Angular derivatives in several complex variables

Notes for a CIME course

Marco Abate

Dipartimento di Matematica, Università di Pisa
Via Buonarroti 2, 56127 pisa
E-mail: abate@dm.unipi.it

September 2002

0. Introduction

A well-known classical result in the theory of one complex variable, due to Fatou [Fa], says that a bounded holomorphic function f defined in the unit disk Δ admits non-tangential limit at almost every point $\sigma \in \partial\Delta$. As satisfying as it is from several points of view, this theorem leaves open the question of whether the function f admits non-tangential limit at a *specific* point $\sigma_0 \in \partial\Delta$.

Of course, one needs to make some assumptions on the behavior of f near the point σ_0; the aim is to find the weakest possible assumptions. In 1920, Julia [Ju1] identified the right hypothesis: assuming, without loss of generality, that the image of the bounded holomorphic function is contained in the unit disk then Julia's assumption is

$$\liminf_{\zeta\to\sigma_0} \frac{1-|f(\zeta)|}{1-|\zeta|} < +\infty. \tag{0.1}$$

In other words, $f(\zeta)$ must go to the boundary as fast as ζ (as we shall show, it cannot go to the boundary any faster, but it might go slower). Then Julia proved the following

Theorem 0.1: (Julia) *Let $f \in \mathrm{Hol}(\Delta,\Delta)$ be a bounded holomorphic function, and take $\sigma \in \partial\Delta$ such that*

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \beta < +\infty$$

for some $\beta \in \mathbb{R}$. Then $\beta > 0$ and f has non-tangential limit $\tau \in \partial\Delta$ at σ.

As we shall see, the proof is just a (clever) application of Schwarz-Pick lemma. The real breakthrough in this theory is due to Wolff [Wo] in 1926 and Carathéodory [C1] in 1929: if f satisfies (0.1) at σ then the derivative f' too admits finite non-tangential limit at σ — and this limit can be computed explicitely. More precisely:

Theorem 0.2: (Wolff-Carathéodory) *Let $f \in \mathrm{Hol}(\Delta,\Delta)$ be a bounded holomorphic function, and take $\sigma \in \partial\Delta$ such that*

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \beta < +\infty$$

for some $\beta > 0$. Then both the incremental ratio

$$\frac{f(\zeta)-\tau}{\zeta-\sigma}$$

and the derivative f' have non-tangential limit $\beta\tau\overline{\sigma}$ at σ, where $\tau \in \partial\Delta$ is the non-tangential limit of f at σ.

Theorems 0.1 and 0.2 are collectively known as the *Julia-Wolff-Carathéodory theorem*. The aim of this survey is to present a possible way to generalize this theorem to bounded holomorphic functions of several complex variables.

The main point to be kept in mind here is that, as first noticed by Korányi and Stein (see, e.g., [St]) and later theorized by Krantz [Kr1], the right kind of limit to consider in studying the boundary behavior of holomorphic functions of several complex variables depends on the geometry of the domain, and it is usually stronger than the non-tangential limit. To better stress this interdependence between analysis and geometry we decided to organize this survey as a sort of template that the reader may apply to the specific cases s/he is interested in.

More precisely, we shall single out a number of geometrical hypotheses (usually expressed in terms of the Kobayashi intrinsic distance of the domain) that when satisfied will imply a Julia-Wolff-Carathéodory theorem. This approach has the advantage to reveal the main ideas in the proofs, unhindered by the technical details needed to verify the hypotheses. In other words, the hard computations are swept under the carpet (i.e., buried in the references), leaving the interesting patterns *over* the carpet free to be examined.

Of course, the hypotheses can be satisfied: for instance, all of them hold for strongly pseudoconvex domains, convex domains with C^ω boundary, convex circular domains of finite type, and in the polydisk; but most of them hold in more general domains too. And one fringe benefit of the approach chosen for this survey is that as soon as somebody proves that the hypotheses hold for a specific domain, s/he gets a Julia-Wolff-Carathéodory theorem in that domain for free. Indeed, this approach has already uncovered new results: to the best of my knowledge, Theorem 4.2 in full generality and Proposition 4.8 have not been proved before.

So in Section 1 of this survey we shall present a proof of the Julia-Wolff-Carathéodory theorem suitable to be generalized to several complex variables. It will consist of three steps:

(a) A proof of Theorem 0.1 starting from the Schwarz-Pick lemma.
(b) A discussion of the Lindelöf principle, which says that if a (K-)bounded holomorphic function has limit restricted to a curve ending at a boundary point then it has the same limit restricted to any non-tangential curve ending at that boundary point.
(c) A proof of the Julia-Wolff-Carathéodory theorem obtained by showing that the incremental ratio and the derivative satisfy the hypotheses of the Lindelöf principle.

Then the next three sections will describe a way of performing the same three steps in a several variables context, providing the template mentioned above.

Finally, a few words on the literature. As mentioned before, Theorem 0.1 first appeared in [Ju1], and Theorem 0.2 in [Wo]. The proof we shall present here is essentially due to Rudin [Ru, Section 8.5]; other proofs and one-variable generalizations can be found in [A3], [Ah], [C1, 2], [J], [Kom], [LV], [Me], [N], [Po], [T] and references therein.

As far as I know, the first several variables generalizations of Theorem 0.1 were proved by Minialoff [Mi] for the unit ball $B^2 \subset \mathbb{C}^2$, and then by Hervé [He] in B^n. The general form we shall discuss originates in [A2]. For some other (finite and infinite dimensional) approaches see [Ba], [M], [W], [R], [Wł1] and references therein.

The one-variable Lindelöf principle has been proved by Lindelöf [Li1, 2]; see also [A3, Theorem 1.3.23], [Ru, Theorem 8.4.1], [Bu, 5.16, 5.56, 12.30, 12.31] and references therein. The first important several variables version of it is due to Čirka [Č]; his approach has been further pursued in [D1, 2], [DZ] and [K]. A different generalization is due to Cima and Krantz [CK] (see also [H1, 2]), and both inspired the presentation we shall give in Section 3 (whose ideas stem from [A2]).

A first tentative extension of the Julia-Wolff-Carathéodory theorem to bounded domains in $\mathbb{C}^2$ is due to Wachs [W]. Hervé [He] proved a preliminary Julia-Wolff-Carathéodory theorem for the unit ball of $\mathbb{C}^n$ using non-tangential limits and considering only incremental ratioes; the full statement for the unit ball is due to Rudin [Ru, Section 8.5]. The Julia-Wolff-Carathéodory theorem for strongly convex domains is in [A2]; for strongly pseudoconvex domains in [A4]; for the polydisk in [A5] (see also Jafari [Ja], even though his statement is not completely correct); for convex domains of finite type in [AT2]. Furthermore, Julia-Wolff-Carathéodory theorems in infinite-dimensional Banach and Hilbert spaces are discussed in [EHRS], [F], [MM], [SW], [Wł2, 3, 4], [Z] and references therein.

Finally, I would also like to mention the shorter survey [AT1], written, as well as the much more substantial paper [AT2], with the unvaluable help of Roberto Tauraso.

1. One complex variable

We already mentioned that Theorem 0.1 is a consequence of the classical Schwarz-Pick lemma. For the sake of completeness, let us recall here the relevant definitions and statements.

Definition 1.1: The *Poincaré metric* on Δ is the complete Hermitian metric κ_Δ^2 of constant Gaussian curvature -4 given by
$$\kappa_\Delta^2(\zeta)=\frac{1}{(1-|\zeta|^2)^2}\,dz\,d\bar z.$$
The *Poincaré distance* ω on Δ is the integrated distance associated to κ_Δ.

It is easy to prove that
$$\omega(\zeta_1,\zeta_2)=\tfrac12\log\frac{1+\left|\frac{\zeta_1-\zeta_2}{1-\overline{\zeta_2}\zeta_1}\right|}{1-\left|\frac{\zeta_1-\zeta_2}{1-\overline{\zeta_2}\zeta_1}\right|}.$$
For us the main property of the Poincaré distance is the classical Schwarz-Pick lemma:

Theorem 1.1: (Schwarz-Pick) *The Poincaré metric and distance are contracted by holomorphic self-maps of the unit disk. In other words, if $f\in\mathrm{Hol}(\Delta,\Delta)$ then*
$$\forall\zeta\in\Delta\qquad f^*(\kappa_\Delta^2)(\zeta)\le\kappa_\Delta^2(\zeta)\tag{1.1}$$
and
$$\forall\zeta_1,\zeta_2\in\Delta\qquad \omega\big(f(\zeta_1),f(\zeta_2)\big)\le\omega(\zeta_1,\zeta_2).\tag{1.2}$$
Furthermore, equality in (1.1) *for some $\zeta\in\Delta$ or in* (1.2) *for some $\zeta_1\ne\zeta_2$ occurs iff f is a holomorphic automorphism of Δ.*

A first easy application of this result is the fact that the $\liminf$ in (0.1) is always positive (or $+\infty$). But let us first give it a name.

Definition 1.2: Let $f\in\mathrm{Hol}(\Delta,\Delta)$ be a holomorphic self-map of Δ, and $\sigma\in\partial\Delta$. Then the *boundary dilation coefficient* $\beta_f(\sigma)$ of f at σ is given by
$$\beta_f(\sigma)=\liminf_{\zeta\to\sigma}\frac{1-|f(\zeta)|}{1-|\zeta|}.$$
If it is finite and equal to $\beta>0$ we shall say that f is *β-Julia* at σ.

Then

Corollary 1.2: *For any $f\in\mathrm{Hol}(\Delta,\Delta)$ we have*
$$\frac{1-|f(\zeta)|}{1-|\zeta|}\ge\frac{1-|f(0)|}{1+|f(0)|}>0\tag{1.3}$$
for all $\zeta\in\Delta$; in particular,
$$\beta_f(\sigma)\ge\frac{1-|f(0)|}{1+|f(0)|}>0$$
for all $\sigma\in\partial\Delta$.

Proof: The Schwarz-Pick lemma yields
$$\omega\big(0,f(\zeta)\big)\le\omega\big(0,f(0)\big)+\omega\big(f(0),f(\zeta)\big)\le\omega\big(0,f(0)\big)+\omega(0,\zeta),$$
that is
$$\frac{1+|f(\zeta)|}{1-|f(\zeta)|}\le\frac{1+|f(0)|}{1-|f(0)|}\cdot\frac{1+|\zeta|}{1-|\zeta|}\tag{1.4}$$
for all $\zeta\in\Delta$. Let $a=(|f(0)|+|\zeta|)/(1+|f(0)||\zeta|)$; then the right-hand side of (1.4) is equal to $(1+a)/(1-a)$. Hence $|f(\zeta)|\le a$, that is
$$1-|f(\zeta)|\ge(1-|\zeta|)\frac{1-|f(0)|}{1+|f(0)||\zeta|}\ge(1-|\zeta|)\frac{1-|f(0)|}{1+|f(0)|}$$
for all $\zeta\in\Delta$, as claimed. □

The main step in the proof of Theorem 0.1 is known as *Julia's lemma*, and it is again a consequence of the Schwarz-Pick lemma:

Theorem 1.3: (Julia) *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and $\sigma \in \partial\Delta$ be such that*

$$\liminf_{\zeta \to \sigma} \frac{1 - |f(\zeta)|}{1 - |\zeta|} = \beta < +\infty.$$

Then there exists a unique $\tau \in \partial\Delta$ such that

$$\frac{|\tau - f(\zeta)|^2}{1 - |f(\zeta)|^2} \le \beta \, \frac{|\sigma - \zeta|^2}{1 - |\zeta|^2}. \tag{1.5}$$

Proof: The Schwarz-Pick lemma yields

$$\left| \frac{f(\zeta) - f(\eta)}{1 - \overline{f(\eta)} f(\zeta)} \right| \le \left| \frac{\zeta - \eta}{1 - \overline{\eta}\zeta} \right|$$

and thus

$$\frac{|1 - \overline{f(\eta)} f(\zeta)|^2}{1 - |f(\zeta)|^2} \le \frac{1 - |f(\eta)|^2}{1 - |\eta|^2} \cdot \frac{|1 - \overline{\eta}\zeta|^2}{1 - |\zeta|^2} \tag{1.6}$$

for all η, $\zeta \in \Delta$. Now choose a sequence $\{\eta_k\} \subset \Delta$ converging to σ and such that

$$\lim_{k \to +\infty} \frac{1 - |f(\eta_k)|}{1 - |\eta_k|} = \beta;$$

in particular, $|f(\eta_k)| \to 1$, and so up to a subsequence we can assume that $f(\eta_k) \to \tau \in \partial\Delta$ as $k \to +\infty$. Then setting $\eta = \eta_k$ in (1.6) and taking the limit as $k \to +\infty$ we obtain (1.5).

