamma(z) = e^{i\theta} z$ and we are in the first case. If $z_0 \neq 0$, then (1.1.18) has (counting multiplicities) exactly two roots, $z_1$ and $z_2$. Moreover,

$$|z_1||z_2| = |z_0/\overline{z_0}| = 1.$$

Therefore either just one of them is in $\Delta$ — and we are again in case (i) —, or both are in $\partial\Delta$, and we are either in case (ii) or (iii), **q.e.d.**

An automorphism of $\Delta$ different from the identity is called *elliptic* if it has a (unique) fixed point in $\Delta$, *parabolic* if it has a unique fixed point on $\partial\Delta$, *hyperbolic* if it has two distinct fixed points on $\partial\Delta$.

Clearly, the same definitions make sense for automorphisms of $H^+$ (remembering to include the point at infinity). In this case, we may tell elliptic, parabolic and hyperbolic automorphisms directly from their representation as elements of $\mathbf{PGL}(2, \mathbf{R})$. Let $\gamma \in \mathbf{PGL}(2, \mathbf{R})$ be represented by $\tilde{\gamma} \in \mathbf{SL}(2, \mathbf{R})$. Then $|\mathrm{tr}\,\tilde{\gamma}|$ clearly depends only on $\gamma$, and not on the particular representative chosen. $|\mathrm{tr}\,\tilde{\gamma}|$ is called the *trace* of $\gamma$; it is obviously invariant under conjugation.

By Proposition 1.1.9, we have also defined the trace of an element of $\mathrm{Aut}(H^+)$. Then:

**Proposition 1.1.11:** *Let* $\gamma \in \mathrm{Aut}(H^+)$, $\gamma \neq \mathrm{id}_{H^+}$. *Then* $\gamma$ *is elliptic (parabolic, hyperbolic) iff its trace is less than* 2 *(equal to* 2, *greater than* 2*).*

*Proof:* Let $\gamma$ be represented by (1.1.17). Then the fixed points equation for $\gamma$ is

$$cw^2 + (d - a)w - b = 0.$$

If $c \neq 0$, then $\gamma$ is elliptic (parabolic, hyperbolic) iff this equation has two distinct complex roots (one double real root, two distinct real roots), i.e., iff $D = (d - a)^2 + 4bc < 0$ (respectively, $D = 0$, $D > 0$). Using the constraint $ad - bc = 1$ we easily compute

$$D = (a + d)^2 - 4,$$

and the assertion follows in this case.

If $c = 0$, then $d = a^{-1}$ and $\gamma$ has a fixed point at $\infty$; in particular, $\gamma$ cannot be elliptic. Hence it is hyperbolic iff it has a fixed point different from $\infty$, i.e., iff $d \neq a$, which is equivalent to $|a + d| > 2$, and we are done, **q.e.d.**

To better understand the different kinds of automorphisms, we use Corollary 1.1.3. If $\gamma \in \mathrm{Aut}(H^+)$ is hyperbolic, up to conjugation we can assume that the fixed points of $\gamma$ are 0 and $\infty$. Hence $\gamma(w) = aw$ for some $a > 0$, $a \neq 1$, and its trace is $a + a^{-1} > 2$. In particular, the group of hyperbolic automorphisms fixing two given points of $\partial H^+$ (or $\partial\Delta$) is isomorphic to $(\mathbf{R}^+, \cdot)$.

Analogously, if $\gamma$ is parabolic we can assume its fixed point is $\infty$, and write $\gamma(w) = w+b$ for some $b \in \mathbf{R}^*$. In particular, the group of parabolic automorphisms fixing one given point of $\partial H^+$ (or $\partial\Delta$) is isomorphic to $(\mathbf{R}, +)$.

Finally, if $\gamma$ is elliptic we can assume its fixed point is $i$, and write

$$\gamma(w) = \frac{w\cos\theta - \sin\theta}{w\sin\theta + \cos\theta} \tag{1.1.19}$$

for some $\theta \in \mathbf{R}$; the trace of $\gamma$ is then $2|\cos\theta|$. In particular, the group of elliptic automorphisms fixing one given point of $H^+$ (or $\Delta$) is isomorphic to $(\mathbf{S}^1, \cdot)$.

Coming back in $\Delta$, an elliptic automorphism $\gamma$ of fixed point 0 is just a rotation about the origin. For this reason the elliptic automorphisms are sometimes called *non-euclidean rotations.*

Clearly, the elliptic automorphisms of trace 0 must be peculiar in one way or another. Indeed we have

**Proposition 1.1.12:** *Let $\gamma \in \mathrm{Aut}(H^+)$ be different from the identity. Then $\gamma$ has trace zero iff $\gamma^2 = \mathrm{id}_{H^+}$.*

*Proof:* Write $\gamma$ as in (1.1.17). Then a computation yields

$$\gamma^2(w) = \frac{(a+d)(aw+b) - w}{(a+d)(cw+d) - 1},$$

where we have used $ad - bc = 1$, and the assertion follows, **q.e.d.**

Now we shall meet for the first time one of the main *leitmotive* of this book: the relation between commuting functions and fixed points. In this first case the connection is easily proved:

**Proposition 1.1.13:** *Let $\gamma_1$, $\gamma_2 \in \mathrm{Aut}(\Delta)$, both different from the identity. Then $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ iff $\gamma_1$ and $\gamma_2$ have the same fixed points.*

*Proof:* Transfer everything on $H^+$. If $\gamma_1$ is parabolic, without loss of generality we can assume $\gamma_1(w) = w + \beta$, with $\beta \in \mathbf{R}^*$. Write $\gamma_2$ as in (1.1.17); then $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ yields

$$\begin{pmatrix} a+\beta c & b+\beta d \\ c & d \end{pmatrix} = \pm \begin{pmatrix} a & \beta a + b \\ c & \beta c + d \end{pmatrix}.$$

Hence $c = 0$, $a = d$ and $ad = 1$, showing that $\gamma_2(w) = w + b$.

If $\gamma_1$ is hyperbolic, without loss of generality we can assume $\gamma_1(w) = \lambda w$ for some $\lambda > 0$, $\lambda \neq 1$. Now $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ yields

$$\begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} \lambda a & b \\ \lambda c & d \end{pmatrix}.$$

Hence $b = c = 0$ and $ad = 1$, showing that $\gamma_2(w) = a^2 w$.

Finally, if $\gamma_1$ is elliptic we can assume without loss of generality that $\gamma_1$ is given by (1.1.19); moreover, since $\gamma_1 \neq \mathrm{id}_{H^+}$, we also have $\sin\theta \neq 0$. This time $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ yields $a = d$ and $b = -c$, showing that $\gamma_2$ is elliptic and fixes $i$, **q.e.d.**

In this book we shall encounter several results of this kind: commuting holomorphic maps must have common fixed points (under suitable hypotheses, of course). For the moment, we shall content ourselves with two corollaries, that we shall need in our study of Riemann surfaces.

**Corollary 1.1.14:** *Let* $\Gamma$ *be a discrete abelian subgroup of* $\mathrm{Aut}(\Delta)$. *Then* $\Gamma$ *is cyclic. In particular, if* $\Gamma$ *does not contain elliptic elements, then* $\Gamma$ *is isomorphic to* $\mathbf{Z}$.

*Proof:* By Proposition 1.1.13, all elements of $\Gamma$ (except the identity) have the same fixed points. Hence $\Gamma$ is (topologically) isomorphic to a discrete subgroup of $\mathbf{R}^+$, $\mathbf{R}$ or $\mathbf{S}^1$, according as $\Gamma$ consists of hyperbolic, parabolic or elliptic automorphisms, and hence it is cyclic, generated by the element nearest to the identity, **q.e.d.**

**Corollary 1.1.15:** *Let* $\Gamma$ *be a non-abelian subgroup of* $\mathrm{Aut}(\Delta)$ *without elliptic elements. Then* $\Gamma$ *contains a hyperbolic automorphism.*

*Proof:* Transfer everything on $H^+$, as usual. Assume there is $\gamma \in \Gamma$ parabolic; we can suppose $\gamma(w) = w + \beta$ for some $\beta$ real and different from 0. Since $\Gamma$ is non-abelian, by Proposition 1.1.13 it contains some other element $\gamma_1$ with different fixed point set. If $\gamma_1$ is hyperbolic, there is nothing to prove. If $\gamma_1$ is parabolic, write it as in (1.1.17); clearly, $c$ must be non-zero. Then $\gamma^k \circ \gamma_1$ has trace $|a + d + k\beta c|$, and so it is hyperbolic for $k$ sufficiently large, **q.e.d.**

We end this section with a technical result we shall need in section 1.1.5.

**Lemma 1.1.16:** (i) *Every* $\gamma \in \mathrm{Aut}(\Delta)$ *sends circular arcs contained in* $\overline{\Delta}$ *in circular arcs contained in* $\overline{\Delta}$.
(ii) *Let* $\gamma \in \mathrm{Aut}(\Delta)$ *be a hyperbolic automorphism with fixed points* $\sigma$, $\tau \in \partial\Delta$. *Then* $\gamma$ *sends any circular arc* $C \subset \overline{\Delta}$ *connecting* $\sigma$ *with* $\tau$ *into itself.*
(iii) *Let* $\gamma \in \mathrm{Aut}(\Delta)$ *be a parabolic automorphism with fixed point* $\tau \in \partial\Delta$. *Then* $\gamma$ *sends any circumference* $C \subset \overline{\Delta}$ *passing through* $\tau$ *into itself.*

*Proof:* (i) This is a straightforward verification.

(ii) By (i), we can assume $\sigma = 1$ and $\tau = -1$ (cf. Corollary 1.1.3). Transfer everything on $H^+$, via the Cayley transform $\Psi$. The image through $\Psi$ of a circular arc connecting $-1$

with 1 is a half-line issuing from 0; since there is $\lambda > 0$ such that $\Psi \circ \gamma \circ \Psi^{-1}(w) = \lambda w$ for all $w \in H^+$, the assertion follows.

(ii) Without loss of generality, we can assume $\tau = 1$. Transfer everything on $H^+$, via the Cayley transform $\Psi$. The image through $\Psi$ of a circumference passing through 1 is a horizontal line. Since there is $b \in \mathbf{R}$ such that $\Psi \circ \gamma \circ \Psi^{-1}(w) = w + b$ for all $w \in H^+$, the assertion follows, **q.e.d.**

### 1.1.3 *Riemann surfaces*

In this section we collect the first fundamental facts concerning Riemann surfaces, up to the classification of Riemann surfaces with abelian fundamental group. As mentioned in the introduction to this chapter, our main tool will be the theory of covering spaces.

For us, a *Riemann surface* is a one-dimensional connected complex manifold. The main fact we shall use without proof is the fundamental *Riemann's uniformization theorem*:

**Theorem 1.1.17:** *Every simply connected Riemann surface is biholomorphic either to the extended complex plane* $\widehat{\mathbf{C}}$*, or to the complex plane* $\mathbf{C}$*, or to the unit disk* $\Delta$*.*

A proof can be found, e.g., in Forster [1981] or in Farkas and Kra [1980].

The importance of this theorem lies in the fact that, as we shall see momentarily, every Riemann surface is the quotient of a simply connected Riemann surface by a group of automorphisms; therefore the study of Riemann surfaces is reduced to the investigation of particular subgroups of $\mathrm{Aut}(\widehat{\mathbf{C}})$, $\mathrm{Aut}(\mathbf{C})$ and $\mathrm{Aut}(\Delta)$. We shall consistently use this approach throughout this section, reverting as needed to a more geometric point of view in subsequent sections.

Our first aim is then to describe any Riemann surface as quotient of a simply connected Riemann surface. We begin with:

**Lemma 1.1.18:** *Let $X$ be a Riemann surface. Then there exists a simply connected Riemann surface $\widetilde{X}$ and a holomorphic covering map $\pi: \widetilde{X} \to X$. Moreover, $\widetilde{X}$ is uniquely determined.*

*Proof:* Let $\pi: \widetilde{X} \to X$ be the universal covering map of $X$; $\widetilde{X}$ is uniquely determined. Using $\pi$ we can endow $\widetilde{X}$ with a unique structure of Riemann surface so that $\pi$ become holomorphic, and the assertion follows, **q.e.d.**

Starting from here, we would like to describe at some extent this covering map. First of all, some definitions. Given a Riemann surface $X$, the simply connected Riemann surface whose existence is proved in Lemma 1.1.18 will be called the *universal covering surface* of $X$. $X$ is said *elliptic* (*parabolic, hyperbolic*) if its universal covering surface is $\widehat{\mathbf{C}}$ (respectively $\mathbf{C}$, $\Delta$).

Let $\pi: \widetilde{X} \to X$ be the universal covering map of a Riemann surface $X$. An *automorphism of the covering* (or *deck transformation*) is an automorphism $\gamma$ of $\widetilde{X}$ such that $\pi \circ \gamma = \pi$. Then standard covering spaces theory (see Forster [1981]) shows that

(a) the group $\Gamma$ of automorphism of the universal covering is isomorphic to the fundamental group $\pi_1(X)$ of the surface;
(b) $\Gamma$ acts transitively on the fibers of the covering, i.e., for any point $z \in X$ and any pair of points $w_1, w_2 \in \pi^{-1}(z)$ there exists $\gamma \in \Gamma$ such that $\gamma(w_1) = w_2$;
(c) $X$ is biholomorphic to the orbit space $\widetilde{X}/\Gamma$.

So we are lead to characterize the subgroups of $\mathrm{Aut}(\widetilde{X})$ that can arise as fundamental group of a Riemann surface covered by $\widetilde{X}$.

Let $X$ be a Riemann surface, and $\Gamma$ a subgroup of $\mathrm{Aut}(X)$. We say that $\Gamma$ acts *freely* on $X$ if no element of $\Gamma$ other than the identity has a fixed point; $\Gamma$ is *properly discontinuous* at a point $z \in X$ if there exists a neighbourhood $U$ of $z$ such that $\{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \emptyset\}$ is finite; $\Gamma$ is *properly discontinuous* tout-court if it is so at every point. Then

**Proposition 1.1.19:** *Let $\pi: \widetilde{X} \to X$ be the universal covering map of a Riemann surface $X$, and let $\Gamma$ denote the automorphism group of the covering. Then $\Gamma$ is properly discontinuous and acts freely on $\widetilde{X}$. Conversely, if $\Gamma$ is a properly discontinuous subgroup of $\mathrm{Aut}(\widetilde{X})$ acting freely on $\widetilde{X}$, then $\widetilde{X}/\Gamma$ has a natural structure of Riemann surface, and the canonical map $\pi: \widetilde{X} \to \widetilde{X}/\Gamma$ is its universal covering.*

*Proof:* Fix $z_0 \in \widetilde{X}$ and a neighbourhood $V$ of $\pi(z_0)$ such that $\pi^{-1}(V)$ is a disjoint union of open subsets of $\widetilde{X}$ biholomorphic through $\pi$ to $V$ — in short, an admissible neighbourhood of $\pi(z_0)$. Let $U$ be the component of $\pi^{-1}(V)$ containing $z_0$; then $\pi|_U$ is a biholomorphism between $U$ and $V$. In particular, if $\gamma \in \Gamma$ is such that $\gamma(U) \cap U \neq \emptyset$, then $\gamma$ is the identity on $U$, and hence everywhere. Since $z_0$ is an arbitrary point of $\widetilde{X}$, this means exactly that $\Gamma$ is properly discontinuous and acts freely on $\widetilde{X}$.

Conversely, if $\Gamma$ is properly discontinuous and acts freely on $\widetilde{X}$, then for every $z_0 \in \widetilde{X}$ there is a neighbourhood $U$ such that $\{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \emptyset\} = \{\mathrm{id}_{\widetilde{X}}\}$. It is then easy to check that the natural projection $\pi: \widetilde{X} \to \widetilde{X}/\Gamma$ is a covering map, and thus induces on $\widetilde{X}/\Gamma$ a natural structure of Riemann surface, **q.e.d.**

So a Riemann surface $X$ is given by its universal covering surface $\widetilde{X}$ and by a properly discontinuous subgroup $\Gamma$ of $\mathrm{Aut}(\widetilde{X})$ acting freely on $\widetilde{X}$ and isomorphic to $\pi_1(X)$. Clearly, two different subgroups of $\mathrm{Aut}(\widetilde{X})$ can give rise to the same Riemann surface. The next proposition tells when this happens:

**Proposition 1.1.20:** *Two Riemann surfaces $X_1$ and $X_2$ are biholomorphic iff they have the same universal covering surface $\widetilde{X}$ and their fundamental groups are conjugated in $\mathrm{Aut}(\widetilde{X})$.*

*Proof:* Let $f: X_1 \to X_2$ be a biholomorphism. Then $X_1$ and $X_2$ have the same universal covering surface $\widetilde{X}$; let $\pi_j: \widetilde{X} \to X_j$ denote the corresponding covering map, and let $\Gamma_j \subset \mathrm{Aut}(\widetilde{X})$ be the automorphism group of $\pi_j$. The biholomorphism $f$ lifts to an automorphism $\varphi$ of $\widetilde{X}$ such that $\pi_2 \circ \varphi = f \circ \pi_1$. Then for every $\gamma_1 \in \Gamma_1$ we have

$$\pi_2 \circ \varphi \circ \gamma_1 = f \circ \pi_1 \circ \gamma_1 = f \circ \pi_1 = \pi_2 \circ \varphi,$$

that is $\varphi \circ \gamma_1 \circ \varphi^{-1} \in \Gamma_2$, and thus $\Gamma_2 = \varphi \Gamma_1 \varphi^{-1}$.

Conversely, assume that $\Gamma_2 = \varphi \Gamma_1 \varphi^{-1}$ for some $\varphi \in \mathrm{Aut}(\widetilde{X})$. Then it is easy to check that $\varphi$ defines a biholomorphism $f\colon X_1 \to X_2$ such that $\pi_2 \circ \varphi = f \circ \pi_1$, **q.e.d.**

During the proof, we lifted a function of a Riemann surface into itself to a map of its universal covering into itself. To fix the terminology, we digress a little to better discuss the argument; a proofs of the following facts can be found, for instance, in Massey [1967].

Let $X$ and $Y$ be two Riemann surfaces, $\pi_X\colon \widetilde{X} \to X$ and $\pi_Y\colon \widetilde{Y} \to Y$ their universal covering maps, and $\Gamma_X \subset \mathrm{Aut}(\widetilde{X})$ and $\Gamma_Y \subset \mathrm{Aut}(\widetilde{Y})$ their fundamental groups. Any function $f \in \mathrm{Hol}(X, Y)$ admits a *lifting*, that is a holomorphic function $\tilde{f} \in \mathrm{Hol}(\widetilde{X}, \widetilde{Y})$ such that $f \circ \pi_X = \pi_Y \circ \tilde{f}$. $\tilde{f}$ is uniquely determined by its value at one point, and any other lifting is of the form $\gamma \circ \tilde{f}$, where $\gamma \in \Gamma_Y$. Furthermore, for every $\gamma \in \Gamma_X$ there exists $\gamma_1 \in \Gamma_Y$ such that

$$\tilde{f} \circ \gamma = \gamma_1 \circ \tilde{f}; \tag{1.1.20}$$

conversely, a function $\tilde{f} \in \mathrm{Hol}(\widetilde{X}, \widetilde{Y})$ is a lifting of a function from $X$ to $Y$ iff (1.1.20) is satisfied. In particular, $\tilde{f}$ is a biholomorphism iff $f$ is so.

Another kind of lifting problem is given $f \in \mathrm{Hol}(X, Y)$ to find $\hat{f} \in \mathrm{Hol}(X, \widetilde{Y})$ such that $f = \pi_Y \circ \hat{f}$. This time the problem is not always solvable; $\hat{f}$ exists iff $f_*\bigl(\pi_1(X)\bigr)$ is trivial, where $f_*\colon \pi_1(X) \to \pi_1(Y)$ is the homomorphism induced by $f$ at the homotopy level. The relations among $f$, $\tilde{f}$ and $\hat{f}$ are summarized by the following commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle \pi_X}\big\downarrow & \overset{\hat{f}}{\nearrow} & \big\downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Another way of looking at this question makes use the notion of automorphic function. A function $g \in \mathrm{Hol}(\widetilde{X}, Y)$ is *automorphic* under $\Gamma_X$ if $g \circ \gamma = g$ for all $\gamma \in \Gamma_X$. In other words, $g$ is automorphic under $\Gamma_X$ iff there is a function $g_o \in \mathrm{Hol}(X, Y)$ such that $g = g_o \circ \pi_X$. This time the explicative commutative diagram is

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{g}_o} & \widetilde{Y} \\ {\scriptstyle \pi_X}\big\downarrow & \overset{g}{\searrow} & \big\downarrow{\scriptstyle \pi_Y} \\ \widetilde{X}/\Gamma_X = X & \xrightarrow{g_o} & Y \end{array}$$

Coming back to our function $f \in \mathrm{Hol}(X, Y)$, we see that $\hat{f} \in \mathrm{Hol}(X, \widetilde{Y})$ exists iff any lifting (and hence all liftings) $\tilde{f} \in \mathrm{Hol}(\widetilde{X}, \widetilde{Y})$ is automorphic under $\Gamma_X$. For future reference, we summarize the situation in the following proposition:

**Proposition 1.1.21:** *Let $X$ and $Y$ be two Riemann surfaces, $\pi_X\colon \widetilde{X} \to X$ and $\pi_Y\colon \widetilde{Y} \to Y$ their universal covering maps, and $\Gamma_X \subset \mathrm{Aut}(\widetilde{X})$ and $\Gamma_Y \subset \mathrm{Aut}(\widetilde{Y})$ their fundamental groups. Choose a function $f \in \mathrm{Hol}(X, Y)$. Then the following statements are equivalent:*

(i) $f_*\bigl(\pi_1(D)\bigr)$ *is trivial;*

(ii) *there exists* $\hat f \in \mathrm{Hol}(X, \widetilde Y)$ *such that* $f = \pi_Y \circ \hat f$;
(iii) *a lifting (and hence any lifting) of* $f$ *is automorphic under* $\Gamma_X$.

The proof can be either considered as an instructive exercise in covering spaces theory or worked out perusing Massey [1967].

The digression is concluded, and now we of course want to know which subgroups of $\mathrm{Aut}(\widehat{\mathbf{C}})$, $\mathrm{Aut}(\mathbf{C})$ and $\mathrm{Aut}(\Delta)$ are properly discontinuous and act freely. The first step is describing the automorphism group of the three simply connected Riemann surfaces. We already know $\mathrm{Aut}(\Delta)$, and the other two are given in

**Proposition 1.1.22:** (i) *Every automorphism* $\gamma$ *of* $\mathbf{C}$ *is of the form*

$$\gamma(z) = az + b, \tag{1.1.21}$$

*for some* $a, b \in \mathbf{C}$, $a \neq 0$.

(ii) *Every automorphism* $\gamma$ *of* $\widehat{\mathbf{C}}$ *is of the form*

$$\gamma(z) = \frac{az+b}{cz+d}, \tag{1.1.22}$$

*where* $a, b, c, d \in \mathbf{C}$ *are such that* $ad - bc = 1$. *The representation is unique up to sign, and* $\mathrm{Aut}(\widehat{\mathbf{C}})$ *is isomorphic to* $\mathbf{PGL}(2,\mathbf{C}) = \mathbf{SL}(2,\mathbf{C})/\{\pm I_2\}$.

*Proof:* (i) An automorphism of $\mathbf{C}$ is an injective entire function, that is a linear polynomial.