We are left to prove the uniqueness of τ. To do so, we need a geometrical interpretation of (1.5).

Definition 1.3: The *horocycle* $E(\sigma, R)$ of *center* σ and *radius* R is the set

$$E(\sigma, R) = \left\{ \zeta \in \Delta \;\middle|\; \frac{|\sigma - \zeta|^2}{1 - |\zeta|^2} < R \right\}.$$

Geometrically, $E(\sigma, R)$ is an euclidean disk of euclidean radius $R/(1 + R)$ internally tangent to $\partial\Delta$ in σ; in particular,

$$|\sigma - \zeta| \le \frac{2R}{1 + R} < 2R \tag{1.7}$$

for all $\zeta \in \overline{E(\sigma, R)}$. A horocycle can also be seen as the limit of Poincaré disks with fixed euclidean radius and centers converging to σ (see, e.g., [Ju2] or [A3, Proposition 1.2.1]).

The formula (1.5) then says that

$$f\big(E(\sigma, R)\big) \subseteq E(\tau, \beta R)$$

for any $R > 0$. Assume, by contradiction, that (1.5) also holds for some $\tau_1 \ne \tau$, and choose $R > 0$ so small that $E(\tau, \beta R) \cap E(\tau_1, \beta R) = \varnothing$. Then we get

$$\varnothing \ne f\big(E(\sigma, R)\big) \subseteq E(\tau, \beta R) \cap E(\tau_1, \beta R) = \varnothing,$$

contradiction. Therefore (1.5) can hold for at most one $\tau \in \partial\Delta$, and we are done. □

In Section 4 we shall need a sort of converse of Julia's lemma:

Lemma 1.4: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$, σ, $\tau \in \partial\Delta$ and $\beta > 0$ be such that*

$$f\big(E(\sigma, R)\big) \subseteq E(\tau, \beta R)$$

for all $R > 0$. Then $\beta_f(\sigma) \leq \beta$.

Proof: For $t \in [0, 1)$ set $R_t = (1 - t)/(1 + t)$, so that $t\sigma \in \partial E(\sigma, R_t)$. Therefore $f(t\sigma) \in \overline{E(\tau, \beta R_t)}$; hence

$$\frac{1 - |f(t\sigma)|}{1 - t} \leq \frac{|\tau - f(t\sigma)|}{1 - t} < 2\beta \frac{R_t}{1 - t} = \frac{2}{1 + t}\beta,$$

by (1.7), and thus

$$\beta_f(\sigma) = \liminf_{\zeta \to \sigma} \frac{1 - |f(\zeta)|}{1 - |\zeta|} \leq \liminf_{t \to 1^-} \frac{1 - |f(t\sigma)|}{1 - t} \leq \beta.$$

□

To complete the proof of Theorem 0.1 we still need to give a precise definition of what we mean by non-tangential limit.

Definition 1.4: Take $\sigma \in \partial\Delta$ and $M \geq 1$; the *Stolz region* $K(\sigma, M)$ of *vertex* σ and *amplitude* M is given by

$$K(\sigma, M) = \left\{ \zeta \in \Delta \,\middle|\, \frac{|\sigma - \zeta|}{1 - |\zeta|} < M \right\}.$$

Geometrically, $K(\sigma, M)$ is an egg-shaped region, ending in an angle touching the boundary of Δ at σ. The amplitude of this angle tends to 0 as $M \to 1^+$, and tends to π as $M \to +\infty$. Therefore we can use Stolz regions to define the notion of non-tangential limit:

Definition 1.5: A function $f\colon \Delta \to \mathbb{C}$ admits *non-tangential limit* $L \in \mathbb{C}$ at the point $\sigma \in \partial\Delta$ if $f(\zeta) \to L$ as ζ tends to σ inside $K(\sigma, M)$ for any $M > 1$.

From the definitions it is apparent that horocycles and Stolz regions are strongly related. For instance, if ζ belongs to $K(\sigma, M)$ we have

$$\frac{|\sigma - \zeta|^2}{1 - |\zeta|^2} = \frac{|\sigma - \zeta|}{1 - |\zeta|} \cdot \frac{|\sigma - \zeta|}{1 + |\zeta|} < M|\sigma - \zeta|,$$

and thus $\zeta \in E(\sigma, M|\sigma - \zeta|)$.

We are then ready for the

Proof of Theorem 0.1: Assume that f is β-Julia at σ, fix $M > 1$ and choose any sequence $\{\zeta_k\} \subset K(\sigma, M)$ converging to σ. In particular, $\zeta_k \in E(\sigma, M|\sigma - \zeta_k|)$ for all $k \in \mathbb{N}$. Then Theorem 1.3 gives a unique $\tau \in \partial\Delta$ such that $f(\zeta_k) \in E(\tau, \beta M|\sigma - \zeta_k|)$. Therefore every limit point of the sequence $\{f(\zeta_k)\}$ must be contained in the intersection

$$\bigcap_{k \in \mathbb{N}} \overline{E(\tau, \beta M|\sigma - \zeta_k|)} = \{\tau\},$$

that is $f(\zeta_k) \to \tau$, and we have proved that f has non-tangential limit τ at σ. □

To prove Theorem 0.2 we need another ingredient, known as *Lindelöf principle.* The idea is that the existence of the limit along a given curve in Δ ending at $\sigma \in \partial\Delta$ forces the existence of the non-tangential limit at σ. To be more precise:

Definition 1.6: Let $\sigma \in \partial\Delta$. A *$\sigma$-curve* in Δ is a continous curve $\gamma\colon [0, 1) \to \Delta$ such that $\gamma(t) \to \sigma$ as $t \to 1^-$. Furthermore, we shall say that a function $f\colon \Delta \to \mathbb{C}$ is *K-bounded* at σ if for every $M > 1$ there exists $C_M > 0$ such that $|f(\zeta)| \leq C_M$ for all $\zeta \in K(\sigma, M)$.

Then Lindelöf [Li2] proved the following

Theorem 1.5: *Let $f\colon\Delta\to\mathbb{C}$ be a holomorphic function, and $\sigma\in\partial\Delta$. Assume there is a σ-curve $\gamma\colon[0,1)\to\Delta$ such that $f\bigl(\gamma(t)\bigr)\to L\in\mathbb{C}$ as $t\to 1^-$. Assume moreover that*

(a) *f is bounded, or that*

(b) *f is K-bounded and γ is non-tangential, that is its image is contained in a K-region $K(\sigma,M_0)$.*

Then f has non-tangential limit L at σ.

Proof: A proof of case (a) can be found in [A3, Theorem 1.3.23] or in [Ru, Theorem 8.4.1]. Since each $K(\sigma,M)$ is biholomorphic to Δ and the biholomorphism extends continuously up to the boundary, case (b) is a consequence of (a). Furthermore, it should be remarked that in case (b) the existence of the limit along γ automatically implies that f is K-bounded ([Li1]; see [Bu, 5.16] and references therein).

However, we shall describe here an easy proof of case (b) when γ is radial, that is $\gamma(t)=t\sigma$, which is the case we shall mostly use.

First of all, without loss of generality we can assume that $\sigma=1$, and then the Cayley transform allows us to transfer the stage to $H^+=\{w\in\mathbb{C}\mid \operatorname{Im} w>0\}$. The boundary point we are interested in becomes ∞, and the curve γ is now given by $\gamma(t)=i(1+t)/(1-t)$.

Furthermore if we denote by $K(\infty,M)\subset H^+$ the image under the Cayley transform of $K(1,M)\subset\Delta$, and by K_ε the truncated cone

$$K_\varepsilon=\bigl\{w\in H^+\mid \operatorname{Im} w>\varepsilon\max\{1,|\operatorname{Re} w|\}\bigr\},$$

we have

$$K(\infty,M)\subset K_{1/(2M)}\qquad\text{and}\qquad K_{1/(2M)}\cap\{w\in H^+\mid \operatorname{Im} w>R\}\subset K(\infty,M'),$$

for every R, $M>1$, where

$$M'=\sqrt{1+4M^2\frac{R+1}{R-1}}.$$

The first inclusion is easy; the second one follows from the formula

$$\left|\frac{1-\zeta}{1-|\zeta|}\right|^2=1+\frac{2}{|\zeta|+\operatorname{Re}\zeta}\left|\frac{\operatorname{Im}\zeta}{1-|\zeta|}\right|^2, \tag{1.8}$$

true for all $\zeta\in\Delta$ with $\operatorname{Re}\zeta>0$.

Therefore we are reduced to prove that if $f\colon H^+\to\mathbb{C}$ is holomorphic and bounded on any K_ε, and $f\circ\gamma(t)\to L\in\mathbb{C}$ as $t\to1^-$, then $f(w)$ has limit L as w tends to ∞ inside K_ε.

Choose $\varepsilon'<\varepsilon$ (so that $K_{\varepsilon'}\supset K_\varepsilon$), and define $f_n\colon K_{\varepsilon'}\to\mathbb{C}$ by $f_n(w)=f(nw)$. Then $\{f_n\}$ is a sequence of uniformly bounded holomorphic functions. Furthermore, $f_n(ir)\to L$ as $n\to+\infty$ for any $r>1$; by Vitali's theorem, the whole sequence $\{f_n\}$ is then converging uniformly on compact subsets to a holomorphic function $f_\infty\colon K_{\varepsilon'}\to\mathbb{C}$. But we have $f_\infty(ir)=L$ for all $r>1$; therefore $f_\infty\equiv L$. In particular, for every $\delta>0$ we can find $N\ge1$ such that $n\ge N$ implies

$$|f_n(w)-L|<\delta\qquad\text{for all } w\in\overline{K_\varepsilon}\text{ such that } 1\le|w|\le2.$$

This implies that for every $\delta>0$ there is $R>1$ such that $w\in\overline{K_\varepsilon}$ and $|w|>R$ implies $|f(w)-L|<\delta$, that is the assertion. Indeed, it suffices to take $R=N$; if $|w|>N$ let $n\ge N$ be the integer part of $|w|$, and set $w'=w/n$. Then $w'\in\overline{K_\varepsilon}$ and $1\le|w'|\le2$, and thus

$$|f(w)-L|=|f_n(w')-L|<\delta,$$

as claimed. □

Example 1.1: It is very easy to provide examples of K-bounded functions which are not bounded: for instance $f(\zeta)=(1+\zeta)^{-1}$ is K-bounded at 1 but it is not bounded in Δ. More generally, every rational function with a pole at $\tau\in\partial\Delta$ and no poles inside Δ is not bounded on Δ but it is K-bounded at every $\sigma\in\partial\Delta$ different from τ.

We are now ready to begin the proof of Theorem 0.2. Let then $f\in\operatorname{Hol}(\Delta,\Delta)$ be β-Julia at $\sigma\in\partial\Delta$, and let $\tau\in\partial\Delta$ be the non-tangential limit of f at σ provided by Theorem 0.1. We would like to show that f' has non-tangential limit $\beta\tau\overline{\sigma}$ at σ; but first we study the behavior of the incremental ratio $\bigl(f(\zeta)-\tau\bigr)/(\zeta-\sigma)$.

Proposition 1.6: *Let $f\in\mathrm{Hol}(\Delta,\Delta)$ be β-Julia at $\sigma\in\partial\Delta$, and let $\tau\in\partial\Delta$ be the non-tangential limit of f at σ. Then the incremental ratio*
$$\frac{f(\zeta)-\tau}{\zeta-\sigma}$$
is K-bounded and has non-tangential limit $\beta\tau\overline{\sigma}$ at σ.

Proof: We shall show that the incremental ratio is K-bounded and that it has radial limit $\beta\tau\overline{\sigma}$ at σ; the assertion will then follow from Theorem 1.5.(b).

Take $\zeta\in K(\sigma,M)$. We have already remarked that we then have $\zeta\in E(\sigma,M|\zeta-\sigma|)$, and thus $f(\zeta)\in E(\tau,\beta M|\zeta-\sigma|)$, by Julia's Lemma. Recalling (1.7) we get
$$|f(\zeta)-\tau|<2\beta M|\zeta-\sigma|,$$
and so the incremental ratio is bounded by $2\beta M$ in $K(\sigma,M)$.

Now given $t\in[0,1)$ set $R_t=(1-t)/(1+t)$, so that $t\sigma\in\partial E(\sigma,R_t)$. Then $f(t\sigma)\in\overline{E(\tau,\beta R_t)}$, and thus
$$1-|f(t\sigma)|\le|\tau-f(t\sigma)|\le 2\beta R_t=2\beta\frac{1-t}{1+t}.$$
Therefore
$$\frac{1-|f(t\sigma)|}{1-t}\le\left|\frac{\tau-f(t\sigma)}{1-t}\right|\le\frac{2}{1+t}\beta=\frac{2}{1+t}\liminf_{\zeta\to\sigma}\frac{1-|f(\zeta)|}{1-|\zeta|};$$
letting $t\to1^-$ we see that
$$\lim_{t\to1^-}\frac{1-|f(t\sigma)|}{1-t}=\lim_{t\to1^-}\left|\frac{\tau-f(t\sigma)}{1-t}\right|=\beta,\tag{1.9}$$
and then
$$\lim_{t\to1^-}\frac{|\tau-f(t\sigma)|}{1-|f(t\sigma)|}=1.\tag{1.10}$$
Since $f(t\sigma)\to\tau$, we know that $\mathrm{Re}\big(\overline{\tau}f(t\sigma)\big)>0$ for t close enough to 1; then (1.8) and (1.10) imply
$$\lim_{t\to1^-}\frac{\tau-f(t\sigma)}{1-|f(t\sigma)|}=\tau,$$
and together with (1.9) we get
$$\lim_{t\to1^-}\frac{f(t\sigma)-\tau}{t\sigma-\sigma}=\beta\tau\overline{\sigma},$$
as desired. □

By the way, the non-tangential limit of the incremental ratio is usually called the *angular derivative* of f at σ, because it represents the limit of the derivative of f inside an angular region with vertex at σ.

We can now complete the

Proof of Theorem 0.2: Again, the idea is to prove that f' is K-bounded and then show that $f'(t\sigma)$ tends to $\beta\tau\overline{\sigma}$ as $t\to1^-$.

Take $\zeta\in K(\sigma,M)$, and choose $\delta_\zeta>0$ so that $\zeta+\delta_\zeta\Delta\subset\Delta$. Therefore we can write
$$\begin{aligned}f'(\zeta)&=\frac{1}{2\pi i}\int_{|\eta|=\delta_\zeta}\frac{f(\zeta+\eta)}{\eta^2}\,d\eta=\frac{1}{2\pi i}\int_{|\eta|=\delta_\zeta}\frac{f(\zeta+\eta)-\tau}{\zeta+\eta-\sigma}\cdot\frac{\zeta+\eta-\sigma}{\eta^2}\,d\eta\\&=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\zeta+\delta_\zeta e^{i\theta})-\tau}{\zeta+\delta_\zeta e^{i\theta}-\sigma}\left[1-\frac{\sigma-\zeta}{\delta_\zeta e^{i\theta}}\right]d\theta.\end{aligned}\tag{1.11}$$
Now, if $M_1>M$ and
$$\delta_\zeta=\frac{1}{M}\frac{M_1-M}{M_1+1}|\sigma-\zeta|,$$

then it is easy to check that $\zeta + \delta_\zeta\Delta \subset K(\sigma, M_1)$; therefore (1.11) and the bound on the incremental ratio yield
$$|f'(\zeta)| \le 2\beta M_1 \left[1 + M\frac{M_1+1}{M_1-M}\right],$$
and so f' is K-bounded.

If $\zeta = t\sigma$, we can take $\delta_{t\sigma} = (1-t)(M-1)/(M+1)$ for any $M > 1$, and (1.11) becomes
$$f'(t\sigma) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(t\sigma + \delta_{t\sigma}e^{i\theta}) - \tau}{t\sigma + \delta_{t\sigma}e^{i\theta} - \sigma}\left[1 - \sigma\frac{M-1}{M+1}e^{-i\theta}\right] d\theta.$$
Since $t\sigma + \delta_{t\sigma}\Delta \subset K(\sigma, M)$, Proposition 1.6 yields
$$\lim_{t\to 1^-} \frac{f(t\sigma + \delta_{t\sigma}e^{i\theta}) - \tau}{t\sigma + \delta_{t\sigma}e^{i\theta} - \sigma} = \beta\tau\overline{\sigma}$$
for any $\theta \in [0, 2\pi]$; therefore we get $f'(t\sigma) \to \beta\tau\bar\sigma$ as well, by the dominated convergence theorem, and we are done. □

It is easy to find examples of function $f \in \mathrm{Hol}(\Delta, \Delta)$ with $\beta_f(1) = +\infty$.

Example 1.2: Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be given by $f(z) = \lambda z^k/k$ where $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$ are such that $k > |\lambda|$. Then $\beta_f(1) = +\infty$ for the simple reason that $|f(1)| = |\lambda|/k < 1$; on the other hand, $f'(1) = \lambda$.

Therefore if $\beta_f(\sigma) = +\infty$ both f and f' might still have finite non-tangential limit at σ, but we have no control on them. However, if we assume that $f(\zeta)$ is actually going to the boundary of Δ as $\zeta \to \sigma$ then the link between the angular derivative and the boundary dilation coefficient is much tighter. Indeed, the final result of this section is

Theorem 1.7: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and $\sigma \in \partial\Delta$ be such that*
$$\limsup_{t\to 1^-} |f(t\sigma)| = 1. \tag{1.12}$$
Then
$$\beta_f(\sigma) = \limsup_{t\to 1^-} |f'(t\sigma)|. \tag{1.13}$$
In particular, f' has finite non-tangential limit at σ iff $\beta_f(\sigma) < +\infty$, and then f has non-tangential limit at σ too.

Proof: If the lim sup in (1.13) is infinite, then $f'(t\sigma)$ cannot converge as $t \to 1^-$, and thus $\beta_f(\sigma) = +\infty$ by Theorem 0.2.

So assume that the lim sup in (1.13) is finite; in particular, there is $M > 0$ such that $|f'(t\sigma)| \le M$ for all $t \in [0, 1)$. We claim that $\beta_f(\sigma)$ is finite too — and then the assertion will follow from Theorem 0.2 again.

For all t_1, $t_2 \in [0, 1)$ we have
$$|f(t_2\sigma) - f(t_1\sigma)| = \left|\int_{t_1}^{t_2} f'(t\sigma)\, dt\right| \le M|t_2 - t_1|. \tag{1.14}$$
Now, (1.12) implies that there is a sequence $\{t_k\} \subset [0, 1)$ converging to 1 and $\tau \in \partial\Delta$ such that $f(t_k) \to \tau$ as $k \to +\infty$. Therefore (1.14) yields
$$|\tau - f(t\sigma)| \le M(1-t)$$
for all $t \in [0, 1)$. Hence
$$\beta_f(\sigma) = \liminf_{\zeta\to\sigma} \frac{1 - |f(\zeta)|}{1 - |\zeta|} \le \liminf_{t\to 1^-} \frac{1 - |f(t\sigma)|}{1-t} \le \liminf_{t\to 1^-} \frac{|\tau - f(t\sigma)|}{1-t} \le M.$$
□

So Julia's condition $\beta_f(\sigma) < +\infty$ is in some sense optimal.

2. Julia's lemma

The aim of this section is to describe a generalization of Julia's lemma to several complex variables, and to apply it to get a several variables version of Theorem 0.1.

As we have seen, the one-variable Julia's lemma is a consequence of the Schwarz-Pick lemma or, more precisely, of the contracting properties of the Poincaré metric and distance. So it is only natural to look first for a generalization of the Poincaré metric.

Among several such generalizations, the most useful for us is the Kobayashi metric, introduced by Kobayashi [Kob1] in 1967.

Definition 2.1: Let X be a complex manifold: the *Kobayashi (pseudo)metric* of X is the function $\kappa_X\colon TX \to \mathbb{R}^+$ defined by

$$\kappa_X(z;v) = \inf\{|\xi| \mid \exists \varphi \in \mathrm{Hol}(\Delta, X) : \varphi(0) = z, d\varphi_0(\xi) = v\}$$

for all $z \in X$ and $v \in T_zX$. Roughly speaking, $\kappa_X(z;v)$ measures the (inverse of) the radius of the largest (not necessarily immersed) holomorphic disk in X passing through z tangent to v.

The Kobayashi pseudometric is an upper semicontinuous (and often continuous) complex Finsler pseudometric, that is it satisfies

$$\kappa_X(z;\lambda v) = |\lambda|\kappa_X(z;v) \tag{2.1}$$

for all $z \in X$, $v \in T_zX$ and $\lambda \in \mathbb{C}$. Therefore it can be used to compute the length of curves:

Definition 2.2: If $\gamma\colon [a,b] \to X$ is a piecewise C^1-curve in a complex manifold X then its *Kobayashi (pseudo)length* is

$$\ell_X(\gamma) = \int_a^b \kappa_X\bigl(\gamma(t);\dot\gamma(t)\bigr)\,dt.$$

The Kobayashi pseudolength of a curve does not depend on the parametrization, by (2.1); therefore we can define the *Kobayashi (pseudo)distance* $k_X\colon X \times X \to \mathbb{R}^+$ by setting

$$k_X(z,w) = \inf\{\ell_X(\gamma)\},$$

where the infimum is taken with respect to all the piecewise C^1-curves $\gamma\colon [a,b] \to X$ with $\gamma(a) = z$ and $\gamma(b) = w$. It is easy to check that k_X is a pseudodistance in the metric space sense. We remark that this is not Kobayashi original definition of k_X, but it is equivalent to it (as proved by Royden [Ro]).

The prefix "pseudo" used in the definitions is there to signal that the Kobayashi pseudometric (and distance) might vanish on nonzero vectors (respectively, on distinct points); for instance, it is easy to see that $\kappa_{\mathbb{C}^n} \equiv 0$ and $k_{\mathbb{C}^n} \equiv 0$.

Definition 2.3: A complex manifold X is *(Kobayashi) hyperbolic* if k_X is a true distance, that is $k_X(z,w) > 0$ as soon as $z \neq w$; it is *complete hyperbolic* if k_X is a complete distance. A related notion has been introduced by Wu [Wu]: a complex manifold is *taut* if $\mathrm{Hol}(\Delta, X)$ is a normal family (and this implies that $\mathrm{Hol}(Y, X)$ is a normal family for any complex manifold Y).

The main general properties of the Kobayashi metric and distance are collected in the following

Theorem 2.1: *Let X be a complex manifold. Then:*

(i) *If X is Kobayashi hyperbolic, then the metric space topology induced by k_X coincides with the manifold topology.*
(ii) *A complete hyperbolic manifold is taut, and a taut manifold is hyperbolic.*
(iii) *All the bounded domains of $\mathbb{C}^n$ are hyperbolic; all bounded convex or strongly pseudoconvex domains of $\mathbb{C}^n$ are complete hyperbolic.*
(iv) *A Riemann surface is Kobayashi hyperbolic iff it is hyperbolic, that is, iff it is covered by the unit disk (and then it is complete hyperbolic).*

(v) *The Kobayashi metric and distance of the unit ball $B^n \subset \mathbb{C}^n$ agree with the Bergmann metric and distance:*

$$\kappa_{B^n}(z;v) = \frac{1}{(1-\|z\|^2)^2}\left[|(z,v)|^2 + (1-\|z\|^2)\|v\|^2\right]$$

for all $z \in B^n$ and $v \in \mathbb{C}^n$, where $(\cdot\,,\cdot)$ denotes the canonical hermitian product in $\mathbb{C}^n$, and

$$k_{B^n}(z,w) = \tfrac{1}{2}\log\frac{1+\|\chi_z(w)\|}{1-\|\chi_z(w)\|} \tag{2.2}$$

for all z, $w \in B^n$, where χ_z is a holomorphic automorphism of B^n sending z into the origin O. In particular, κ_Δ and k_Δ are the Poincaré metric and distance of the unit disk.

(vi) *The Kobayashi metric and distance are contracted by holomorphic maps: if $f\colon X \to Y$ is a holomorphic map between complex manifolds, then*

$$\kappa_Y\bigl(f(z); df_z(v)\bigr) \le \kappa_X(z;v)$$

for all $z \in X$ and $v \in T_zX$, and

$$k_Y\bigl(f(z), f(w)\bigr) \le k_X(z,w)$$

for all z, $w \in X$. In particular, biholomorphisms are isometries for the Kobayashi metric and distance.

For comments, proofs and much much more see, e.g., [A3, JP, Kob2] and references therein.

For us, the most important property of Kobayashi metric and distance is clearly the last one: the Kobayashi metric and distance have a built-in Schwarz-Pick lemma. So it is only natural to try and use them to get a several variables version of Julia's lemma. To do so, we need ways to express Julia's condition (0.1) and to define horocycles in terms of Kobayashi distance and metric.