(ii) It is clear that every $\gamma \in \mathrm{Hol}(\widehat{\mathbf{C}}, \widehat{\mathbf{C}})$ given by (1.1.22) is an automorphism of $\widehat{\mathbf{C}}$; moreover, they act transitively on $\widehat{\mathbf{C}}$. Therefore, it suffices to show that the isotropy group of $\infty$ consists of automorphisms of the given form. But it is exactly $\mathrm{Aut}(\mathbf{C})$, and the assertion follows from (i), **q.e.d.**

To determine the right subgroups to consider, we need

**Lemma 1.1.23:** *Let* $\Gamma$ *be a group of automorphisms of a Riemann surface* $X$, *properly discontinuous at some point of* $X$. *Then* $\Gamma$ *is discrete.*

*Proof:* Assume $\Gamma$ is not discrete. Then there is an infinite sequence of distinct elements $\gamma_\nu \in \Gamma$ converging to an element $\gamma \in \Gamma$. Therefore $\gamma^{-1} \circ \gamma_\nu \to \mathrm{id}_X$, and $\Gamma$ cannot be properly discontinuous at any point of $X$, **q.e.d.**

In the next section, we shall see (Proposition 1.1.48) that if $X$ is a hyperbolic Riemann surface, then every discrete subgroup of $\mathrm{Aut}(X)$ is everywhere properly discontinuous. For the moment, Lemma 1.1.23 is enough to describe the properly discontinuous subgroups of $\mathrm{Aut}(\widehat{\mathbf{C}})$ and $\mathrm{Aut}(\mathbf{C})$ acting freely on $\widehat{\mathbf{C}}$, respectively $\mathbf{C}$, that is all the elliptic and parabolic Riemann surfaces:

**Corollary 1.1.24:** (i) *No non-trivial subgroup of* $\mathrm{Aut}(\widehat{\mathbf{C}})$ *acts freely on* $\widehat{\mathbf{C}}$*. In particular, the unique elliptic Riemann surface is* $\widehat{\mathbf{C}}$*.*
(ii) *The properly discontinuous subgroups of* $\mathrm{Aut}(\mathbf{C})$ *acting freely on* $\mathbf{C}$ *are, up to conjugation,* $\{\mathrm{id}_{\mathbf{C}}\}$*,* $\{\gamma(z) = z + n \mid n \in \mathbf{Z}\}$ *and* $\Gamma_\tau = \{\gamma(z) = z + m + n\tau \mid m, n \in \mathbf{Z}\}$*, where* $\tau \in H^+$*. In particular, the parabolic Riemann surfaces are* $\mathbf{C}$*,* $\mathbf{C}^*$ *and the tori* $\mathbf{C}/\Gamma_\tau$*.*

*Proof:* (i) Every element of $\mathrm{Aut}(\widehat{\mathbf{C}})$ has a fixed point in $\widehat{\mathbf{C}}$, by (1.1.22), and thus no subgroup of $\mathrm{Aut}(\widehat{\mathbf{C}})$ can act freely on $\widehat{\mathbf{C}}$.

(ii) By (1.1.21), the only elements of $\mathrm{Aut}(\mathbf{C})$ without fixed points are the translations $\gamma(z) = z + b$. In particular, by Lemma 1.1.23, the fundamental group of a parabolic surface is isomorphic to a discrete subgroup of $\mathbf{C}$.

The discrete subgroups of $\mathbf{C}$ are easily described: up to conjugation they are $\{0\}$, $\mathbf{Z}$ and $\mathbf{Z} \oplus \tau\mathbf{Z}$, where $\tau \in H^+$. They are properly discontinuous, and the assertion follows. The universal covering map $\pi: \mathbf{C} \to \mathbf{C}^*$ is given by

$$\pi(z) = \exp(2\pi i z), \tag{1.1.23}$$

**q.e.d.**

Later on we shall see (Corollary 1.1.49) that the properly discontinuous subgroups of $\mathrm{Aut}(\Delta)$ acting freely on $\Delta$ are exactly the discrete subgroups without elliptic elements.

A scrupulous reader may now properly ask for examples of hyperbolic Riemann surfaces. A first list of examples is provided by topology. Riemann surfaces with non-abelian fundamental group must be hyperbolic, for the non-hyperbolic Riemann surfaces always have abelian fundamental group. This get rid of compact Riemann surfaces; furthermore, it is clear that the non-compact Riemann surfaces not biholomorphic to a plane domain must be hyperbolic.

How about plane domains? First of all, again by topological considerations, a domain $D \subset \widehat{\mathbf{C}}$ with fundamental group non-trivial and not biholomorphic to $\mathbf{Z}$ must be hyperbolic; for instance, $\widehat{\mathbf{C}}$ minus three points is hyperbolic. Furthermore, every bounded domain must be hyperbolic, by Liouville's theorem.

At this point, the above mentioned scrupulous reader may begin suspecting that most of the plane domains are hyperbolic. In fact, almost all of them are so; this is a corollary of the following observation showing how non-hyperbolic and hyperbolic Riemann surfaces live in completely separated realms:

**Proposition 1.1.25:** *Every holomorphic function* $f: X \to Y$ *from an elliptic or parabolic Riemann surface* $X$ *into a hyperbolic Riemann surface* $Y$ *is constant.*

*Proof:* Let $\tilde{f}: \widetilde{X} \to \Delta$, where $\widetilde{X} = \widehat{\mathbf{C}}$ or $\mathbf{C}$, be a lifting of $f$ to the universal covering. By Liouville's theorem, $\tilde{f}$ is constant, and thus $f$ itself is constant, **q.e.d.**

**Corollary 1.1.26:** *Every domain* $D \subset \widehat{\mathbf{C}}$ *such that* $\widehat{\mathbf{C}} \setminus D$ *contains at least three points is a hyperbolic Riemann surface.*

*Proof:* In fact, by definition there exists a holomorphic immersion of $D$ into $\widehat{\mathbf{C}}$ minus three points; therefore $D$ must be hyperbolic by Proposition 1.1.25, **q.e.d.**

By the way, if we take $X = \mathbf{C}$ in Proposition 1.1.25 we recover the *little Picard theorem:* an entire function missing two values is constant.

Another corollary is the commonest form of the *Riemann's mapping theorem:*

**Corollary 1.1.27:** *Every simply connected domain $D \subset \mathbf{C}$ different from $\mathbf{C}$ is biholomorphic to $\Delta$.*

*Proof:* Indeed, if $D \ne \mathbf{C}$ then $\widehat{\mathbf{C}} \setminus D$ must contain at least three points, and the assertion follows from Corollary 1.1.26, **q.e.d.**

Later on we shall need a very natural (and quite difficult to prove) extension of Riemann's mapping theorem, the *Osgood-Taylor-Carathéodory theorem:*

**Theorem 1.1.28:** *Let $D \subset \mathbf{C}$ be a simply connected bounded domain such that $\partial D$ is a Jordan curve. Then every biholomorphism $f\colon D \to \Delta$ extends continuously to a homeomorphism between $\overline{D}$ and $\overline{\Delta}$. If moreover $\partial D$ is a $C^1$ curve, then $f$ extends to a $C^1$ diffeomorphism between $\overline{D}$ and $\overline{\Delta}$.*

A proof can be found, e.g., in Burckel [1979] for the Jordan case, and in Goluzin [1969] for the $C^1$ case.

For obvious reasons, a plane domain $D \subset \widehat{\mathbf{C}}$ such that $\widehat{\mathbf{C}} \setminus D$ contains at least three points will be called *hyperbolic.* Note that if $D \subset \widehat{\mathbf{C}}$ is hyperbolic then we can always assume $\infty \notin D$. More generally, a *hyperbolic domain* of a compact Riemann surface $\widehat{X}$ is a non-compact domain $D \subset \widehat{X}$ which is hyperbolic as Riemann surface. For the sake of brevity, we shall often say "a hyperbolic domain $D \subset \widehat{X}$" instead of "a hyperbolic domain $D$ of a compact Riemann surface $\widehat{X}$".

Now it is clear that most of Riemann surfaces are hyperbolic, and starting from the next section most of our theorems will be about hyperbolic Riemann surfaces. On the other hand, elliptic and parabolic Riemann surfaces share a particular feature: they all have abelian fundamental groups. This suggests that hyperbolic Riemann surfaces with abelian fundamental group must be special in some sense. In fact, they are very few. The classification is given in the following:

**Theorem 1.1.29:** *Let $X$ be a Riemann surface with abelian fundamental group. Then either*
(i) *$\pi_1(X)$ is trivial, and $X$ is either $\widehat{\mathbf{C}}$, $\mathbf{C}$ or $\Delta$, or*
(ii) *$\pi_1(X) \cong \mathbf{Z}$, and $X$ is either $\mathbf{C}^*$, $\Delta^* = \Delta \setminus \{0\}$ or $A(r,1) = \{z \in \mathbf{C} \mid r < |z| < 1\}$ for some $0 < r < 1$, or*
(iii) *$\pi_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $X$ is a torus.*

*Proof:* By Corollary 1.1.24, we can limit ourselves to hyperbolic Riemann surfaces. Realize $\pi_1(X)$ as a subgroup $\Gamma$ of $\mathrm{Aut}(\Delta)$; by Lemma 1.1.23 and Corollary 1.1.14 $\Gamma$ is infinite cyclic, that is $\pi_1(X) \cong \mathbf{Z}$, and it is generated by a either parabolic or hyperbolic element $\gamma$.

Transfer everything on $H^+$, as usual. If $\gamma$ is parabolic, without loss of generality we can assume $\gamma(w) = w + 1$. Then $H^+/\Gamma$ is biholomorphic to $\Delta^*$, and the covering map $\pi\colon H^+ \to \Delta^*$ is given by

$$\pi(w) = \exp(2\pi i w). \tag{1.1.24}$$

If $\gamma$ is hyperbolic, without any loss of generality we can assume $\gamma(w) = aw$ for some $a > 1$. Then $H^+/\Gamma$ is biholomorphic to $A(r, 1)$, where $r = \exp(-2\pi^2/\log a)$. The covering map $\pi\colon H^+ \to A(r, 1)$ is given by

$$\pi(w) = \exp\left(2\pi i \frac{\log w}{\log a}\right), \tag{1.1.25}$$

where $\log w$ is the principal branch of the logarithm in $H^+$. Note that if $a_1 \neq a_2$, with both $a_1$ and $a_2$ greater than 1, then the group generated by $\gamma_1(w) = a_1 w$ is not conjugated to the group generated by $\gamma_2(w) = a_2 w$, and the two corresponding Riemann surfaces are not biholomorphic (by Proposition 1.1.20), **q.e.d.**

We recall that a Riemann surface $X$ which is not simply connected is usually called *multiply connected*; if $\pi_1(X) \cong \mathbf{Z}$, it is called *doubly connected*. Then

**Corollary 1.1.30:** *Every doubly connected Riemann surface $X$ is biholomorphic either to $\mathbf{C}^*$, $\Delta^*$ or to an annulus $A(r, 1)$ for some $0 < r < 1$.*

In this setting, we can also describe the automorphism group of a Riemann surface:

**Proposition 1.1.31:** *Let $\pi\colon \widetilde{X} \to X$ be the universal covering of a Riemann surface $X$. Realize the fundamental group of $X$ as the automorphism group $\Gamma$ of the covering, and let $N(\Gamma)$ be the normalizer of $\Gamma$ in $\mathrm{Aut}(\widetilde{X})$. Then $\mathrm{Aut}(X) \cong N(\Gamma)/\Gamma$.*

*Proof:* Arguing exactly as in Proposition 1.1.20 we see that an automorphism of $X$ gives rise to an automorphism of $\widetilde{X}$ normalizing $\Gamma$, and *vice versa*, **q.e.d.**

As an application, we may compute the automorphism group of the Riemann surfaces listed in Theorem 1.1.29:

**Proposition 1.1.32:** (i) *Every $\gamma \in \mathrm{Aut}(\mathbf{C}^*)$ is of the form $\gamma(z) = \lambda z^{\pm 1}$, for some $\lambda \in \mathbf{C}^*$. In particular, the connected component at the identity of $\mathrm{Aut}(\mathbf{C}^*)$ is isomorphic to $\mathbf{C}^*$.*
(ii) *Let $\tau \in H^+$, and let $X_\tau$ be the torus $\mathbf{C}/\Gamma_\tau$. Then $\mathrm{Aut}(X_\tau)$ is isomorphic to $(\mathbf{R}^2/\mathbf{Z}^2) \times \mathbf{Z}_6$ if $\tau = e^{i\pi/3}$ or $e^{2i\pi/3}$, to $(\mathbf{R}^2/\mathbf{Z}^2) \times \mathbf{Z}_4$ if $\tau = i$, and to $(\mathbf{R}^2/\mathbf{Z}^2) \times \mathbf{Z}_2$ otherwise. In particular, the connected component at the identity of $\mathrm{Aut}(X_\tau)$ is always isomorphic to $\mathbf{R}^2/\mathbf{Z}^2$.*
(iii) *Every $\gamma \in \mathrm{Aut}(\Delta^*)$ is of the form $\gamma(z) = e^{i\theta} z$ for some $\theta \in \mathbf{R}$.*
(iv) *Every $\gamma \in \mathrm{Aut}\big(A(r, 1)\big)$ is either of the form $\gamma(z) = e^{i\theta} z$ or of the form $\gamma(z) = e^{i\theta} r z^{-1}$ for some $\theta \in \mathbf{R}$.*

*Proof:* (i) The fundamental group of $\mathbf{C}^*$ is generated by $\gamma_0(z) = z+1$. Then the normalizer of $\Gamma$ in $\mathrm{Aut}(\mathbf{C})$ is composed by the functions $\gamma(z) = \pm(z + b)$ with $b \in \mathbf{C}$. Using the universal covering map (1.1.23) to read the automorphisms in $\mathbf{C}^*$, the assertion follows.