A way to proceed is suggested by metric space theory (and its applications to real differential geometry of negatively curved manifolds; see, e.g., [BGS]). Let X be a locally compact complete metric space with distance d. We may define an embedding $\iota\colon X \to C^0(X)$ of X into the space $C^0(X)$ of continuous functions on X mapping $z \in X$ into the function $d_z = d(z,\cdot)$. Now identify two continuous functions on X differing only by a constant; let $\overline{X}$ be the image of the closure of $\iota(X)$ in $C^0(X)$ under the quotient map π, and set $\partial X = \overline{X} \setminus \pi\bigl(\iota(X)\bigr)$. It is easy to check that $\overline{X}$ and ∂X are compact in the quotient topology, and that $\pi \circ \iota\colon X \to \overline{X}$ is a homeomorphism with the image. The set ∂X is the *ideal boundary* of X.

Any element $h \in \partial X$ is a continuous function on X defined up to a constant. Therefore the sublevels of h are well-defined: they are the *horospheres* centered at the boundary point h. Now, a preimage $h_0 \in C^0(X)$ of $h \in \partial X$ is the limit of functions of the form d_{z_k} for some sequence $\{z_k\} \subset X$ without limit points in X. Since we are interested in $\pi(d_{z_k})$ only, we can force h_0 to vanish at a fixed point $z_0 \in X$. This amounts to defining the horospheres centered in h by

$$E(h,R) = \{z \in X \mid \lim_{k\to\infty}[d(z,z_k) - d(z_0,z_k)] < \tfrac{1}{2}\log R\} \tag{2.3}$$

(see below for the reasons suggesting the appearance of $\frac{1}{2}\log$). Notice that, since d is a complete distance and $\{z_k\}$ is without limit points, $d(z,z_k) \to +\infty$ as $k \to +\infty$. On the other hand, $|d(z,z_k) - d(z_0,z_k)| \le d(z,z_0)$ is always finite. So, in some sense, the limit in (2.3) computes one-half the logarithm of a (normalized) distance of z from the boundary point h, and the horospheres are a sort of distance balls centered in h.

In our case, this suggests the following approach:

Definition 2.4: Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain in $\mathbb{C}^n$. The *(small) horosphere* of *center* $x \in \partial D$, *radius* $R > 0$ and *pole* $z_0 \in D$ is the set

$$E^D_{z_0}(x,R) = \{z \in D \mid \limsup_{w\to x}[k_D(z,w) - k_D(z_0,w)] < \tfrac{1}{2}\log R\}. \tag{2.4}$$

A few remarks are in order.

Remark 2.1: One clearly can introduce a similar notion of *large horosphere* replacing the lim sup by a lim inf in the previous definition. Large horospheres and small horospheres are actually different iff the geometrical boundary $\partial D \subset \mathbb{C}^n$ is smaller than the ideal boundary discussed above. It can be proved (see [A2] or [A3, Corollary 2.6.48]) that if D is a strongly convex C^3 domain then the lim sup in (2.4) actually is a limit, and thus the ideal boundary and the geometrical boundary coincide (as well as small and large horospheres).

Remark 2.2: The $\frac{1}{2}\log$ in the definition appears to recover the classical horocycles in the unit disk. Indeed, if we take $D = \Delta$ and $z_0 = 0$ it is easy to check that

$$\omega(\zeta,\eta) - \omega(0,\eta) = \tfrac{1}{2}\log\left(\frac{|1-\overline{\eta}\zeta|^2}{1-|\zeta|^2}\right) + \log\left(\frac{1+\left|\frac{\eta-\zeta}{1-\overline{\eta}\zeta}\right|}{1+|\eta|}\right);$$

therefore for all $\sigma \in \partial\Delta$ we have

$$\lim_{\eta\to\sigma}[\omega(\zeta,\eta) - \omega(0,\eta)] = \tfrac{1}{2}\log\left(\frac{|\sigma-\zeta|^2}{1-|\zeta|^2}\right),$$

and thus $E(\sigma,R) = E_0^\Delta(\sigma,R)$.

In a similar way one can explicitely compute the horospheres in another couple of cases:

Example 2.1: It is easy to check that the horospheres in the unit ball (with pole at the origin) are the classical horospheres (see, e.g., [Kor]) given by

$$E_O^{B^n}(x,R) = \left\{ z \in B^n \;\middle|\; \frac{|1-(z,x)|^2}{1-\|z\|^2} < R \right\}$$

for all $x \in \partial B^n$ and $R > 0$. Geometrically, $E_O^{B^n}(x,R)$ is an ellipsoid internally tangent to ∂B^n in x, and it can be proved (arguing as in [A2, Propositions 1.11 and 1.13]) that horospheres in strongly pseudoconvex domains have a similar shape.

Example 2.2: On the other hand, the shape of horospheres in the unit polydisk $\Delta^n \subset \mathbb{C}^n$ is fairly different (see [A5]):

$$E_O^{\Delta^n}(x,R) = \left\{ z \in \Delta^n \;\middle|\; \max_{|x_j|=1}\left\{\frac{|x_j - z_j|^2}{1-|z_j|^2}\right\} < R \right\} = E_1 \times \cdots \times E_n$$

for all $x \in \partial\Delta^n$ and $R > 0$, where $E_j = \Delta$ if $|x_j| < 1$ and $E_j = E(x_j,R)$ if $|x_j| = 1$.

Now we need a sensible replacement of Julia's condition (0.1). Here the key observation is that $1-|\zeta|$ is exactly the (euclidean) distance of $\zeta \in \Delta$ from the boundary. Keeping with the interpretation of the lim sup in (2.3) as a (normalized) Kobayashi distance of $z \in D$ from $x \in \partial D$, one is then tempted to consider something like

$$\inf_{x\in\partial D}\limsup_{w\to x}[k_D(z,w) - k_D(z_0,w)] \tag{2.5}$$

as a sort of (normalized) Kobayashi distance of $z \in D$ from the boundary. If we compute in the unit disk we find that

$$\inf_{\sigma\in\partial\Delta}\limsup_{\eta\to\sigma}[\omega(\zeta,\eta) - \omega(0,\eta)] = \tfrac{1}{2}\log\frac{1-|\zeta|}{1+|\zeta|} = -\omega(0,\zeta).$$

So we actually find $\frac{1}{2}\log$ of the euclidean distance from the boundary (up to a harmless correction), confirming our ideas. But, even more importantly, we see that the natural lower bound $-k_D(z_0,z)$ of (2.5) measures exactly the same quantity.

Another piece of evidence supporting this idea comes from the boundary estimates of the Kobayashi distance. As it can be expected, it is very difficult to compute explicitly the Kobayashi distance and metric of a complex manifold; on the other hand, it is not as difficult (and very useful) to estimate them. For instance, we have the following (see, e.g., [A3, section 2.3.5] or [Kob2, section 4.5] for strongly pseudoconvex domains, [AT2] for convex C^2 domains, and [A3, Proposition 2.3.5] or [Kob2, Example 3.1.24] for convex circular domains):

Theorem 2.2: *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and take $z_0 \in D$. Assume that*
(a) *D is strongly pseudoconvex, or*
(b) *D is convex with C^2 boundary, or*
(c) *D is convex circular.*
Then there exist c_1, $c_2 \in \mathbb{R}$ such that

$$c_1 - \tfrac{1}{2}\log d(z, \partial D) \le k_D(z_0, z) \le c_2 - \tfrac{1}{2}\log d(z, \partial D) \tag{2.6}$$

for all $z \in D$, where $d(\cdot, \partial D)$ denotes the euclidean distance from the boundary.

This is the first instance of the template phenomenon mentioned in the introduction. In the sequel, very often we shall not need to know the exact shape of the boundary of the domain under consideration; it will be enough to have estimates like the ones above on the boundary behavior of the Kobayashi distance. Let us then introduce the following template definition:

Definition 2.5: We shall say that a domain $D \subset \mathbb{C}^n$ *has the one-point boundary estimates* if for one (and hence every) $z_0 \in D$ there are c_1, $c_2 \in \mathbb{R}$ such that

$$c_1 - \tfrac{1}{2}\log d(z, \partial D) \le k_D(z_0, z) \le c_2 - \tfrac{1}{2}\log d(z, \partial D)$$

for all $z \in D$. In particular, D is complete hyperbolic.

So, again, if a domain has the one-point boundary estimates the Kobayashi distance from an interior point behaves exactly as half the logarithm of the euclidean distance from the boundary. We are then led to the following definition:

Definition 2.6: Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain, $x \in \partial D$ and $\beta > 0$. We shall say that a holomorphic function $f\colon D \to \Delta$ is *β-Julia* at x (with respect to a pole $z_0 \in D$) if

$$\liminf_{z \to x} \bigl[k_D(z_0, z) - \omega\bigl(0, f(z)\bigr)\bigr] = \tfrac{1}{2}\log\beta < +\infty.$$

The previous computations show that when $D = \Delta$ we recover the one-variable definition exactly. Furthermore, if the lim inf is finite for one pole then it is finite for all poles (even though β possibly changes). Moreover, the lim inf cannot ever be $-\infty$, because

$$k_D(z_0, z) - \omega\bigl(0, f(z)\bigr) \ge \omega\bigl(f(z_0), f(z)\bigr) - \omega\bigl(0, f(z)\bigr) \ge -\omega\bigl(0, f(z_0)\bigr),$$

and so $\beta > 0$ always. Finally, we explicitely remark that we might use a similar approach for holomorphic *maps* from D into another complete hyperbolic domain; but for the sake of simplicity in this survey we shall restrict ourselves to bounded holomorphic functions (see [A2, 4, 5] for more on the general case).

We have now enough tools to prove our several variables Julia's lemma:

Theorem 2.3: *Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain, and let $f \in \mathrm{Hol}(D, \Delta)$ be β-Julia at $x \in \partial D$ with respect to a pole $z_0 \in D$, that is assume that*

$$\liminf_{z \to x} \bigl[k_D(z_0, z) - \omega\bigl(0, f(z)\bigr)\bigr] = \tfrac{1}{2}\log\beta < +\infty.$$

Then there exists a unique $\tau \in \partial\Delta$ such that

$$f\bigl(E^D_{z_0}(x, R)\bigr) \subset E(\tau, \beta R)$$

for all $R > 0$.

Proof: Choose a sequence $\{w_k\} \subset D$ converging to x such that

$$\lim_{k \to +\infty} \bigl[k_D(z_0, w_k) - \omega\bigl(0, f(w_k)\bigr)\bigr] = \liminf_{z \to x} \bigl[k_D(z_0, z) - \omega\bigl(0, f(z)\bigr)\bigr].$$

We can also assume that $f(w_k) \to \tau \in \overline{\Delta}$. Being D complete hyperbolic, we know that $k_D(z_0, w_k) \to +\infty$; therefore we must have $\omega\bigl(0, f(w_k)\bigr) \to +\infty$, and so $\tau \in \partial\Delta$. Now take $z \in E^D_{z_0}(x, R)$. Then using the contracting property of the Kobayashi distance we get

$$\begin{aligned}
\frac{1}{2}\log\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} &= \lim_{\eta\to\tau}\bigl[\omega\bigl(f(z), \eta\bigr) - \omega(0, \eta)\bigr] = \lim_{k\to+\infty}\bigl[\omega\bigl(f(z), f(w_k)\bigr) - \omega\bigl(0, f(w_k)\bigr)\bigr]\\
&\le \limsup_{k\to+\infty}\bigl[k_D(z, w_k) - \omega\bigl(0, f(w_k)\bigr)\bigr]\\
&= \limsup_{k\to+\infty}[k_D(z, w_k) - k_D(z_0, w_k)] + \lim_{k\to+\infty}\bigl[k_D(z_0, w_k) - \omega\bigl(0, f(w_k)\bigr)\bigr]\\
&= \limsup_{w\to x}[k_D(z, w) - k_D(z_0, w)] + \frac{1}{2}\log\beta < \frac{1}{2}\log(\beta R),
\end{aligned}$$

and $f(z) \in E(\tau, \beta R)$. The uniqueness of τ follows as in the proof of Theorem 1.3. □

The next step consists in introducing a several variables version of the classical non-tangential limit. Korányi [Kor] has been the first one to notice that in several complex variables the obvious notion of non-tangential limit (i.e., limit inside cone-shaped approach regions) is not the right one for studying the boundary behavior of holomorphic functions. Indeed, in B^n he introduced the *admissible approach region* $K(x, M)$ of *vertex* $x \in \partial B^n$ and *amplitude* $M > 1$ defined by

$$K(x, M) = \left\{ z \in B^n \;\middle|\; \frac{|1 - (z, x)|}{1 - \|z\|} < M \right\}, \tag{2.7}$$

and said that a function $f\colon B^n \to \mathbb{C}$ had *admissible limit* $L \in \mathbb{C}$ at $x \in \partial B^n$ if $f(z) \to L$ as $z \to x$ inside $K(x, M)$ for any $M > 1$. Admissible regions are a clear generalization of one-variable Stolz regions, but the shape is different: though they are cone-shaped in the normal direction to ∂B^n at x (more precisely, the intersection with the complex line $\mathbb{C}x$ is exactly a Stolz region), they are tangent to ∂B^n in complex tangential directions. Nevertheless, Korányi was able to prove a Fatou theorem in the ball: any bounded holomorphic function has admissible limit at almost every point of ∂B^n, which is a much stronger statement than asking only for the existence of the non-tangential limit.

Later, Stein [St] (see also [KS]) generalized Korányi results to any C^2 domain $D \subset\subset \mathbb{C}^n$ defining the admissible limit using the euclidean approach regions

$$A(x, M) = \bigl\{ z \in D \bigm| |(z - x, \mathbf{n}_x)| < M\delta_x(z),\ \|z - x\|^2 < M\delta_x(z) \bigr\},$$

where $x \in \partial D$, $M > 1$, $\mathbf{n}_x$ is the outer unit normal vector to ∂D at x, and

$$\delta_x(z) = \min\bigl\{ d(z, \partial D), d(z, x + T_x\partial D) \bigr\};$$

notice that $A(x, M) \subseteq K(x, M)$ if $D = B^n$. Furthermore, in the same period Čirka [Č] introduced another kind of approach regions, depending on the order of contact of complex submanifolds with the boundary of the domain.

Both Stein's and Čirka's approach regions are defined in euclidean terms, and so are not suited for our arguments casted in terms of invariant distances. Another possibility is provided by the approach regions introduced by Cima and Krantz [CK] (see also [Kr1, 2]):

$$\mathcal{A}(x, M) = \{ z \in D \mid k_D(z, N_x) < M \},$$

where N_x is the set of points in D of the form $x - t\mathbf{n}_x$, with $t \in \mathbb{R}$, and $\mathbf{n}_x$ is the outer unit normal vector to ∂D at x. The approach regions $\mathcal{A}(x, M)$ in strongly pseudoconvex domains are comparable to Stein's and Čirka's approach regions — and thus yield the same notion of admissible limit. Unfortunately, the presence of the euclidean normal vector $\mathbf{n}_x$ is again unsuitable for our needs, and so we are forced to introduce a different kind of approach regions.

As we discussed before, the horospheres can be interpreted as sublevels of a sort of "distance" from the point x in the boundary, distance normalized using a fixed pole z_0. It turns out that a good way to define approach regions is taking the sublevels of the average between the Kobayashi distance from the pole z_0 and the "distance" from x. More precisely:

Definition 2.7: Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain. The *(small) K-region* $K_{z_0}^D(x, M)$ of *vertex* $x \in \partial D$, *amplitude* $M > 1$ and *pole* $z_0 \in D$ is the set

$$K_{z_0}^D(x, M) = \bigl\{ z \in D \bigm| \limsup_{w \to x} [k_D(z, w) - k_D(z_0, w)] + k_D(z_0, z) < \log M \bigr\}, \tag{2.8}$$

and we say that a function $f\colon D \to \mathbb{C}$ has *K-limit* $L \in \mathbb{C}$ at $x \in \partial D$ if $f(z) \to L$ as $z \to x$ inside $K_{z_0}^D(x, M)$ for any $M > 1$.

As usual, a few remarks and examples are in order.

Remark 2.3: Replacing the lim sup by a lim inf one obtains the definition of large K-regions, that we shall not use in this paper but that are important in the study of this kind of questions for holomorphic maps (instead of functions).

Remark 2.4: Changing the pole in K-regions amounts to a shifting in the amplitudes, and thus the notion of K-limit does not depend on the pole.

Example 2.3: It is easy to check that in the unit ball we recover Korányi's admissible regions exactly: $K_O^{B^n}(x, M) = K(x, M)$ for all $x \in \partial B^n$ and $M > 1$. More generally, it is not difficult to check (see [A2]) that in strongly pseudoconvex domains our K-regions are comparable with Stein's and Čirka's admissible regions, and so our K-limit is equivalent to their admissible limit.

Example 2.4: On the other hand, our K-regions are defined even in domains whose boundary is not smooth; for instance, in the polydisk we have ([A5])

$$K_O^{\Delta^n}(x, M) = \left\{ z \in \Delta^n \;\middle|\; \frac{1 + \|z\|}{1 - \|z\|} \max_{|x_j|=1} \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \right\} < M^2 \right\} \tag{2.9}$$

for all $x \in \partial \Delta^n$, where $\|z\| = \max\{|z_j|\}$; in particular, if $z \to x$ inside some $K_O^{\Delta^n}(x, M)$ then $z_j \to x_j$ non-tangentially if $|x_j| = 1$ while $z_j \to x_j$ without restrictions if $|x_j| < 1$.

Of course, one would like to compare K-regions with cone-shaped regions, that is K-limits with non-tangential limits. To do so, we again need to know something on the boundary behavior of the Kobayashi distance. More precisely, we need the following template definition:

Definition 2.8: We say that a domain $D \subset \mathbb{C}^n$ *has the two-points upper boundary estimate* at $x \in \partial D$ if there exist $\varepsilon > 0$ and $C > 0$ such that

$$k_D(z_1, z_2) \le \tfrac{1}{2} \sum_{j=1}^{2} \log \left(1 + \frac{\|z_1 - z_2\|}{d(z_j, \partial D)} \right) + C \tag{2.10}$$

for all z_1,$z_2 \in B(x, \varepsilon) \cap D$, where $B(x, \varepsilon)$ is the euclidean ball of center x and radius ε.

Forstneric and Rosay [FR] have proved that C^2 domains have the two-points upper estimate, and a similar proof shows that this is true for convex circular domains too.

Assume then that $D \subset \mathbb{C}^n$ has the one-point boundary estimates and the two-points boundary estimate at $x \in \partial D$ (e.g., D is strongly pseudoconvex, or C^2 convex, or convex circular). Then if $z \in D$ is close enough to x we have

$$\limsup_{w \to x} [k_D(z, w) - k_D(z_0, w)] + k_D(z_0, z) \le \tfrac{1}{2} \log \left(1 + \frac{\|z - x\|}{d(z, \partial D)} \right) + \tfrac{1}{2} \log \frac{\|z - x\|}{d(z, \partial D)} + C - c_1 + c_2,$$

and thus cones with vertex at x are contained in K-regions. This means that the existence of a K-limit is stronger than the existence of a non-tangential limit, and that $x \in \overline{K(x, M)} \cap \partial D$, even though the latter intersection can be strictly larger than $\{x\}$ (this happens, for instance, in the polydisk).

Going back to our main concern, definition (2.8) allows us to immediately relate horospheres and K-regions: for instance it is clear that

$$z \in K_{z_0}^D(x, M) \quad \Longrightarrow \quad z \in E_{z_0}^D\bigl(x, M^2/R(z)\bigr), \tag{2.11}$$

where $R(z) > 0$ is such that $k_D(z_0, z) = \frac{1}{2} \log R(z)$. We are then able to prove a several variables generalization of Theorem 0.1:

Theorem 2.4: *Let $D\subset\mathbb{C}^n$ be a complete hyperbolic domain, and let $f\in\mathrm{Hol}(D,\Delta)$ be β-Julia at $x\in\partial D$ with respect to a pole $z_0\in D$, that is assume that*

$$\liminf_{z\to x}\bigl[k_D(z_0,z)-\omega\bigl(0,f(z)\bigr)\bigr]=\tfrac{1}{2}\log\beta<+\infty.$$

Assume moreover that $x\in\overline{K^D_{z_0}(x,M)}$ for some (and then for all large enough) $M>1$. Then there exists a unique $\tau\in\partial\Delta$ such that f has K-limit τ at x.

Proof: It suffices to prove that if $y\in\overline{K^D_{z_0}(x,M)}\cap\partial D$ and $z\to y$ inside $K^D_{z_0}(x,M)$ then $f(z)\to\tau$, where $\tau\in\partial D$ is the point provided by Theorem 2.3. But indeed if $z\in K^D_{z_0}(x,M)$ then we just remarked that $z\in E^D_{z_0}\bigl(x,M^2/R(z)\bigr)$; therefore Theorem 2.3 yields $f(z)\in E\bigl(\tau,\beta M^2/R(z)\bigr)$. Since when z tends to the boundary of D we have $R(z)\to+\infty$, we get $f(z)\to\tau$ and we are done. □

Actually, this proof yields slightly more than what is stated: it shows that $f(z)\to\tau$ as soon as z tends to any point in $\overline{K^D_{z_0}(x,M)}\cap\partial D$, even though this intersection might be strictly larger than $\{x\}$ (for instance in the polydisk).

3. Lindelöf principles

The next step in our presentation consists in proving a Lindelöf principle in several complex variables. As first noticed by Čirka [Č], neither the non-tangential limit nor the K-limit (or admissible limit) are the right one to consider: the former is too weak, the latter too strong. But let us be more precise.

Definition 3.1: Let $D\subset\mathbb{C}^n$ be a domain in $\mathbb{C}^n$, and $x\in\partial D$. An *x-curve* in D is again a continuous curve $\gamma:[0,1)\to D$ such that $\gamma(t)\to x$ as $t\to 1^-$. Then, for us, a *Lindelöf principle* is a statement of the following form:

> “There are two classes $\mathcal{S}$ and $\mathcal{R}$ of x-curves in D such that: if $f:D\to\mathbb{C}$ is a bounded holomorphic function such that $f\bigl(\gamma^o(t)\bigr)\to L\in\mathbb{C}$ as $t\to 1^-$ for one curve $\gamma^o\in\mathcal{S}$ then $f\bigl(\gamma(t)\bigr)\to L$ as $t\to 1^-$ for all $\gamma\in\mathcal{R}$.”

In the classical Lindelöf principle, $\mathcal{S}$ is the set of all σ-curves in Δ, while $\mathcal{R}$ is the set of all non-tangential σ-curves. Remembering the previous section, one can be tempted to conjecture that in several variables one could take as $\mathcal{S}$ again the set of all x-curves, and as $\mathcal{R}$ the set of all x-curves contained in a K-region (or in an admissible region). But this is not true even in the ball, as remarked by Čirka:

Example 3.1: Take $f:B^2\to\Delta$ given by

$$f(z,w)=\frac{w^2}{1-z^2},$$

and $x=(1,0)$. Then if $\gamma_0(t)=(t,0)$ we have $f\circ\gamma_0\equiv 0$, and indeed it is not difficult to prove that f has non-tangential limit 0 at x. On the other hand, for any $c\in\Delta$ we can consider the x-curve $\gamma_c:[0,1)\to B^2$ given by $\gamma_c(t)=\bigl(t,c(1-t)^{1/2}\bigr)$; then

$$f\bigl(\gamma_c(t)\bigr)=\frac{c^2(1-t)}{1-t^2}=\frac{c^2}{1+t}\to\frac{c^2}{2}.$$

So the existence of the limit along such a curve does not imply that f has the same radial limit. Conversely, the existence of the radial limit does not imply that f has the same limit along a curve γ_c even though such a curve is contained in a K-region: indeed,

$$\frac{\bigl|1-\bigl(\gamma_c(t),x\bigr)\bigr|}{1-\|\gamma_c(t)\|}\le\frac{2\bigl|1-\bigl(\gamma_c(t),x\bigr)\bigr|}{1-\|\gamma_c(t)\|^2}=\frac{2}{1+t-|c|^2}\le\frac{2}{1-|c|^2},$$

and so the image of γ_c is contained in $K\bigl(x,2/(1-|c|^2)\bigr)$. Finally, if $1>\alpha>1/2$ and $c\in\Delta$ we can consider the curve $\gamma_{c,\alpha}(t)=\bigl(t,c(1-t)^\alpha\bigr)$. This is not a non-tangential curve, because

$$\frac{\|\gamma_{c,\alpha}(t)-x\|}{1-\|\gamma_{c,\alpha}(t)\|}\ge\frac{\|\gamma_{c,\alpha}(t)-x\|}{1-\|\gamma_{c,\alpha}(t)\|^2}=\frac{1}{(1-t)^{1-\alpha}}\frac{\bigl((1-t)^{2(1-\alpha)}+|c|^2\bigr)^{1/2}}{1+t-|c|^2(1-t)^{2\alpha-1}}\to+\infty$$

as $t \to 1^-$. However,

$$f\big(\gamma_{c,\alpha}(t)\big) = \frac{c^2(1-t)^{2\alpha}}{1-t^2} = \frac{c^2(1-t)^{2\alpha-1}}{1+t} \to 0$$

as $t \to 1^-$, and so $f \circ \gamma(t)$ tends to zero for x-curves γ belonging to a family strictly larger than the one of non-tangential curves.