(ii) The fundamental group $\Gamma_\tau$ of $X_\tau$ is generated by $\gamma_1(z) = z + 1$ and $\gamma_\tau(z) = z + \tau$. Let $\sigma(z) = az + b$ be an automorphism of $\mathbf{C}$, with inverse $\sigma^{-1}(z) = a^{-1}(z - b)$. Then $\sigma \in N(\Gamma_\tau)$ iff $\sigma^{-1}\Gamma_\tau\sigma = \sigma\Gamma_\tau\sigma^{-1} = \Gamma_\tau$. In other words, $\sigma \in N(\Gamma_\tau)$ iff $a^{-1}(m + n\tau)$ and $a(m + n\tau)$ belong to $\mathbf{Z} \oplus \tau\mathbf{Z}$ for all $m$, $n \in \mathbf{Z}$. This clearly implies either $a = \pm 1$ or

$a = \pm\tau$. The former possibility clearly defines elements of $N(\Gamma_\tau)$. On the contrary, the latter possibility defines elements of $N(\Gamma_\tau)$ iff $\tau^2 = p_1 + q_1\tau$ and $\tau^{-1} = p_2 + q_2\tau$ for suitable $p_1$, $p_2$, $q_1$, $q_2 \in \mathbf{Z}$, that is iff

$$q_1\tau + p_1 = \frac{1}{q_2}(1 - p_2\tau).$$

Recalling that $\operatorname{Im}\tau > 0$, we obtain $q_1 = -p_2/q_2$ and $p_1 = 1/q_2$. Hence $\tau$ is a root of the polynomial $t^2 - \varepsilon p_2 t + \varepsilon$ with positive imaginary part, where $\varepsilon = \pm 1$. This implies $\varepsilon = 1$ and $p_2^2 < 4$, and so the only possibilities are

$$\tau_0 = i, \qquad \tau_1 = \frac{1 + i\sqrt{3}}{2} = e^{i\pi/3} \qquad \text{and} \qquad \tau_{-1} = \frac{-1 + i\sqrt{3}}{2} = e^{2i\pi/3}.$$

Summing up, if $\tau \neq \tau_0$, $\tau_1$, $\tau_{-1}$ then $N(\Gamma_\tau) = \bigl\{\gamma(z) = \pm(z + b) \bigm| b \in \mathbf{C}\bigr\}$, whereas if $\tau$ is one of the exceptional values we have

$$N(\Gamma_{\tau_0}) = \bigl\{\gamma(z) = \pm(z + b) \bigm| b \in \mathbf{C}\bigr\} \cup \bigl\{\gamma(z) = \pm(iz + b) \bigm| b \in \mathbf{C}\bigr\}$$

and

$$N(\Gamma_{\tau_{\pm 1}}) = \bigl\{\gamma(z) = \pm(z + b) \bigm| b \in \mathbf{C}\bigr\} \cup \bigl\{\gamma(z) = \pm(\tau_{\pm 1} z + b) \bigm| b \in \mathbf{C}\bigr\},$$

and the assertion follows.

(iii) The fundamental group of $\Delta^*$ is generated by $\gamma_0(w) = w + 1$. Then the normalizer of $\Gamma$ in $\operatorname{Aut}(H^+)$ is given by $N(\Gamma) = \{\gamma(w) = w + b \mid b \in \mathbf{R}\}$. Using the universal covering map (1.1.24), the assertion follows. Alternatively, note that, by Riemann's extension theorem, every automorphism of $\Delta^*$ is the restriction of an automorphism of $\Delta$ leaving the origin fixed, and apply Schwarz's lemma.

(iv) The fundamental group of $A(r, 1)$ is generated by $\gamma_0(w) = a_0 w$, where

$$a_0 = \exp(-2\pi^2/\log r).$$

Then $N(\Gamma)$ is generated by the functions $\gamma(w) = aw$ for $a > 0$ and by $\gamma(w) = -1/w$. Using the universal covering map (1.1.25), the assertion again follows, **q.e.d.**

For sake of better place, we end this section recalling some classical facts of complex analysis we shall need later.

For any $z_0 \in \mathbf{C}$ and $r > 0$ we shall denote by $D(z_0, r) \subset \mathbf{C}$ the open euclidean disk of center $z_0$ and radius $r$. Our rewiew is based on a simplified version of *Rouché's theorem*:

**Theorem 1.1.33:** *Let $f$ and $g$ be functions holomorphic in a neighbourhood of a closed disk $\overline{D(z_0, r)} \subset \mathbf{C}$ and such that $|f - g| < |g|$ on $\partial D(z_0, r)$. Then $f$ and $g$ have the same number of zeroes in $D(z_0, r)$.*

*Proof:* Let $f_\lambda = g + \lambda(f - g)$ for $\lambda \in [0, 1]$. Then on $\partial D(z_0, r)$

$$0 < |g(z)| - |f(z) - g(z)| \le |g(z)| - \lambda|f(z) - g(z)| \le |f_\lambda(z)|.$$

Let $a_\lambda$ denote the number of roots of $f_\lambda$ in $D(z_0, r)$. By the logarithmic indicator theorem,

$$a_\lambda = \frac{1}{2\pi i} \int\limits_{\partial D(z_0,r)} \frac{f_\lambda'(\zeta)}{f_\lambda(\zeta)}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\partial D(z_0,r)} \frac{g'(\zeta) + \lambda\bigl(f'(\zeta) - g'(\zeta)\bigr)}{g(\zeta) + \lambda\bigl(f(\zeta) - g(\zeta)\bigr)}\, d\zeta.$$

Since $a_\lambda$ is integer and depends continuosly on $\lambda$, it is constant. In particular $a_0$, the number of zeroes of $g$, is equal to $a_1$, the number of zeroes of $f$, **q.e.d.**

A first corollary, that will later be considerably generalized and proved in a different way (see Proposition 1.3.14 and Corollary 2.1.32), is

**Corollary 1.1.34:** *Let $f\colon \Delta \to \Delta$ be holomorphic and such that $f(\Delta)$ is relatively compact in $\Delta$. Then $f$ has a fixed point.*

*Proof:* Let $r < 1$ be such that $|f(z)| < r$ for all $z \in \Delta$. Then on $\partial D(0,r)$

$$\big|z - \big(z - f(z)\big)\big| = |f(z)| < r = |z|.$$

Hence we may apply Theorem 1.1.33 to $\mathrm{id}_\Delta$ and $f - \mathrm{id}_\Delta$, and $f$ has exactly one fixed point in $D(0,r)$, **q.e.d.**

The second application is a list of three results, collectively known as *Hurwitz's theorems*:

**Corollary 1.1.35:** *Let $\{f_\nu\}$ be a sequence of functions holomorphic in a given neighbourhood of a closed disk $\overline{D(z_0,r)} \subset \mathbf{C}$. Assume that $f_\nu \to f$ uniformly on $\overline{D(z_0,r)}$, and that $f$ vanishes nowhere on $\partial D(z_0,r)$. Then for $\nu$ large enough $f_\nu$ has the same number of zeroes as $f$ in $D(z_0,r)$.*

*Proof:* Let $m = \inf\big\{|f(z)| \bigm| z \in \partial D(z_0,r)\big\} > 0$. Since $f_\nu \to f$ uniformly, for $\nu$ large enough we have $|f_\nu - f| < m \le |f|$ on $\partial D(z_0,r)$. Hence we can apply Theorem 1.1.33 to $f_\nu$ and $f$, **q.e.d.**

**Corollary 1.1.36:** *Let $X$ and $Y$ be two Riemann surfaces, and $\{f_\nu\} \subset \mathrm{Hol}(X,Y)$ a sequence of holomorphic functions converging, uniformly on compact sets, to a non-constant function $f \in \mathrm{Hol}(X,Y)$. Choose $w_0 \in f(X)$, and assume $f^{-1}(w_0)$ contains at least $p$ distinct points. Then for all $\nu$ large enough $f_\nu^{-1}(w_0)$ contains at least $p$ distinct points.*

*Proof:* Take $z_1, \ldots, z_p \in f^{-1}(w_0)$. Choose a neighbourhood $V \subset\subset Y$ of $w_0$ and $r_0 > 0$ such that there is a biholomorphism $\varphi\colon V \to D(0,r_0)$ with $\varphi(w_0) = 0$. Since $f_\nu \to f$ uniformly on compact sets, and $f$ is not constant, for all $j = 1, \ldots, p$ we can find a neighbourhood $U_j \subset\subset X$ of $z_j$ such that

(a) there are $r_j > 0$ and a biholomorphism $\psi_j\colon D(0,2r_j) \to U_j$ with $\psi_j(0) = z_j$;
(b) $f_\nu(U_j) \subset V$ for all $\nu$ large enough;
(c) $U_j \cap f^{-1}(w_0) = \{z_j\}$ for all $j = 1, \ldots, p$;
(d) $U_h \cap U_k = \emptyset$ if $h \ne k$.

Then Corollary 1.1.35 applied to $\varphi \circ f_\nu \circ \psi_j\colon D(0,r_j) \to D(0,r_0)$ shows that $f_\nu^{-1}(w_0) \cap U_j$ contains at least one point for $\nu$ large enough (and all $j$), and we are done, **q.e.d.**

**Corollary 1.1.37:** *Let $X$ and $Y$ be two Riemann surfaces, and $\{f_\nu\} \subset \mathrm{Hol}(X,Y)$ a sequence of injective holomorphic functions converging, uniformly on compact sets, to a function $f \in \mathrm{Hol}(X,Y)$. Then $f$ is either constant or injective.*

*Proof:* Assume $f^{-1}(z_0)$ contains at least two points for some $z_0 \in Y$. Then, by Corollary 1.1.36, either $f \equiv z_0$ or $f_\nu^{-1}(z_0)$ contains at least two points for every $\nu$ large enough, impossible, **q.e.d.**

### 1.1.4 *Hyperbolic Riemann surfaces and Montel's theorem*

In this section, which is probably the most important of this chapter, we shall use the geometrical structure induced by the Poincaré distance on any hyperbolic Riemann surface to derive a direct proof of Montel's theorem. The significance of this approach is twofold. On one side, it is a beatiful example of the correlation between geometrical and functional aspects of the theory of holomorphic functions. On the other side, the constructions involved here have been the original motivation of the more general arguments we shall describe in the second part of this book.

Our first aim is to transfer the Poincaré distance from $\Delta$ to any hyperbolic Riemann surface. Let $X$ be a hyperbolic Riemann surface, and denote by $\pi_X\colon \Delta \to X$ its universal covering map. Then the *Poincaré distance* $\omega_X\colon X \times X \to \mathbf{R}^+$ on $X$ is given by

$$\forall z,\, w \in X \qquad \omega_X(z,w) = \inf\bigl\{\omega(\tilde z, \tilde w) \bigm| \tilde z \in \pi_X^{-1}(z),\ \tilde w \in \pi_X^{-1}(w)\bigr\}. \tag{1.1.26}$$

We called $\omega_X$ a distance, but this requires a proof.

**Proposition 1.1.38:** *Let $X$ be a hyperbolic Riemann surface. Then $\omega_X$ is a distance on $X$ inducing the standard topology.*

*Proof:* The main fact here is that

$$\forall z,\, w \in X \qquad\qquad \omega_X(z,w) = \inf\bigl\{\omega(\tilde z, \tilde w) \bigm| \tilde w \in \pi_X^{-1}(w)\bigr\}, \tag{1.1.27}$$

where $\tilde z$ is *any* point in the fiber $\pi_X^{-1}(z)$. Indeed, (1.1.27) follows from the fact that the automorphism group of the covering acts transitively on the fibers. Using (1.1.26) and (1.1.27) it is now a routine matter to prove that $\omega_X$ is a distance.

For the last statement, note that $\pi_X$ is a local isometry between $(\Delta,\omega)$ and $(X,\omega_X)$, i.e., every point $z \in \Delta$ has a neighbourhood $U$ such that $\pi_X|_U\colon (U,\omega) \to (\pi_X(U),\omega_X)$ is an isometry. From this it easily follows that $\pi_X$ is continuous and open from $\Delta$ to $(X,\omega_X)$, and hence $\omega_X$ induces the standard topology, **q.e.d.**

From a differential geometric point of view, $\omega_X$ is but the distance induced by the Riemannian structure defined on $X$ by means of the Poincaré metric on $\Delta$ and the covering map $\pi_X$.