The conclusion of this example is that in general both classes $\mathcal{S}$ and $\mathcal{R}$ in a Lindelöf principle might not coincide with the classes of non-tangential curves, or of curves contained in a K-region, or of all x-curves. For instance, let us describe one of the Lindelöf principles proved by Čirka [Č]. Assume that the boundary ∂D is of class C^1 in a neighbourhood of a point $x \in \partial D$, and denote by $\mathbf{n}_x$ the outer unit normal vector to ∂D in x. Furthermore, let $H_x(\partial D) = T_x(\partial D) \cap iT_x(\partial D)$ be the holomorphic tangent space to ∂D at x, set $N_x = x + \mathbb{C}\mathbf{n}_x$, and let $\pi_x\colon \mathbb{C}^n \to N_x$ be the complex-linear projection parallel to $H_x(\partial D)$, so that $z - \pi_x(z) \in H_x(\partial D)$ for all $z \in \mathbb{C}^n$. Finally, set $H_z = z + H_x(\partial D)$. Then Čirka proved a Lindelöf principle taking: as $\mathcal{S}$ the set of x-curves γ such that the image of $\pi_x \circ \gamma$ is contained in D and such that

$$\lim_{t\to 1^-} \frac{\big\|\gamma(t) - \pi_x \circ \gamma(t)\big\|}{d\big(\pi_x \circ \gamma(t), H_{\pi_x\circ\gamma(t)} \cap \partial D\big)} = 0; \tag{3.1}$$

and as $\mathcal{R}$ the set of curves $\gamma \in \mathcal{S}$ such that $\pi_x \circ \gamma$ approaches x non-tangentially in $D \cap N_x$. Notice that $\mathcal{R}$ contains properly the set of all non-tangential curves. For instance, it is easy to check that if $D = B^2$ and $x = (1,0)$ then $\gamma_{c,\alpha} \in \mathcal{R}$ iff $\alpha > 1/2$.

Some years later another kind of Lindelöf principle (valid for normal holomorphic functions, not only for bounded ones) has been proved by Cima and Krantz [CK]. They supposed ∂D smooth at $x \in \partial D$, and used: as $\mathcal{S}$ the set of non-tangential x-curves; and as $\mathcal{R}$ the set of x-curves γ such that

$$\lim_{t\to 1^-} k_D\big(\gamma(t), \Gamma_x\big) = 0$$

for some cone Γ_x of vertex x inside D. Now, notice that, by continuity, $\gamma \in \mathcal{R}$ iff we can find a cone Γ_x and an x-curve γ_x whose image is contained in Γ_x such that

$$\lim_{t\to 1^-} k_D\big(\gamma(t), \gamma_x(t)\big) = 0. \tag{3.2}$$

If in Čirka's setting we take $\gamma_x = \pi_x \circ \gamma$ it is not difficult to see that (3.1) implies (3.2). Indeed, let $r(t) > 0$ be the largest r such that the image of $\Delta_r = r\Delta$ through the map $\psi(\zeta) = \gamma_x(t) + \zeta\big(\gamma(t) - \gamma_x(t)\big)$ is contained in D. Clearly,

$$r(t)\|\gamma(t) - \gamma_x(t)\| \geq d(\gamma_x(t), H_{\gamma_x(t)} \cap \partial D);$$

in particular, (3.1) implies $r(t) \to +\infty$. But then

$$k_D\big(\gamma(t), \gamma_x(t)\big) \leq k_{\Delta_{r(t)}}(1,0) = \omega\left(0, \frac{1}{r(t)}\right) \to 0,$$

and so (3.2) holds.

It turns out that as soon as we have something like (3.2), to prove a Lindelöf principle it is just a matter of applying the one-variable Lindelöf principle and the contracting property of the Kobayashi distance. These considerations suggested in [A2] the introduction of a very general setting producing Lindelöf principles.

Definition 3.2: Let $D \subset \mathbb{C}^n$ be a domain in C^n. A *projection device* at $x \in \partial D$ is given by the following data:

(a) a neighbourhood U of x in $\mathbb{C}^n$;
(b) a holomorphically embedded disk $\varphi_x\colon \Delta \to U \cap D$ such that $\lim_{\zeta\to 1} \varphi_x(\zeta) = x$;
(c) a family $\mathcal{P}$ of x-curves in $U \cap D$;
(d) a device associating to every x-curve $\gamma \in \mathcal{P}$ a 1-curve $\tilde{\gamma}_x$ in Δ — or, equivalently, a x-curve $\gamma_x = \varphi_x \circ \tilde{\gamma}_x$ in $\varphi_x(\Delta) \subset U \cap D$.

When we have a projection device, we shall *always* use $\varphi_x(0)$ as pole for horospheres and K-regions.

It is very easy to produce examples of projection devices. For instance:

Example 3.2: The *trivial projection device.* Take $U=\mathbb{C}^n$, choose as φ_x any holomorphically embedded disk satisfying the hypotheses, as $\mathcal{P}$ the set of all x-curves, and to any $\gamma\in\mathcal{P}$ associate the radial curve $\tilde\gamma_x(t)=1-t$.

Example 3.3: The *euclidean projection device.* This is the device used by Čirka. Assume that ∂D is of class C^1 at x, let $\mathbf{n}_x$ be the outer unit normal at x, and set $N_x=x+\mathbb{C}\mathbf{n}_x$ as before. Choose U so that $(U\cap D)\cap N_x$ is simply connected with continuous boundary, and let $\varphi_x\colon\Delta\to(U\cap D)\cap N_x$ be a biholomorphism extending continuously up to the boundary with $\varphi_x(1)=x$. Let again $\pi_x\colon\mathbb{C}^n\to N_x$ be the orthogonal projection, and choose as $\mathcal{P}$ the set of x-curves γ in $U\cap D$ such that the image of $\pi_x\circ\gamma$ is still contained in $U\cap D$. Then for every $\gamma\in\mathcal{P}$ we set $\gamma_x=\pi_x\circ\gamma$.

Example 3.4: This is a slight variation of the previous one. If D is convex and of class C^1 in a neighbourhood of x, both the projection $\pi_x(D)$ and the intersection $D\cap N_x$ are convex domains in N_x; therefore there is a biholomorphism $\psi\colon\pi_x(D)\to D\cap N_x$ extending continuously to the boundary so that $\psi(x)=x$. Then we can take $U=\mathbb{C}^n$, φ_x as in Example 3.3, $\mathcal{P}$ as the set of all x-curves, and set $\gamma_x=\psi\circ\pi_x\circ\gamma$.

To describe the next projection device, that it will turn out to be the most useful, we need a new definition:

Definition 3.3: A holomorphic map $\varphi\colon\Delta\to X$ in a complex manifold X is a *complex geodesic* if it is an isometry between the Poincaré distance ω and the Kobayashi distance k_X.

Complex geodesics have been introduced by Vesentini [V1], and deeply studied by Lempert [Le] and Royden-Wong [RW]. In particular, they proved that if D is a bounded convex domain then for every z_0, $z\in D$ there exists a complex geodesic $\varphi\colon\Delta\to D$ passing through z_0 and z, that is such that $\varphi(0)=z_0$ and $z\in\varphi(\Delta)$. Moreover, there also exists a left-inverse of φ, that is a bounded holomorphic function $\tilde p\colon D\to\Delta$ such that $\tilde p\circ\varphi=\mathrm{id}_\Delta$ (see [A3, Chapter 2.6] or [Kob2, sections 4.6–4.8] for complete proofs). Furthermore, if there exists $z_0\in D$ such that for every $z\in D$ we can find a complex geodesic φ continuous up to the boundary passing through z_0 and z (this happens, for instance, if D is strongly convex with C^3-boundary [Le], if it is convex of finite type [AT2], or if it is convex circular and $z_0=O$ [V2]) then it is easy to prove (see, e.g., [A1]) that for any $x\in\partial D$ there is a complex geodesic φ continuous up to the boundary such that $\varphi(0)=z_0$ and $\varphi(1)=x$.

Example 3.5: The *canonical projection device.* Let $D\subset\subset\mathbb{C}^n$ be a bounded convex domain, and let $x\in\partial D$ be such that there is a complex geodesic $\varphi_x\colon\Delta\to D$ so that $\varphi_x(\zeta)\to x$ as $\zeta\to 1$. Then the canonical projection device is obtained taking $U=\mathbb{C}^n$, $\mathcal{P}$ as the set of all x-curves, and setting $\tilde\gamma_x=\tilde p_x\circ\gamma$, where $\tilde p_x\colon D\to\Delta$ is the left-inverse of φ_x. Notice that the canonical projection device is defined only using the Kobayashi distance; therefore it will be particularly well-suited for our aims.

Example 3.6: This is a far-reaching generalization of the previous example. Let D be any domain, $x\in\partial D$ any point, and $\varphi_x\colon\Delta\to D$ any holomorphically embedded disk satisfying the hypotheses. Choose a bounded holomorphic function $h\colon D\to\Delta$ such that $h(z)\to 1$ as $z\to x$ in D. Then we have a projection device just by choosing $U=\mathbb{C}^n$, $\mathcal{P}$ as the set of all x-curves, and setting $\tilde\gamma_x=h\circ\gamma$.

Example 3.7: All the previous examples can be localized: if there is a neighbourhood U of $x\in\partial D$ such that we can define a projection device at x for $U\cap D$, we clearly have a projection device at x for D. In particular, if D is locally biholomorphic to a convex domain in x (e.g., if D is strongly pseudoconvex in x), then we can localize the projection devices of Examples 3.3, 3.4 and 3.5.

We can now define the right kind of limit for Lindelöf principles.

Definition 3.4: Let $D\subset\mathbb{C}^n$ be a domain equipped with a projection device at $x\in\partial D$. We shall say that a curve $\gamma\in\mathcal{P}$ is *special* if

$$\lim_{t\to 1^-}k_{D\cap U}\bigl(\gamma(t),\gamma_x(t)\bigr)=0;\tag{3.3}$$

and that it is *restricted* if $\tilde\gamma_x$ is a non-tangential 1-curve. Let $\mathcal{S}$ denote the set of special x-curves, and $\mathcal{R}\subset\mathcal{S}$ denote the set of special restricted x-curves. We shall say that a function $f\colon D\to\mathbb{C}$ has *restricted K-limit* $L\in\mathbb{C}$ at x if $f\bigl(\gamma(t)\bigr)\to L$ as $t\to 1^-$ for all $\gamma\in\mathcal{R}$.

Remark 3.1: We could have defined the notion of special curve using k_D instead of $k_{D\cap U}$ in (3.3), and the following proofs would have worked anyway with a possibly larger set of curves. However, the chosen definition stresses the local nature of the projection device (as it should be, because it is a tool born to deal with local phenomena), allowing to replace D by $D \cap U$ everywhere. Furthermore, if D has the one-point boundary estimates, the two-two points upper boundary estimate at x and also the *two-points lower boundary estimate,* that is for any pair of distinct points $x_1 \neq x_2 \in \partial D$ there esist $\varepsilon > 0$ and $K \in \mathbb{R}$ such that

$$k_D(z_1, z_2) \geq -\tfrac{1}{2} \log d(z_1, \partial D) - \tfrac{1}{2} \log d(z_2, \partial D) + K$$

as soon as $z_1 \in B(x_1, \varepsilon) \cap D$ and $z_2 \in B(x_2, \varepsilon) \cap D$, then ([A3, Theorem 2.3.65])

$$\lim_{\substack{z,w \to x \\ z \neq w}} \frac{k_D(z,w)}{k_{D \cap U}(z,w)} = 1,$$

and so the two definitions of special curves coincide. Examples of domains having the two-points lower boundary estimate include strongly pseudoconvex domains ([FR], [A3, Corollary 2.3.55]).

The whole point of the definition of projection device is that the arguments used in [Č] and [CK] boil down to the following very general Lindelöf principle:

Theorem 3.1: *Let $D \subset \mathbb{C}^n$ be a domain equipped with a projection device at $x \in \partial D$. Let $f\colon D \to \mathbb{C}$ be a bounded holomorphic function such that $f\bigl(\gamma^o(t)\bigr) \to L \in \mathbb{C}$ as $t \to 1^-$ for one special curve $\gamma^o \in \mathcal{S}$. Then f has restricted K-limit L at x.*

Proof: We can assume that $f(D) \subset\subset \Delta$. If $\gamma \in \mathcal{S}$ we have

$$\omega\bigl(f\bigl(\gamma(t)\bigr), f\bigl(\gamma_x(t)\bigr)\bigr) \leq k_{D \cap U}\bigl(\gamma(t), \gamma_x(t)\bigr) \to 0$$

as $t \to 1^-$; therefore f has limit along γ iff it does along γ_x. In particular, it has limit L along γ^o_x; the classical Lindelöf principle applied to $f \circ \varphi_x$ shows then that f has limit L along γ_x for all restricted γ. But in turn this implies that f has limit L along all $\gamma \in \mathcal{R}$, and we are done. □

The same proof, adapted as in [CK], works for normal functions too, not necessarily bounded.

Of course, the interest of such a result is directly proportional to how large the set $\mathcal{R}$ is. Let us see a few examples.

Example 3.8: The first one is a negative one: if $D = \Delta$ and $x = 1$ then the class $\mathcal{R}$ for the trivial projection device contains only 1-curves tangent in 1 to the radius, and so in this case Theorem 3.1 is even weaker than the classical Lindelöf principle. In other words, the trivial projection device probably is not that useful.

Example 3.9: The next one is much better: if $D \subset \mathbb{C}^n$ is of class C^1 we already remarked that all x-curves satisfying Čirka's condition (3.1) are special; therefore Theorem 3.1 recovers Čirka's result.

Example 3.10: If D is strongly convex, it is not difficult to check (see [A4]) that for the euclidean projection device a restricted x-curve is special iff

$$\lim_{t \to 1^-} \frac{\|\gamma(t) - \gamma_x(t)\|}{d\bigl(\gamma_x(t), \partial D\bigr)} = 0; \tag{3.4}$$

in particular, all non-tangential x-curves are special and restricted. A much harder computation (see [A3, Proposition 2.7.1]) shows that if D is strongly convex of class C^3 then for the canonical projection device a restricted x-curve γ is special again iff (3.4) holds (but this time γ_x is given by the canonical projection device, not by the euclidean one). Using this characterization it is possible to prove ([A3, Lemma 2.7.12]) that non-tangential x-curves are special and restricted for the canonical projection device too.

Example 3.11: In [A5] it is shown that in the polydisk Δ^n, if we use the canonical projection device associated to the complex geodesic $\varphi_x(\zeta) = \zeta x$, an x-curve γ is special iff

$$\lim_{t \to 1^-} \max_{|x_j|=1} \left\{ \frac{|\gamma_j(t) - (\gamma_x)_j(t)|}{1 - \|\gamma_x(t)\|} \right\} = 0.$$

Unfortunately, as already happened in one-variable, to prove the Julia-Wolff-Carathéodory theorem one needs a Lindelöf principle for not necessarily bounded holomorphic functions.

Definition 3.5: We shall say that a function $f\colon D \to \mathbb{C}$ is *K-bounded* at $x \in \partial D$ if for every $M > 1$ there exists $C_M > 1$ such that $|f(z)| < C_M$ for all $z \in K^D_{z_0}(x, M)$; it is clear that this condition does not depend on the pole.

Then we shall need a Lindelöf principle for K-bounded functions. As it can be expected, such a Lindelöf principle does not hold for any projection device: we need some connection between the projection device and K-regions. We shall express this connection in a template form.

Definition 3.6: Let $D \subset \mathbb{C}^n$ be a domain. A projection device at $x \in \partial D$ is *good* if
(i) for any $M > 1$ there is $\tilde{M} > 1$ such that $\varphi_x\big(K(1, M)\big) \subset K^{D\cap U}_{z_0}(x, \tilde{M})$;
(ii) if $\gamma \in \mathcal{R}$ then there exists $M_1 = M_1(\gamma)$ such that

$$\lim_{t\to 1^-} k_{K^{D\cap U}_{z_0}(x,M_1)}\big(\gamma(t), \gamma_x(t)\big) = 0.$$

Condition (i) actually is almost automatic. Indeed, it suffices that φ_x sends non-tangential 1-curves in non-tangential x-curves (this happens for instance if $\varphi'_x(1)$ exists and it is transversal to ∂D), and that non-tangential approach regions are contained in K-regions (as happens if D has the one-point boundary estimates and the two-point upper boundary estimate at x, as already noticed in the previous section).

Now, condition (i) implies that for any $\gamma \in \mathcal{R}$ there is $M > 1$ such that $\gamma(t)$, $\gamma_x(t) \in K^{D\cap U}_{z_0}(x, M)$ for all t close enough to 1. Indeed, since γ is restricted, $\tilde{\gamma}_x(t)$ belongs to some Stolz region in Δ for t close enough to 1. Therefore, by condition (i), $\gamma_x(t)$ belongs to some K-region at x, and, being γ special, $\gamma(t)$ belongs to a slightly larger K-region for all t close enough to 1.

In particular, condition (ii) makes sense; but it is harder to verify. The usual approach is the following: for $t \in [0, 1)$ set

$$\psi_t(\zeta) = \gamma_x(t) + \zeta\big(\gamma(t) - \gamma_x(t)\big),$$

and

$$r(t, M_1) = \sup\big\{r > 0 \bigm| \psi_t(\Delta_r) \subset K^{D\cap U}_{z_0}(x, M_1)\big\}.$$

The contracting property of the Kobayashi distance yields

$$k_{K^{D\cap U}_{z_0}(x,M_1)}\big(\gamma(t), \gamma_x(t)\big) \le \omega\left(0, \frac{1}{r(t, M_1)}\right);$$

therefore to prove (ii) it suffices to show that for any $\gamma \in \mathcal{R}$ there exists $M_1 > 1$ such that $r(t, M_1) \to +\infty$ as $t \to 1^-$. And to prove this latter assertion we need informations on the shape of $K^{D\cap U}_{z_0}(x, M)$ near the boundary, that is on the boundary behavior of the Kobayashi distance (and of the projection device).

Example 3.12: Both the euclidean and the canonical projection device are good on strongly convex domains (see [A2, 4]) — and thus their localized versions are good in strongly pseudoconvex domains. Furthermore, the canonical projection device is good in convex domains with C^ω boundary (or more generally in strictly linearly convex domains of finite type [AT2]) and in the polydisk [A5]; as far as I know, it is still open the question of whether the canonical projection device associated to the complex geodesic $\varphi_x(\zeta) = \zeta x$ is good in any convex circular domain.

Adapting the proof of Theorem 3.1 we get a Lindelöf principle for K-bounded functions:

Theorem 3.2: *Let $D \subset \mathbb{C}^n$ be a domain equipped with a good projection device at $x \in \partial D$. Let $f\colon D \to \mathbb{C}$ be a K-bounded holomorphic function such that $f\big(\gamma^o(t)\big) \to L \in \mathbb{C}$ as $t \to 1^-$ for one special restricted curve $\gamma^o \in \mathcal{R}$. Then f has restricted K-limit L at x.*

Proof: Take $\gamma \in \mathcal{R}$, and let $M_1 > 1$ be given by condition (ii). Then if f is bounded by C on $K^D_{z_0}(x, M_1)$ we have

$$k_{\Delta_C}\Big(f\big(\gamma(t)\big), f\big(\gamma_x(t)\big)\Big) \le k_{K^{D\cap U}_{z_0}(x,M_1)}\big(\gamma(t), \gamma_x(t)\big) \to 0$$

as $t \to 1^-$; therefore f has limit along γ iff it does along γ_x. In particular, it has limit L along γ^o_x; the classical Lindelöf principle for K-bounded functions in the disk Theorem 1.5.(b) applied to $f \circ \varphi_x$ (which is K-bounded thanks to condition (i) in the definition of good projection devices) shows then that f has limit L along γ_x for all restricted γ. But in turns this implies that f has limit L along all $\gamma \in \mathcal{R}$, and we are done. $\square$

As the proof makes clear, replacing K-regions by other kinds of approach regions (and changing conditions (i) and (ii) accordingly) one gets similar results. Let us describe a possible variation, which is useful for instance in convex domains of finite type.

Definition 3.7: A projection device is *geometrical* if there is a holomorphic function $\tilde{p}_x: D \cap U \to \Delta$ such that $\tilde{p}_x \circ \varphi_x = \mathrm{id}_\Delta$ and $\tilde{\gamma}_x = \tilde{p}_x \circ \gamma$ for all $\gamma \in \mathcal{P}$.

Example 3.13: The canonical projection device is a geometrical projection device.

Remark 3.2: In a geometrical projection device the map φ_x always is a complex geodesic of $U \cap D$: indeed

$$k_{U\cap D}\bigl(\varphi_x(\zeta_1), \varphi_x(\zeta_2)\bigr) \le \omega(\zeta_1, \zeta_2) = \omega\bigl(\tilde{p}_x\bigl(\varphi_x(\zeta_1)\bigr), \tilde{p}_x\bigl(\varphi_x(\zeta_2)\bigr)\bigr) \le k_{U\cap D}\bigl(\varphi_x(\zeta_1), \varphi_x(\zeta_2)\bigr)$$

for all ζ_1, $\zeta_2 \in \Delta$.

Definition 3.8: Let $D \subset \mathbb{C}^n$ be a domain equipped with a geometrical projection device at $x \in \partial D$. The *T-region* of *vertex* x, *amplitude* $M > 1$ and *girth* $\delta \in (0,1)$ is the set

$$T(x, M, \delta) = \bigl\{ z \in U \cap D \bigm| \tilde{p}_x(z) \in K(1, M),\, k_{U\cap D}\bigl(z, p_x(z)\bigr) < \omega(0, \delta) \bigr\},$$

where $p_x = \varphi_x \circ \tilde{p}_x$. A function $f: D \to \mathbb{C}$ is *T-bounded* at x if there is $0 < \delta_0 < 1$ such that f is bounded on each $T(x, M, \delta_0)$, with the bound depending on M as usual.

By construction, $\varphi_x\bigl(K(1, M)\bigr) \subset T(x, M, \delta)$ for all $0 < \delta < 1$. Furthermore, if $\gamma \in \mathcal{R}$ there always are $M > 1$ and $\delta \in (0,1)$ such that $\gamma(t) \in T(x, M, \delta)$ for all t close enough to 1. Therefore we can introduce the following definition:

Definition 3.9: A geometrical projection device at $x \in \partial D$ is *T-good* if for any $\gamma \in \mathcal{R}$ then there exist $M = M(\gamma) > 1$ and $\delta = \delta(\gamma) > 0$ such that

$$\lim_{t \to 1^-} k_{T(x,M,\delta)}\bigl(\gamma(t), \gamma_x(t)\bigr) = 0.$$

Example 3.14: The canonical projection device is T-good in all convex domains of finite type ([AT2]).

Then arguing as before we get

Theorem 3.3: *Let $D \subset \mathbb{C}^n$ be a domain equipped with a T-good geometrical projection device at $x \in \partial D$. Let $f: D \to \mathbb{C}$ be a T-bounded holomorphic function such that $f\bigl(\gamma^o(t)\bigr) \to L \in \mathbb{C}$ as $t \to 1^-$ for one special restricted curve $\gamma^o \in \mathcal{R}$. Then f has restricted K-limit L at x.*

Of course, to compare such a result with Theorem 3.2 one would like to know whether K-bounded functions are T-bounded or not — and this boils down to compare K-regions and T-regions. It turns out that to make the comparison we need another property of the projection device:

Definition 3.10: We shall say that a projection device at $x \in \partial D$ *preserves horospheres* if

$$\varphi_x\bigl(E(1, R)\bigr) \subset E_{z_0}^{U\cap D}(x, R)$$

for all $R > 0$.

Proposition 3.4: *Let $D \subset \mathbb{C}^n$ be a domain equipped with a geometrical projection device at $x \in \partial D$ preserving horospheres. Then*

$$T(x, M, \delta) \subset K_{z_0}^{U\cap D}\bigl(x, M(1+\delta)/(1-\delta)\bigr), \tag{3.5}$$

for all $M > 1$ and $0 < \delta < 1$. Furthermore, a T-good geometrical projection device preserving horospheres is automatically good.

Proof: First of all, we claim that

$$\varphi_x\bigl(K(1, M)\bigr) \subset K_{z_0}^{U\cap D}(x, M)$$

for all $M > 1$. Indeed, if $\zeta \in K(1, M)$ we have $\zeta \in E\big(1, M^2/R(\zeta)\big)$, where $R(\zeta)$ satisfies

$$\tfrac{1}{2}\log R(\zeta) = \omega(0, \zeta) = k_{U\cap D}\big(z_0, \varphi_x(\zeta)\big)$$

as usual. But then $\varphi_x(\zeta) \in E^{U\cap D}_{z_0}\big(x, M^2/R(\zeta)\big)$, which immediately implies $\varphi_x(\zeta) \in K^{U\cap D}_{z_0}(x, M)$, as claimed.