The first question about a distance is: is it complete? In our case the answer is positive:

**Proposition 1.1.39:** *Let $X$ be a hyperbolic Riemann surface. Then every $\omega_X$-bounded subset of $X$ is relatively compact. In particular, $\omega_X$ is a complete distance.*

*Proof:* It clearly suffices to show that a generic closed ball $\overline{B} = \{w \in X \mid \omega_X(z,w) \le r\}$ is compact, where $z \in X$ and $r > 0$.

Choose $\tilde z \in \Delta$ such that $\pi_X(\tilde z) = z$, and fix $\varepsilon > 0$. We claim that $\overline{B}$ is contained in $\pi_X\Bigl(\overline{B_\omega(\tilde z, r+\varepsilon)}\Bigr)$. Indeed, take $w \in \overline{B}$; then, by (1.1.27), there is $\tilde w \in \pi_X^{-1}(w)$ such that $\omega(\tilde z, \tilde w) < r + \varepsilon$, and so $\tilde w \in \overline{B_\omega(\tilde z, r+\varepsilon)}$. Since $\overline{B_\omega(\tilde z, r+\varepsilon)}$ is compact, the assertion follows, **q.e.d.**

The main property of the Poincarè distance on $\Delta$ was the Schwarz-Pick lemma. Accordingly, the main property of the Poincaré distance on an arbitrary Riemann surface is:

**Theorem 1.1.40:** *Let $X$ and $Y$ be two hyperbolic Riemann surfaces, and $f: X \to Y$ a holomorphic function. Then*

$$\forall z_1,\, z_2 \in X \qquad \omega_Y\big(f(z_1), f(z_2)\big) \le \omega_X(z_1, z_2).$$

*Proof:* Lift $f$ to a holomorphic function $\tilde{f}: \Delta \to \Delta$ such that $\pi_Y \circ \tilde{f} = f \circ \pi_X$. Now take $z_1,\, z_2 \in X$, and fix $\varepsilon > 0$. Choose $\tilde{z}_1,\, \tilde{z}_2 \in \Delta$ so that $\pi_X(\tilde{z}_j) = z_j$ for $j = 1, 2$ and $\omega(\tilde{z}_1, \tilde{z}_2) < \omega_X(z_1, z_2) + \varepsilon$. Then the Schwarz-Pick lemma yields

$$\omega_Y\big(f(z_1), f(z_2)\big) \le \omega\big(\tilde{f}(\tilde{z}_1), \tilde{f}(\tilde{z}_2)\big) \le \omega(\tilde{z}_1, \tilde{z}_2) \le \omega_X(z_1, z_2) + \varepsilon.$$

Since $\varepsilon$ is arbitrary, the assertion follows, **q.e.d.**

It should be noticed that if we set $\omega_X \equiv 0$ for a non-hyperbolic Riemann surface $X$, then (recalling Proposition 1.1.25) Theorem 1.1.40 holds for holomorphic functions between any pair of Riemann surfaces. The real significance of this observation will be revealed in the second part of this book: $\omega_X$ is just the Kobayashi distance on $X$.

An important consequence of Theorem 1.1.40 is that the family of holomorphic functions between two hyperbolic Riemann surfaces is equicontinuous. In particular,

**Corollary 1.1.41:** *Let $X$ and $Y$ be hyperbolic Riemann surfaces. Then the topology of pointwise convergence on* $\mathrm{Hol}(X, Y)$ *coincides with the compact-open topology.*

*Proof:* Since $\mathrm{Hol}(X, Y)$ is equicontinuous, we can quote Kelley [1955], p. 232, **q.e.d.**

From now on, then, we shall indifferently consider $\mathrm{Hol}(X, Y)$ endowed with the topology of pointwise convergence, or with the compact-open topology.

But now we had better to describe the concepts involved in Montel's theorem. Let $X_1$ and $X_2$ be two Riemann surfaces. A sequence of holomorphic maps $\{f_\nu\} \subset \mathrm{Hol}(X_1, X_2)$ is *compactly divergent* if for every pair of compact sets $K_1 \subset X_1$ and $K_2 \subset X_2$ there is $\nu_0 \in \mathbf{N}$ such that $f_\nu(K_1) \cap K_2 = \emptyset$ for all $\nu \ge \nu_0$. If $X_2 = \mathbf{C}$, we shall sometimes say that $\{f_\nu\}$ *diverges to infinity*, uniformly on compact sets.

A family $\mathcal{F} \subset \mathrm{Hol}(X_1, X_2)$ of holomorphic maps is said *normal* if every sequence in $\mathcal{F}$ admits either a convergent subsequence or a compactly divergent subsequence. For instance, $\{z^k\} \subset \mathrm{Hol}(\mathbf{C}, \mathbf{C})$ is not a normal family, whereas $\{z^k\} \subset \mathrm{Hol}(\Delta, \Delta)$ is.

Normality is a sort of compactness condition; for instance if $Y$ is a compact Riemann surface, then a family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ is normal iff it is relatively compact. So normality is naturally linked to the *Ascoli-Arzelá theorem,* that we shall use in the following form:

**Theorem 1.1.42:** *Let $X$ be a locally compact metric space, and $Y$ a metric space. Then a family $\mathcal{F} \subset C^0(X, Y)$ is relatively compact iff*
(i) *$\mathcal{F}(x) = \{f(x) \mid f \in \mathcal{F}\}$ is relatively compact in $Y$ for every $x \in X$, and*
(ii) *$\mathcal{F}$ is equicontinuous.*

For a proof see, e.g., Kelley [1955], p. 233.

Finally we can state and prove *Montel's theorem:*

**Theorem 1.1.43:** *Let $X$ and $Y$ be hyperbolic Riemann surfaces. Then* $\mathrm{Hol}(X,Y)$ *is a normal family.*

*Proof:* Let $\{f_\nu\}$ be a sequence of holomorphic functions from $X$ to $Y$; we have to prove that if $\{f_\nu\}$ is not compactly divergent then it admits a convergent subsequence.

Assume $\{f_\nu\}$ is not compactly divergent; then there are compact sets $K_1 \subset X$ and $K_2 \subset Y$ such that $f_\nu(K_1) \cap K_2 \neq \emptyset$ for infinitely many $\nu$; clearly, up to a subsequence we can assume $f_\nu(K_1) \cap K_2 \neq \emptyset$ for all $\nu$.

We claim that $\{f_\nu(z)\}$ is relatively compact in $Y$ for any $z \in X$. In fact, fix $z_0 \in X$ and $w_0 \in K_2$; furthermore, let $\delta_1$ denote the $\omega_X$-diameter of $K_1$, and $\delta_2$ the $\omega_Y$-diameter of $K_2$. Then Theorem 1.1.40 yields

$$\forall \nu \in \mathbf{N} \qquad \omega_Y\big(f_\nu(z_0), w_0\big) \le \min\{\omega_X(z_0, z) \mid z \in K_1\} + \delta_1 + \delta_2.$$

Hence $\{f_\nu(z_0)\}$ is $\omega_Y$-bounded and thus relatively compact in $Y$, by Proposition 1.1.39, and the claim is proved.

To finish the proof it suffices now to invoke the Ascoli-Arzelà theorem: its hypotheses are fulfilled (again by Theorem 1.1.40), and so $\{f_\nu\}$ is relatively compact. In particular, we can extract a converging subsequence, and we are done, **q.e.d.**

So we have completely traced the way from the seemingly innocuous Schwarz lemma to the all powerful Montel theorem. The novice reader will soon learn to appreciate the strength of Montel's theorem (the experienced reader appreciates it yet, we hope); a first sample is given by the striking consequences it has on the topology of $\mathrm{Hol}(X,Y)$. For instance, we have

**Corollary 1.1.44:** *Let $X$ and $Y$ be two hyperbolic Riemann surfaces. Then* $\mathrm{Hol}(X,Y)$ *is locally compact.*

*Proof:* Take $f \in \mathrm{Hol}(X,Y)$, fix $z_0 \in X$ and let $U \subset Y$ be a relatively compact open neighbourhood of $f(z_0)$; it suffices to show that the neighbourhood

$$W(z_0, U) = \{g \in \mathrm{Hol}(X,Y) \mid g(z_0) \in U\}$$

of $f$ is relatively compact in $\mathrm{Hol}(X,Y)$. But indeed no sequence in $W(z_0,U)$ can be compactly divergent, and the assertion follows from Theorem 1.1.43, **q.e.d.**

But the first really important consequence is that the convergence of a sequence of functions in the usual compact-open topology is assured by very weak hypotheses, as shown in *Vitali's theorem*:

**Theorem 1.1.45:** *Let $X$ and $Y$ be hyperbolic Riemann surfaces, and let $\{f_\nu\}$ be a sequence of functions in* $\mathrm{Hol}(X,Y)$. *Assume there is a set $A \subset X$ with at least one accumulation point such that $\{f_\nu(z)\}$ converges for every $z \in A$. Then $\{f_\nu\}$ converges uniformly on compact subsets of $X$.*

*Proof:* Clearly the sequence $\{f_\nu\}$ cannot contain compactly divergent subsequences; hence, by Theorem 1.1.43, it suffices to show that it has only one limit point in $\mathrm{Hol}(X,Y)$. Let $f$, $g \in \mathrm{Hol}(X,Y)$ be two limit points of $\{f_\nu\}$. Since $\{f_\nu(z)\}$ converges for every $z \in A$, it follows that $f \equiv g$ on $A$, and hence everywhere, by the identity principle, **q.e.d.**

We can also apply Montel's theorem to the investigation of the topological structure of the automorphism group of a hyperbolic Riemann surface:

**Proposition 1.1.46:** (i) *Let* $\{f_\nu\}$ *be a sequence of automorphisms of a hyperbolic Riemann surface* $X$ *converging to a holomorphic function* $f \in \mathrm{Hol}(X,X)$. *Then* $f \in \mathrm{Aut}(X)$.
(ii) *Let* $\{f_\nu\}$ *be a sequence of automorphisms of a hyperbolic domain* $D \subset \widehat{X}$ *converging to a holomorphic function* $f\colon D \to \widehat{X}$. *Then either* $f$ *is a constant belonging to* $\partial D$ *or* $f \in \mathrm{Aut}(D)$.

*Proof:* To reduce part (ii) to part (i), it suffices to remark that if $f(D) \cap \partial D \neq \emptyset$, then $f$ cannot be open, and so it is a constant belonging to $\partial D$. Therefore we can directly assume $f \in \mathrm{Hol}(D,D)$ and prove only part (i).

Assume $f$ is not constant. Set $g_\nu = f_\nu^{-1}$; by Theorem 1.1.43, up to a subsequence we can assume that either $g_\nu \to g \in \mathrm{Hol}(X,X)$ or $\{g_\nu\}$ is compactly divergent. Since for any $z \in X$ we have $f_\nu(z) \to f(z) \in X$ and $g_\nu\big(f_\nu(z)\big) = z$, $\{g_\nu\}$ cannot diverge. Then

$$\forall z \in X \qquad g\big(f(z)\big) = \lim_{\nu\to\infty} g_\nu\big(f_\nu(z)\big) = z,$$

and, since $g(X) \subset X$,

$$\forall z \in X \qquad f\big(g(z)\big) = \lim_{\nu\to\infty} f_\nu\big(g_\nu(z)\big) = z;$$

therefore $g = f^{-1}$.

Finally, assume $f \equiv z_0 \in X$. Again, up to a subsequence, we can suppose that the sequence $\{f_\nu^{-1}\}$ converges to a function $g \in \mathrm{Hol}(X,X)$. But then for all $z \in X$ we should have $z = \lim_{\nu\to\infty} f_\nu^{-1}\big(f_\nu(z)\big) = g(z_0)$, impossible, **q.e.d.**

**Corollary 1.1.47:** *Let* $X$ *be a hyperbolic Riemann surface. Then* $\mathrm{Aut}(X)$ *is closed in* $\mathrm{Hol}(X,X)$, *and hence it is locally compact. Furthermore, the isotropy group* $\mathrm{Aut}_{z_0}(X)$ *is compact for all* $z_0 \in X$.

*Proof:* The closure is Proposition 1.1.46; the local compactness is Corollary 1.1.44; the compactness follows remarking that no sequence of elements of $\mathrm{Aut}_{z_0}(X)$ can be compactly divergent, **q.e.d.**

Now we can keep an old promise:

**Proposition 1.1.48:** *Let* $X$ *be a hyperbolic Riemann surface. Then every discrete subgroup* $\Gamma$ *of* $\mathrm{Aut}(X)$ *is everywhere properly discontinuous.*

*Proof:* If $\Gamma$ is not discontinuous at some point $z_0 \in X$, there exists an infinite sequence of distinct elements $\gamma_\nu \in \Gamma$ and a sequence $\{z_\nu\} \subset X$ converging to $z_0$ such that $\gamma_\nu(z_\nu) \to z_0$. Up to a subsequence, we can assume that $\gamma_\nu$ tends to a function $\gamma\colon X \to X$, for $\{\gamma_\nu\}$ cannot have compactly divergent subsequences. By Proposition 1.1.46, $\gamma \in \mathrm{Aut}(X)$; hence $\gamma \in \Gamma$, for $\Gamma$ is closed, and this is impossible because $\Gamma$ is discrete, **q.e.d.**

**Corollary 1.1.49:** *The properly discontinuous subgroups of* $\mathrm{Aut}(\Delta)$ *acting freely on* $\Delta$ *are the discrete subgroups without elliptic elements.*

We shall now describe another application of Theorem 1.1.40, giving a proof of the Big Picard Theorem.

First of all we shall need some information about $\omega_{\Delta^*}$. For every $r \in (0,1)$ set

$$\delta(r) = \max\bigl\{\omega_{\Delta^*}(re^{i\theta_1}, re^{i\theta_2}) \bigm| \theta_1,\, \theta_2 \in \mathbf{R}\bigr\}.$$

Using (1.1.24), (1.1.27) and (1.1.14) it is easy to check that

$$\delta(r) = \tfrac{1}{2}\log\frac{\sqrt{1+(\log r)^2/\pi^2}+1}{\sqrt{1+(\log r)^2/\pi^2}-1}.$$

In particular

$$\lim_{r\to 0}\delta(r) = 0. \tag{1.1.28}$$

This has an interesting consequence:

**Lemma 1.1.50:** *Let* $X$ *be a hyperbolic Riemann surface, and take* $f \in \mathrm{Hol}(\Delta^*, X)$ *such that there is a sequence* $\{z_\nu\} \subset \Delta^*$ *converging to* $0$ *so that* $f(z_\nu) \to w_0 \in X$ *as* $\nu \to +\infty$. *Then* $f_*\bigl(\pi_1(\Delta^*)\bigr)$ *is trivial.*

*Proof:* Set $r_\nu = |z_\nu|$ and let $\sigma_\nu\colon [0,1] \to \Delta^*$ be the curve $\sigma_\nu(t) = r_\nu e^{2\pi i t}$. Clearly, it suffices to show that the curve $f\circ\sigma_\nu$ is homotopic to a point in $X$ for $\nu$ large enough.

Choose a contractible neighbourhood $U$ of $w_0$, and $r_0 > 0$ such that the $\omega_X$-disk of radius $2r_0$ and center $w_0$ is contained in $U$. Now we have $\omega_X\bigl(f(z_\nu), w_0\bigr) < r_0$ eventually; hence, by (1.1.28), the image of $f\circ\sigma_\nu$ is contained in $U$ for $\nu$ large enough, and we are done, **q.e.d.**

The astonishing fact is that the *Big Picard Theorem* now follows:

**Theorem 1.1.51:** *Let* $X$ *be a hyperbolic Riemann surface contained in a compact Riemann surface* $\widehat{X}$. *Then every* $f \in \mathrm{Hol}(\Delta^*, X)$ *extends holomorphically to a function* $\tilde{f} \in \mathrm{Hol}(\Delta, \widehat{X})$.

*Proof:* Clearly, we can assume $\widehat{X}\setminus X$ is a finite set (containing at least three points if $\widehat{X} = \widehat{\mathbf{C}}$, at least one point if $\widehat{X}$ is a torus, even empty if $\widehat{X}$ itself is hyperbolic). Suppose first there is a sequence $\{z_\nu\} \subset \Delta^*$ converging to $0$ such that $f(z_\nu) \to w_0 \in X$. By Lemma 1.1.50, $f_*\bigl(\pi_1(\Delta^*)\bigr)$ is trivial; hence, by Proposition 1.1.21, $f$ lifts to a function $\hat{f}\colon \Delta^* \to \Delta$ such that $f = \pi\circ\hat{f}$, where $\pi\colon \Delta \to X$ is the universal covering map of $X$. Then Riemann's removable singularity theorem implies that $\hat{f}$ extends holomorphically to a function $f_1 \in \mathrm{Hol}(\Delta,\Delta)$, and clearly $\tilde{f} = \pi\circ f_1 \in \mathrm{Hol}(\Delta, X)$ is the required extension of $f$.

So assume now that for no sequence $\{z_\nu\} \subset \Delta^*$ converging to $0$ the sequence $\{f(z_\nu)\}$ converges to a point of $X$. This means that for every neighbourhood $U$ of $\widehat{X}\setminus X$ in $\widehat{X}$ there is a small disk $V$ about $0$ such that $f(V\setminus\{0\}) \subset U$. But if we take $U$ biholomorphic to a (not necessarily connected) bounded open set in $\mathbf{C}$ (it is possible because $\widehat{X}\setminus X$ is finite), another application of Riemann's removable singularity theorem shows that $f$ extends holomorphically across $0$ to a function $\tilde{f} \in \mathrm{Hol}(\Delta, \widehat{X})$, **q.e.d.**

We end this section showing that Theorem 1.1.43 characterizes hyperbolic Riemann surfaces:

**Proposition 1.1.52:** *Let $X$ be a Riemann surface. Then $X$ is hyperbolic iff* $\mathrm{Hol}(X,X)$ *is normal.*

*Proof:* By Theorem 1.1.43 it suffices to show that if $X$ is not hyperbolic then $\mathrm{Hol}(X,X)$ is not normal.

Since $\mathrm{Hol}(\widehat{\mathbf{C}},\widehat{\mathbf{C}})$, $\mathrm{Hol}(\mathbf{C},\mathbf{C})$ and $\mathrm{Hol}(\mathbf{C}^*,\mathbf{C}^*)$ contain the sequence $\{z^k\}$, it is clear that neither of them is normal. To deal with the tori, for every $p\in\mathbf{N}$ let $\tilde{\mu}_p\colon\mathbf{C}\to\mathbf{C}$ be given by $\tilde{\mu}_p(z)=pz$. Then every $\tilde{\mu}_p$ induces a holomorphic function $\mu_p\in\mathrm{Hol}(\mathbf{C}/\Gamma_\tau,\mathbf{C}/\Gamma_\tau)$, where $\Gamma_\tau=\mathbf{Z}\oplus\tau\mathbf{Z}$ and $\tau\in H^+$. It is easy to check that $\{\mu_p\}$ does not have converging subsequences, and we are done, **q.e.d.**

## 1.1.5 *Boundary behavior of the universal covering map*

In this section we want to study more accurately the boundary behavior of the universal covering map of a particular kind of multiply connected hyperbolic domains. A (always non-compact) domain $D$ of a compact Riemann surface $\widehat{X}$ is *of regular type* if

(a) every connected component of $\partial D$ is either a Jordan curve (that is a closed simple continuous curve), or an isolated point, and
(b) for every connected component $\Sigma$ of $\partial D$ there exists a neighbourhood $V$ of $\Sigma$ such that $V\cap\partial D=\Sigma$.

Hyperbolic domains of regular type form a large class of Riemann surfaces which are sufficiently well behaved for our needs and not too much specific. The results we prove will be used in section 1.3.3.

So let $D\subset\widehat{X}$ be a multiply connected hyperbolic domain of regular type, and $\pi\colon\Delta\to D$ its universal covering map; realize $\pi_1(D)$ as a subgroup $\Gamma$ of $\mathrm{Aut}(\Delta)$, as usual. We recall the definition of the isomorphism $\mu\colon\pi_1(D)\to\Gamma$. Let $[\sigma]$ be an element of $\pi_1(D)$, and for every $z_0\in\Delta$ choose a loop $\sigma_{z_0}\colon[0,1]\to D$ representing $[\sigma]$ such that $\sigma_{z_0}(0)=\pi(z_0)$; two such $\sigma_{z_0}$ are always homotopic. Let $\tilde{\sigma}_{z_0}\colon[0,1]\to\Delta$ be the unique lifting of $\sigma_{z_0}$ such that $\tilde{\sigma}_{z_0}(0)=z_0$. Then

$$\mu[\sigma](z_0)=\tilde{\sigma}_{z_0}(1).$$

It is clear that $\mu$ is well defined, and it is not difficult to check that it is an isomorphism (see Forster [1981], p. 34).

Now let $\Sigma$ be a connected component of $\partial D$; we shall say that $\Sigma$ is a *point component* if it is an isolated point, a *Jordan component* otherwise. To every component $\Sigma$ we associate the element $[\sigma_\Sigma]$ in $\pi_1(D)$ — and hence $\gamma_\Sigma=\mu[\sigma_\Sigma]\in\Gamma$ — represented by any simple loop $\sigma_\Sigma$ in $D$ separating $\Sigma$ from $\partial D\setminus\Sigma$, leaving $\Sigma$ on its left side.

For future reference, we now officially state the following triviality:

**Lemma 1.1.53:** *Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type. Then $\partial D$ has a finite number of connected components, and $\pi_1(D)$ is finitely generated.*

*Proof:* Assume, by contradiction, that $\{\Sigma_\nu\}$ is an infinite sequence of distinct connected components of $\partial D$. Take $z_\nu \in \Sigma_\nu$; up to a subsequence, $\{z_\nu\}$ converges to a point $w_0 \in \partial D$. But then the connected component of $\partial D$ containing $w_0$ cannot be separated from the other components, contradiction.

Finally, if $\Sigma_0, \ldots, \Sigma_p$ are the connected components of $\partial D$, then $\{[\sigma_{\Sigma_1}], \ldots, [\sigma_{\Sigma_p}]\}$ is a set of generators of $\pi_1(D)$, **q.e.d.**

In particular, therefore, hyperbolic domains of regular type are of finite topological characteristic (if you know what this means).

We would like to study the relationship between $\Sigma$ and $\gamma_\Sigma$. As first guiding example, take $D = A(r,1) \subset \mathbf{C}$, with $0 < r < 1$. Then denoting by $\Sigma_0$ the outer boundary and by $\Sigma_1$ the inner boundary, $\gamma_{\Sigma_0}$ and $\gamma_{\Sigma_1}$ are hyperbolic automorphisms of $\Delta$ with fixed points 1 and $-1$, and the universal covering map of $D$ extends continuously to $\partial\Delta \setminus \{\pm 1\}$.

As second example, take $D = \Delta^*$, and set $\Sigma_0 = \{0\}$ and $\Sigma_1 = \partial\Delta$. Then this time $\gamma_{\Sigma_0}$ and $\gamma_{\Sigma_1}$ are parabolic automorphisms of $\Delta$ with fixed point 1, and the universal covering map $\pi$ of $D$ extends continuously to $\partial\Delta \setminus \{1\}$. Furthermore, $\pi(z)$ tends to 0 as $z$ goes to 1 non-tangentially.

The aim of this section is to show that the previous two examples illustrate the general phenomena. A main tool for our job will be a weak version of Fatou's uniqueness theorem, proved by means of the following result, the so-called *two constants theorem:*

**Theorem 1.1.54:** *Let $f\colon \Delta \to \mathbf{C}$ be holomorphic and such that $|f| \le M$ for a suitable $M \ge 0$. Assume that on a given arc $A \subset \partial\Delta$ of length $\alpha$ we have*

$$\forall \tau \in A \qquad \limsup_{z \to \tau} |f(z)| \le m,$$

*for a suitable $m \le M$. For all $0 < r < 1$ set*

$$\lambda(r) = \frac{\alpha}{2\pi} \frac{1-r}{1+r}.$$

*Then for any $r \in (0,1)$ we have*

$$\forall z \in \overline{D(0,r)} \qquad |f(z)| \le m^{\lambda(r)} M^{1-\lambda(r)}. \tag{1.1.29}$$

*Proof:* Assume first that $f$ is continuous in $\overline{\Delta}$, and define $\varphi$, $h\colon \overline{\Delta} \to \mathbf{R}$ by

$$\varphi(z) = \log\bigl(\max\{|f(z)|, m\}\bigr),$$

and

$$h(z) = \frac{1}{2\pi i} \int_{\partial\Delta} P(z,\zeta)\varphi(\zeta)\, d\zeta,$$

where

$$P(z,\zeta) = \operatorname{Re}\left[\frac{\zeta+z}{\zeta-z}\right]$$

is the Poisson kernel. Since $\varphi$ is subharmonic on $\Delta$, $h$ is harmonic on $\Delta$ and $\varphi = h$ on $\partial\Delta$, we have $\varphi \le h$ on $\overline{\Delta}$.

After a rotation, we can assume that $\overline{A} = \{e^{i\theta} \mid 0 \le \theta \le \alpha\}$. On $A$ we have $|f(z)| \le m$, and so $\varphi(z) = \log m$. Therefore

$$\begin{aligned} h(z) &\le \frac{1}{2\pi}\int_0^\alpha P(z,e^{i\theta})\log m\,d\theta + \frac{1}{2\pi}\int_\alpha^{2\pi} P(z,e^{i\theta})\log M\,d\theta \\ &= \log M - \psi(z)\bigl[\log M - \log m\bigr], \end{aligned}$$

where

$$\psi(z) = \frac{1}{2\pi}\int_0^\alpha P(z,e^{i\theta})\,d\theta \ge \frac{1}{2\pi}\int_0^\alpha \frac{1-|z|}{1+|z|}\,d\theta = \lambda(|z|).$$

Hence

$$h(z) \le \log M - \lambda(|z|)[\log M - \log m] = \lambda(|z|)\log m + \bigl(1-\lambda(|z|)\bigr)\log M.$$

Since $|f| \le \max\{|f|, m\} = e^\varphi \le e^h$, we obtain

$$|f(z)| \le m^{\lambda(|z|)} M^{1-\lambda(|z|)}, \tag{1.1.30}$$

and (1.1.29) in this case.

If $f$ is not necessarily continuous on $\overline{\Delta}$, for every $s \in (0,1)$ define $f_s \colon \overline{\Delta} \to \mathbf{C}$ by $f_s(z) = f(sz)$. If $s > r$, (1.1.30) together with the maximum principle yields

$$\forall z \in \overline{D(0,r)} \qquad |f(z/s)| \le \bigl(\sup_{\tau\in\overline{A}} |f_s(\tau)|\bigr)^{\lambda(r)} M^{1-\lambda(r)}.$$

Letting $s \to 1$ we obtain (1.1.29), **q.e.d.**

Then our version of *Fatou's uniqueness theorem* is:

**Corollary 1.1.55:** *Let $D$ be a domain in a Riemann surface $X$ such that $\partial D$ is a Jordan curve, $Y$ another Riemann surface, and $f \colon D \to Y$ holomorphic. Assume there is a non-void open arc $A \subset \partial D$ and $y_0 \in Y$ such that*

$$\forall \tau \in A \qquad \lim_{z\to\tau} f(z) = y_0.$$

*Then $f \equiv y_0$.*

*Proof:* Fix a neighbourhood $U$ of $y_0$ in $Y$ such that there is a biholomorphism $\phi \colon U \to \Delta$ with $\phi(y_0) = 0$. By continuity we can find a neighbourhood $V$ in $X$ of a point $\tau_0 \in A$, a biholomorphism $\psi \colon V \to \Delta$ and a simply connected subdomain $D_1 \subset D$ bounded by a Jordan curve such that $D_1 \subset\subset V$, $\partial D_1 \cap \partial D \subset A$ and $f(D_1) \subset U$.

By Theorem 1.1.28 there is a biholomorphism $\eta \colon \psi(D_1) \to \Delta$ which extends to a homeomorphism of the closures. Then if we apply Theorem 1.1.54 to $\phi \circ f \circ (\eta \circ \psi)^{-1}$ we find $f \equiv y_0$ on $D_1$, and thus everywhere, **q.e.d.**

Now let $\Sigma$ be a connected component of the boundary of a hyperbolic domain $D \subset \widehat{X}$ of regular type. Choose a simple loop $\sigma$ in $D$ separating $\Sigma$ from $\partial D \setminus \Sigma$, and leaving $\Sigma$ on the left. Denote by $D_1 \subset D$ the doubly connected domain bounded by $\sigma$ and $\Sigma$.

Let $\tilde{\sigma}\colon [0,1] \to \Delta$ be a lifting of $\sigma$; we know that $\gamma_\Sigma\bigl(\tilde{\sigma}(0)\bigr) = \tilde{\sigma}(1)$. Let $\tau_1$, $\tau_2 \in \partial\Delta$ be the fixed points of $\gamma_\Sigma$ (possibly $\tau_1 = \tau_2$). We can enclose the image of $\tilde{\sigma}$ in a lens $L$ bounded by two arcs of circumference connecting $\tau_1$ and $\tau_2$. By Lemma 1.1.16, $L$ is invariant under $\gamma_\Sigma$; hence the image $\overline{\sigma}$ of $\tilde{\sigma}\bigl([0,1]\bigr)$ under the action of the cyclic group $\Gamma_\Sigma$ generated by $\gamma_\Sigma$ is a Jordan arc contained in $L$ and connecting $\tau_1$ and $\tau_2$ (see Figures 1.1 and 1.2).

***Figure 1.1*** *The point component case.*

***Figure 1.2*** *The Jordan component case.*

Now let $H\colon [0,1] \times [0,1] \to \overline{D_1}$ be an isotopy, i.e., an injective homotopy between $\sigma$ and $\Sigma$ (that is $H$ sends one-to-one $(0,1) \times [0,1)$ into $D_1$, $H(\cdot,0) = H(\cdot,1)$, $H(0,\cdot) = \sigma$ and $H(1,\cdot) = \Sigma$). We can lift $H$ to a homotopy $\widetilde{H}\colon [0,1) \times [0,1] \to \Delta$ such that $\widetilde{H}(0,\cdot) = \tilde{\sigma}$; using the action of $\Gamma_\Sigma$, we can extend $\widetilde{H}$ to a homotopy $\overline{H}\colon [0,1) \times (-\infty,+\infty) \to \Delta$ such that $\overline{H}(0,\cdot) = \overline{\sigma}$, $\lim\limits_{t\to-\infty} \overline{H}(\cdot,t) \equiv \tau_1$ and $\lim\limits_{t\to+\infty} \overline{H}(\cdot,t) \equiv \tau_2$. The image of $\overline{H}$ is one component $\Delta_1$ of $\Delta \setminus \overline{\sigma}$, and $\pi(\Delta_1) = H\bigl([0,1) \times [0,1]\bigr) = D_1$.

$\Delta_1$ is bounded by the image of $\overline{\sigma}$ and by a closed subarc (possibly reduced to a point) of $\partial\Delta$; the open arc (possibly void) will be denoted by $C_\Sigma$ and called the *principal arc associated* to $\Sigma$. If $\overline{\Delta_1} \cap \partial\Delta$ contains only one point, i.e., if $C_\Sigma = \emptyset$ — it can happen only if $\gamma_\Sigma$ is parabolic —, that point will be denoted by $\tau_\Sigma$ and called the *principal point associated* to $\Sigma$. Any open arc $\gamma(C_\Sigma)$ and any point $\gamma(\tau_\Sigma)$, with $\gamma \in \Gamma$, will be said *associated* to $\Sigma$.

Now we have introduced the terminology we need, and we can begin proving theorems. First of all, we study the situation for a point component:

**Theorem 1.1.56:** *Let $\Sigma = \{a\}$ be a point component of the boundary of a multiply connected hyperbolic domain $D \subset \widehat{X}$ of regular type, and denote by $\pi\colon\Delta \to D$ the universal covering map of $D$. Then:*
(i) *$\gamma_\Sigma$ is parabolic;*
(ii) *$C_\Sigma$ is empty;*
(iii) *if $\tau \in \partial\Delta$ is associated to $\Sigma$, then $\pi(z)$ tends to $a$ as $z \to \tau$ non-tangentially.*

*Proof:* We retain the notations introduced so far. Assume, by contradiction, that $C_\Sigma$ is not empty, and let $\{z_\nu\} \subset \Delta$ be any sequence converging to a point of $C_\Sigma$. Then $z_\nu \in \Delta_1$ eventually, and so every limit point of $\{\pi(z_\nu)\}$ must belong to $\overline{D_1} \cap \partial D = \Sigma$; therefore $\pi(z_\nu) \to a$ as $\nu \to \infty$. In other words, we have shown that $\pi(z)$ tends to $a$ as $z$ tends to $C_\Sigma$; Corollary 1.1.55 then implies that $\pi$ is constant, impossible.

Thus $C_\Sigma$ is empty; hence $\gamma_\Sigma$ is parabolic, and we have proved (i) and (ii). To prove (iii) we can assume $\tau = \tau_\Sigma$; the statements are invariant under the action of $\Gamma$.

If we take any euclidean disk $\Delta'$ internally tangent to $\partial\Delta$ in $\tau_\Sigma$ and containing (or not containing) the image of $\tilde{\sigma}$, then $\Delta'$ still contains (or does not contain) the image of $\overline{\sigma}$, by Lemma 1.1.16. In particular, we can find $\Delta' \subset \Delta_1$. Now let $\{z_\nu\}$ be a sequence converging to $\tau_\Sigma$ non-tangentially. Then $z_\nu \in \Delta' \subset \Delta_1$ eventually, and again we find that $\pi(z_\nu)$ must converge to $a$, **q.e.d.**

*Mutatis mutandis,* an analogous argument works for Jordan components:

**Theorem 1.1.57:** *Let $\Sigma$ be a Jordan component of the boundary of a multiply connected hyperbolic domain $D \subset \widehat{X}$ of regular type, denote by $\pi\colon\Delta \to D$ the universal covering map of $D$, and realize the fundamental group of $D$ as a subgroup $\Gamma$ of $\mathrm{Aut}(\Delta)$. Then:*
(i) *$C_\Sigma$ is not empty;*
(ii) *if $C \subset \partial\Delta$ is an open arc associated to $\Sigma$, then $\pi(z)$ extends continuosly to $C$, the image of $C$ through this extension is exactly $\Sigma$ and $\Gamma$ is properly discontinuous at every point of $C$.*

*Proof:* We still retain the notations introduced so far. For every $\varepsilon > 0$, set

$$D_1^\varepsilon = H\big((0,1) \times (\varepsilon, 1-\varepsilon)\big).$$

Fix $\varepsilon > 0$; then the function $f = \widetilde{H} \circ H^{-1}\colon D_1^\varepsilon \to \Delta_1$ is holomorphic, being an inverse of $\pi$. If $C_\Sigma$ were empty, and thus $\gamma_\Sigma$ parabolic, the same argument used in the proof of

Theorem 1.1.56 would show that $f(z)$ would tend to the unique fixed point of $\gamma_\Sigma$ as $z$ goes to $\Sigma \cap \overline{D_1^\varepsilon}$, and this is impossible, again by Corollary 1.1.55.

To prove (ii), we can assume $C = C_\Sigma$. Fix a point $\tau \in C_\Sigma$; then we can choose the homotopy $\widetilde{H}$ and $\varepsilon > 0$ in such a way that $\tau$ is in the boundary of $f(D_1^\varepsilon)$ — see Figure 1.2. Then $f$ is a biholomorphism between $D_1^\varepsilon$ and $f(D_1^\varepsilon)$, with inverse $\pi$, and both $D_1^\varepsilon$ and $f(D_1^\varepsilon)$ are simply connected domains bounded by Jordan curves (for $H$ is an isotopy). Therefore, by Theorem 1.1.28, $\pi$ extends continuously to a neighbourhood (in $\overline{\Delta}$) of $\tau$. Furthermore, $\pi$ is then locally injective at $\tau$; hence $\Gamma$ must be properly discontinuous at $\tau$, and we are done, **q.e.d.**

It would be nice if the automorphism associated to a Jordan component were hyperbolic. This is true with only one exception:

**Proposition 1.1.58:** *Let $D \subset \widehat{X}$ be a multiply connected hyperbolic domain of regular type, not biholomorphic to $\Delta^*$. Then a connected component $\Sigma$ of $\partial D$ is Jordan iff $\gamma_\Sigma$ is hyperbolic.*

*Proof:* If $\gamma_\Sigma$ is hyperbolic, $C_\Sigma$ is not empty and thus $\Sigma$ must be Jordan, by Theorem 1.1.56. Conversely, let $\Sigma$ be a Jordan component, and assume, by contradiction, that $\gamma_\Sigma$ is parabolic. Then since $C_\Sigma$ must be not empty, $C_\Sigma = \partial\Delta \setminus \{\tau_\Sigma\}$, where $\tau_\Sigma$ is the unique fixed point of $\gamma_\Sigma$.

Let $\gamma \in \Gamma$ be different from the identity. Clearly, $\Gamma$ cannot be properly discontinuous at a fixed point of $\gamma$; hence, by Theorem 1.1.57.(ii), $\gamma$ must be parabolic with fixed point $\tau_\Sigma$. But then, by Proposition 1.1.13, $\Gamma$ is abelian and composed only by parabolic elements; therefore $D$ is biholomorphic to $\Delta^*$, by Theorem 1.1.29, **q.e.d.**

We end this section with an observation regarding the boundary behavior of the Poincaré distance we shall need in section 1.3.3:

**Proposition 1.1.59:** *Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type. Take $\tau_0 \in \partial D$, and a sequence $\{z_\nu\} \subset D$ converging to $\tau_0$. Let $\{w_\nu\} \subset D$ be another sequence, and assume there is $M > 0$ so that*

$$\forall \nu \in \mathbf{N} \qquad \omega_D(z_\nu, w_\nu) \le M. \tag{1.1.31}$$

*Then $w_\nu \to \tau_0$ as $\nu \to +\infty$.*

*Proof:* Assume first $D = \Delta$. Then (1.1.31) is equivalent to the existence of $\delta < 1$ such that

$$\forall \nu \in \mathbf{N} \qquad \left| \frac{z_\nu - w_\nu}{1 - \overline{w}_\nu z_\nu} \right| \le \delta < 1. \tag{1.1.32}$$

Let $\{w_{\nu_j}\}$ be a subsequence converging to $\sigma_0 \in \overline{\Delta}$. If $\sigma_0 \ne \tau_0$, (1.1.32) would yield

$$1 = \left| \frac{\tau_0 - \sigma_0}{1 - \bar{\sigma}_0 \tau_0} \right| = \lim_{j \to \infty} \left| \frac{z_{\nu_j} - w_{\nu_j}}{1 - \overline{w}_{\nu_j} z_{\nu_j}} \right| \le \delta,$$

contradiction; hence $\sigma_0 = \tau_0$, and $w_\nu \to \tau_0$.

In the disk, (1.1.31) says something more. Assume that $\{z_\nu\}$ tends non-tangentially to $\tau_0$; in other words, assume there is $M_1 > 0$ such that

$$\forall \nu \in \mathbf{N} \qquad \frac{|\tau_0 - z_\nu|}{1 - |z_\nu|} < M_1. \tag{1.1.33}$$

Now, a computation shows that

$$\forall z,\, w \in \Delta \qquad \frac{|\tau_0 - w|}{1 - |w|} \le \frac{|\tau_0 - z|}{1 - |z|} \cdot e^{2\omega(z,w)};$$

hence (1.1.33) yields

$$\forall \nu \in \mathbf{N} \qquad \frac{|\tau_0 - w_\nu|}{1 - |w_\nu|} \le M_1 e^{2M},$$

that is $w_\nu \to \tau_0$ non-tangentially.

Now let $D$ be any hyperbolic domain of regular type, and denote by $\pi\colon \Delta \to D$ its universal covering map. We shall first consider the case of $\tau_0$ belonging to a Jordan component of $\partial D$. By Theorem 1.1.57, we can find $\tilde\tau_0 \in \partial\Delta$ and $\{\tilde z_\nu\} \subset \Delta$ so that $\pi(\tilde z_\nu) = z_\nu$, $\pi(\tilde\tau_0) = \tau_0$ and $\tilde z_\nu \to \tilde\tau_0$ as $\nu \to +\infty$. The definition of the Poincaré distance on $D$ — namely, (1.1.27) — implies that we can find $\{\widetilde w_\nu\} \subset \Delta$ so that $\pi(\widetilde w_\nu) = w_\nu$ and $\omega(\tilde z_\nu, \widetilde w_\nu) \le M + 1$ for all $\nu \in \mathbf{N}$. But then, by the first part of the proof, $\widetilde w_\nu \to \tilde\tau_0$, and so $w_\nu \to \tau_0$.

Finally, assume $\{\tau_0\}$ is a point component of $\partial D$. Since $\overline{D}$ is compact, it suffices to show that if a subsequence $\{w_{\nu_j}\}$ of $\{w_\nu\}$ converges to $\sigma_0 \in \overline{D}$, then $\sigma_0 = \tau_0$. Up to a subsequence, we can assume (cf. the proof of Theorem 1.1.56) that $\{z_{\nu_j}\}$ converges to $\tau_0$ in such a way that we can find a point $\tilde\tau_0 \in \partial\Delta$ associated to $\{\tau_0\}$ and a sequence $\{\tilde z_j\} \subset \Delta$ converging non-tangentially to $\tilde\tau_0$ so that $\pi(\tilde z_j) = z_{\nu_j}$ for all $j \in \mathbf{N}$. As before, choose $\{\widetilde w_j\} \subset \Delta$ so that $\pi(\widetilde w_j) = w_{\nu_j}$ and $\omega_\Delta(\tilde z_j, \widetilde w_j) \le M + 1$ for all $j \in \mathbf{N}$. But then we saw that this forces $\widetilde w_j \to \tilde\tau_0$ non-tangentially and hence, by Theorem 1.1.56, $w_{\nu_j} \to \tau_0$, **q.e.d.**

## Notes

In 1869, Schwarz [1869] proved the following result: let $f\colon \Delta \to \Delta$ be a holomorphic injective function continuous up to the boundary such that $f(0) = 0$ and $\rho_1 \le |f(\tau)| \le \rho_2$ for all $\tau \in \partial\Delta$; then $\rho_1|z| \le |f(z)| \le \rho_2|z|$ for all $z \in \overline{\Delta}$. Only forty-three years later Carathéodory [1912] recognized the real significance of this result, and gave it the statement, the proof (inspired by E. Schmidt; see also Poincaré [1884] and Carathéodory [1905]) and the name we know today. It is interesting to notice that both Poincaré and Carathéodory used Schwarz's lemma essentially to show that $\mathrm{Aut}(\Delta)$ acts simply transitively on line elements.

The fully invariant version Theorem 1.1.6 calling in the Poincaré metric is due to Pick [1915a, b], though Carathéodory [1912] already knew Corollary 1.1.4. The Poincaré metric itself was first investigated by Riemann [1854], as an example (in modern terminology) of metric with constant Gaussian curvature. The first one to use the Poincaré metric to study non-euclidean geometry and to devise the disk model of the Lobačevski hyperbolic plane was Beltrami [1868a, b]. Only on 1882 Poincaré [1882] began to deal with the metric that now bears his name, both in $\Delta$ and in $H^+$, using it for his work on Fuchsian groups. More on the geometry of the Poincaré metric can be found, e.g., in Bianchi [1927].

The whole section 1.1.2 as well as the approach we used in our study of Riemann surfaces is essentially taken from the theory of Fuchsian and Kleinian groups, as developed by Poincaré [1882, 1883, 1884]. A modern introduction to this beatiful theory is Kra [1972].

The Uniformization Theorem 1.1.17 was first stated by Riemann [1851], but his proof had some gaps. The complete proof evolved in more than half a century, together with the modern concept of Riemann surface and $n$-dimensional manifold, through the works of many people. The most important are Poincaré [1883] (who introduced new powerful methods and gave a partial proof of the existence of the universal covering map), Osgood [1900] (who proved the theorem for plane domains), Hilbert [1904] (who rigorously proved Dirichlet's principle, a main tool), Poincaré [1907a] and Kœbe [1907] (who proved the complete statement) and Weyl [1913] (who put the result in the today perspective). A direct proof or Riemann's mapping theorem Corollary 1.1.27 can be found, e.g., in Rudin [1966].

The Little Picard Theorem was first proved by Picard [1879a].

The Osgood-Taylor-Carathéodory Theorem 1.1.28 was conjectured by Osgood in 1901, and proved almost simultaneously by Osgood and Taylor [1913] and Carathéodory [1913a]. Carathéodory himself, in his papers [1913b, c], with his theory of prime ends made definitive investigations about the boundary behavior of the universal covering map of arbitrary simply connected domains; a modern account can be found in Pommerenke [1975]. The $C^1$ part of Theorem 1.1.28 is due to Kellogg [1912].

Theorem 1.1.33 is the easiest version of a general result of Rouché [1862], where the disk is replaced by any bounded domain of $\mathbf{C}$. Our proof follows Saks and Zygmund [1971], and it is inspired by Cohn [1922]. Corollaries 1.1.35, 1.1.36 and 1.1.37 are in Hurwitz [1889].

A different generalization of Schwarz's lemma is due to Lindelöf [1909]: if $G^X$ is the Green function of a hyperbolic Riemann surface $X$, then $G^Y_{f(z)} \circ f \geq G^X_z$ for every holomorphic function $f$ between two hyperbolic Riemann surfaces $X$ and $Y$ and every $z \in X$.

Montel's Theorem 1.1.43 was first proved by Montel [1912], at least for hyperbolic domains. Montel's proof used the modular function, i.e., the universal covering map of $\mathbf{C} \setminus \{0,1\}$. Later on other proofs were devised, relying only on more elementary facts; see for instance Gerretsen and Sansone [1969] or Burckel [1979]. These proofs use Schottky's theorem (due to Schottky [1904, 1906]), that states that for any holomorphic function $f\colon \Delta \to \mathbf{C} \setminus \{0,1\}$ we have

$$\log |f(z)| < (\pi + \log^+ |f(0)|) \frac{1+|z|}{1-|z|},$$

where $\log^+$ is the positive part of the logarithm (this form of Schottky's theorem is due to Hayman [1947]; cf. also Ahlfors [1938]). A proof in our spirit of Schottky's theorem is described in Kobayashi [1970]: it relies on an explicit metric of negative Gaussian curvature on $\mathbf{C} \setminus \{0,1\}$ (constructed by Grauert and Rieckziegel [1965]), and on Ahlfors' differential geometric version of Schwarz's lemma (see Ahlfors [1938, 1973]). Our approach to Montel's theorem is motivated by the several variables theory we shall develop in the second part of this book; using the terminology we shall discuss there, Montel's theorem says that every hyperbolic Riemann surface is taut. Our proof is adapted from a more general argument due to Wu [1967].

Usually, the name Montel's theorem is ascribed to another result, previously proved by Montel himself (Montel [1907]; but cf. also Stieltjes [1894] and Kœbe [1908]): every sequence of holomorphic functions from a Riemann surface $X$ into a *bounded* domain $D \subset\subset \mathbf{C}$ has a subsequence converging in $\mathrm{Hol}(X, \mathbf{C})$; in other words, $\mathrm{Hol}(X, D)$ is relatively compact in $\mathrm{Hol}(X, \mathbf{C})$. This is now an immediate consequence of our Montel's theorem: it suffices to imbed $D$ into a bigger bounded domain $D_1 \supset\supset D$, and then to invoke the normality of $\mathrm{Hol}(X, D_1)$. As we shall see later on, this says that a bounded domain is tautly imbedded in $\mathbf{C}$.

Theorem 1.1.45 is in Vitali [1903, 1904] (see also Stieltjes [1894], Porter [1904] and Carathéodory and Landau [1911]).

Corollary 1.1.47 is the first step toward the proof of a general fact stating that the automorphism group of a complex manifold equipped with a distance contracted by holomorphic maps is a finite dimensional Lie group (see H. Cartan [1935], Wu [1967] and Kobayashi [1970]).

The Big Picard Theorem is originally appeared in Picard [1879b], and subsequently strengthened by Julia [1924]. Our proof is modelled on Huber [1953] and Kobayashi [1970]. For generalizations see Kobayashi [1976], Lang [1987] and references therein.

Section 1.1.5 is adapted from Julia [1934], and form a short account of the works of Schottky [1877, 1897], Picard [1913], Kœbe [1916, 1918], de la Vallée-Poussin [1930, 1931] and Julia [1932] himself on the uniformization theory of multiply connected domains. For further information on this matter, consult Goluzin [1969].

The two constants Theorem 1.1.54 is due to Ostrowsky [1922]; our proof is taken from Burckel [1979]. Fatou's uniqueness theorem states that if $f\colon \Delta \to \mathbf{C}$ is a bounded holomorphic function such that $\lim_{r\to 1} f(re^{i\theta}) = 0$ for all $e^{i\theta}$ belonging to a subset of $\partial\Delta$ of positive measure, then $f \equiv 0$ (Fatou [1906]). A modern proof can be found, e.g., in Rudin [1966].