Now take $z \in T(x, M, \delta)$. Then we have

$$\begin{aligned} k_{U\cap D}\big(z, w\big) - k_{U\cap D}(z_0, w) + k_{U\cap D}(z_0, z) \\ &\le 2k_{U\cap D}\big(z, p_x(z)\big) + k_{U\cap D}(p_x(z), w) - k_{U\cap D}(z_0, w) + k_{U\cap D}\big(z_0, p_x(z)\big) \\ &< 2\omega(0, \delta) + k_{U\cap D}(p_x(z), w) - k_{U\cap D}(z_0, w) + k_{U\cap D}\big(z_0, p_x(z)\big) \end{aligned}$$

for all $w \in U \cap D$. Now, $\tilde{p}_x(z) \in K(1, M)$; therefore $p_x(z) \in K^{U\cap D}_{z_0}(x, M)$ and

$$\limsup_{w\to x}\big[k_{U\cap D}(z, w) - k_{U\cap D}(z_0, w)\big] + k_{U\cap D}(z_0, z) < \log\left(\frac{1+\delta}{1-\delta}\right) + \log M,$$

that is (3.5). Then condition (ii) in the definition of good projection device follows immediately by the contraction property of the Kobayashi distance. □

So if the projection device preserves horospheres we see that K-bounded functions are always T-bounded, and thus the hypotheses on the function in Theorem 3.3 are weaker than the hypotheses in Theorem 3.2.

Of course, one would like to know when a geometrical projection device preserves horospheres. It is easy to prove that φ_x sends horocycles into *large* horospheres for any geometrical projection device; so if large and small horospheres coincide (as it happens in strongly convex domains, for instance), we are done.

Another sufficient condition is the following:

Proposition 3.5: *Let $D \subset \mathbb{C}^n$ be a domain equipped with a geometrical projection device at $x \in \partial D$. Assume there is a neighbourhood $V \subseteq U$ of x and a family $\Psi\colon V \cap D \to \mathrm{Hol}(\Delta, U \cap D)$ of holomorphic disks in $U \cap D$ such that, writing $\psi_z(\zeta)$ for $\Psi(z)(\zeta)$, the following holds:*
(a) *$\psi_z(0) = z_0 = \varphi_x(0)$ for all $z \in V \cap D$;*
(b) *for all $z \in V \cap D$ there is $r_z \in [0, 1)$ such that $\psi_z(r_z) = z$;*
(c) *ψ_z converges to ϕ_x, uniformly on compact subsets, as $z \to x$ in $V \cap D$;*
(d) *$k_{U\cap D}(z_0, z) - \omega(0, r_z)$ tends to 0 as $z \to x$ in $V \cap D$.*
Then the projection device preserves horospheres.

Proof: Since we always have

$$\begin{aligned} \lim_{t\to 1^-}[\omega(\zeta, t) - \omega(0, t)] &= \lim_{t\to 1^-}\big[k_{U\cap D}\big(\varphi_x(\zeta), \varphi_x(t)\big) - k_{U\cap D}\big(z_0, \varphi_x(t)\big)\big] \\ &\le \limsup_{w\to x}\big[k_{U\cap D}(\varphi_x(\zeta), w) - k_{U\cap D}(z_0, w)\big], \end{aligned}$$

it suffices to prove the reverse inequality. In other words, it suffices to prove that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$k_{U\cap D}(\varphi_x(\zeta), w) - k_{U\cap D}(z_0, w) \le \lim_{t\to 1^-}[\omega(\zeta, t) - \omega(0, t)] + \varepsilon$$

as soon as $w \in B(x, \delta) \cap D \subset V$.

First of all, we claim that for any $z \in U \cap D$ and $\psi \in \mathrm{Hol}(\Delta, U \cap D)$ the function

$$t \mapsto k_{U\cap D}\big(z, \psi(t)\big) - \omega(0, t)$$

is not increasing. Indeed, if $t_1 \le t_2$ we have

$$\begin{aligned} k_{U\cap D}\big(z, \psi(t_1)\big) - \omega(0, t_1) &= k_{U\cap D}\big(z, \psi(t_1)\big) + \omega(t_1, t_2) - \omega(0, t_2) \\ &\ge k_{U\cap D}\big(z, \psi(t_1)\big) + k_{U\cap D}\big(\psi(t_1), \psi(t_2)\big) - \omega(0, t_2) \\ &\ge k_{U\cap D}\big(z, \psi(t_2)\big) - \omega(0, t_2). \end{aligned}$$

In particular, given $\zeta \in \Delta$ we can find $t_0 < 1$ so that

$$k_{U\cap D}\bigl(\varphi_x(\zeta), \varphi_x(t)\bigr) - \omega(0,t) \le \lim_{t\to 1^-} [\omega(\zeta,t) - \omega(0,t)] + \varepsilon/3$$

when $t \ge t_0$.

Now choose $\delta > 0$ such that when $w \in B(x,\delta) \cap D \subset V$ we have

$$\bigl|k_{U\cap D}\bigl(\varphi_x(\zeta), \psi_w(t_0)\bigr) - k_{U\cap D}\bigl(\varphi_x(\zeta), \varphi_x(t_0)\bigr)\bigr| \le k_{U\cap D}\bigl(\varphi_x(t_0), \psi_w(t_0)\bigr) \le \varepsilon/3$$

and

$$\omega(0, r_w) \ge k_{U\cap D}(z_0, w) \ge \max\{\omega(0, r_w) - \varepsilon/3, \omega(0, t_0)\}.$$

Then

$$\begin{aligned} k_{U\cap D}\bigl(\varphi_x(\zeta), w\bigr) - k_{U\cap D}(z_0, w) &\le k_{U\cap D}\bigl(\varphi_x(\zeta), \psi_w(r_w)\bigr) - \omega(0, r_w) + \varepsilon/3 \\ &\le k_{U\cap D}\bigl(\varphi_x(\zeta), \psi_w(t_0)\bigr) - \omega(0, t_0) + \varepsilon/3 \\ &\le k_{U\cap D}\bigl(\varphi_x(\zeta), \varphi_x(t_0)\bigr) - \omega(0, t_0) + 2\varepsilon/3 \\ &\le \lim_{t\to 1^-} [\omega(\zeta,t) - \omega(0,t)] + \varepsilon, \end{aligned}$$

and we are done. □

So a geometrical projection device preserves horospheres if φ_x is embedded in a continuous family of "almost geodesic" disks sweeping a one-sided neighbourhood of x.

Example 3.15: In strongly convex domains ([Le]) and in strictly linearly convex domains of finite type ([AT2]) the conditions of the previous proposition can be satisfied using complex geodesics. In convex circular domains, it suffices to use linear disks (when φ_x is linear, as usual in this case). In particular, then, the canonical projection device is good in strictly linearly convex domains of finite type and in convex circular domains of finite type. But it might well be possible that the conditions in Proposition 3.5 are satisfied in other classes of domains too.

It follows that, at present, we can apply Theorem 3.3 to all convex domains of finite type, because we know that there the canonical projection device is T-good, whereas we can apply Theorem 3.2 only to strictly linearly convex (or convex circular) domains of finite type, because of the previous two Propositions — and to the polydisk, because we can prove directly that the canonical projection device is good there. So Theorems 3.2 and 3.3 are applicable to different classes of domains, and this is the reason we presented both. Anyway, it is very natural to conjecture that the canonical projection device is good in all convex domains of finite type and in all convex circular domains.

4. The Julia-Wolff-Carathéodory theorem

We are almost ready to prove the several variables version of the Wolff-Carathéodory part of the Julia-Wolff-Carathéodory theorem. We only need to introduce another couple of concepts.

Definition 4.1: A geometrical projection device at $x \in \partial D$ is *bounded* if $d(z,\partial D)/|1-\tilde{p}_x(z)|$ is bounded in $U \cap D$, and the reciprocal quotient $|1-\tilde{p}_x(z)|/d(z,\partial D)$ is K-bounded in $U \cap D$.

Notice that this condition is local, because $d(z,\partial D) = d\bigl(z, \partial(D\cap U)\bigr)$ if $z \in D$ is close enough to x. Since the results we are seeking are local too, when using geometrical projection devices from now on *we shall assume* $U = \mathbb{C}^n$, so that $\tilde{p}_x$ is defined on the whole of D, effectively identifying D with $D \cap U$. For instance, results proved for strongly convex domains will apply immediately to strongly pseudoconvex domains.

Actually, geometric projection devices are very often bounded:

Lemma 4.1: *Let $D \subset \mathbb{C}^n$ be a domain having the one-point boundary estimates. Then every geometrical projection device in D is bounded.*

Proof: Let us first show that $d(z,\partial D)/|1-\tilde{p}_x(z)|$ is bounded in D. Indeed we have

$$-\tfrac{1}{2}\log|1-\tilde{p}_x(z)| \le -\tfrac{1}{2}\log(1-|\tilde{p}_x(z)|) \le \omega\bigl(0, \tilde{p}_x(z)\bigr) \le k_D(z_0, z) \le c_2 - \tfrac{1}{2}\log d(z,\partial D),$$

and thus $d(z,\partial D)/|1-\tilde p_x(z)|\le \exp(2c_2)$ for all $z\in D$.

To prove K-boundedness of the reciprocal, we first of all notice that $\tilde p_x$ is 1-Julia at x. Indeed,

$$\liminf_{z\to x}\bigl[k_D(z_0,z)-\omega\bigl(0,\tilde p_x(z)\bigr)\bigr]\le \liminf_{\zeta\to 1}\bigl[k_D\bigl(\varphi_x(0),\varphi_x(\zeta)\bigr)-\omega(0,\zeta)\bigr]=0.$$

Therefore we can apply Theorem 2.3, with $\tau=1$ because $\tilde p_x\circ\varphi_x=\mathrm{id}$. Take $z\in K^D_{z_0}(x,M)$, and define $R(z)$ by $k_D(z_0,z)=\frac12\log R(z)$. Then (2.11) and Theorem 2.3 yield $\tilde p_x(z)\in E\bigl(1,M^2/R(z)\bigr)$, and so

$$\tfrac12\log|1-\tilde p_x(z)|\le \tfrac12\log(2M^2)-k_D(z_0,z)\le \tfrac12\log(2M^2)+\tfrac12\log d(x,\partial D)-c_1,$$

that is $|1-\tilde p_x(z)|/d(z,\partial D)\le 2M^2\exp(-2c_1)$ for all $z\in K^D_{z_0}(x,M)$, as claimed. □

Now let D be a complete hyperbolic domain, and let $f\colon D\to\Delta$ be a bounded holomorphic function. If f is β-Julia at $x\in\partial D$, we know that it has K-limit $\tau\in\partial\Delta$ at x, and we want to discuss the boundary behavior of the partial derivatives of f.

If $v\in\mathbb{C}^n$, $v\ne O$, we shall write

$$\frac{\partial f}{\partial v}=\sum_{j=1}^n v_j\frac{\partial f}{\partial z_j}$$

for the partial derivative of f in the direction v. The idea is that the behavior of $\partial f/\partial v$ depends on the boundary behavior of the Kobayashi metric in the direction v. To be more specific, let us introduce the following definition:

Definition 4.2: Let $D\subset\mathbb{C}^n$ be a domain in $\mathbb{C}^n$, and $x\in\partial D$. The *Kobayashi class* $\mathbf{s}_x(v)$ and the *Kobayashi type* $s_x(v)$ of a nonzero vector $v\in\mathbb{C}^n$ at x are defined by

$$\mathbf{s}_x(v)=\{s\ge 0\mid d(z,\partial D)^s\kappa_D(z;v)\text{ is }K\text{-bounded at }x\},\quad\text{and}\quad s_x(v)=\inf\mathbf{s}_x(v). \tag{4.1}$$

Example 4.1: If ∂D is of class C^2 near x, it is easy to prove (see, e.g., [AT2]) that $s_x(v)\le 1$ for all $v\in\mathbb{C}^n$.

Example 4.2: If D is strongly pseudoconvex, Graham's estimates [G] show that $s_x(v)=1$ if v is transversal to ∂D at x, that is if v is not orthogonal to $\mathbb{C}\mathbf{n}_x$ (where $\mathbf{n}_x$ is the outer unit normal vector to ∂D at x, as before). On the other hand, $s_x(v)=1/2$ if $v\ne O$ is complex-tangential to ∂D at x, and in both cases $s_x(v)\in\mathbf{s}_x(v)$.

Example 4.3: If D is convex of finite type $L\ge 2$ at x then (see [AT2]) we have $s_x(v)=1\in\mathbf{s}_x(v)$ if v is transversal to ∂D, and $1/L\le s_x(v)\le 1-1/L$ if $v\ne O$ is complex tangential to ∂D.

Example 4.4: If $D\subset\mathbb{C}^2$ is pseudoconvex of finite type $L\ge 2$, then Catlin's estimates [Ca] show that $1/L\le s_x(v)\le 1/2$ when v is complex tangential, and $s_x(v)=1\in\mathbf{s}_x(v)$ as always if v is transversal.

Example 4.5: The Kobayashi metric of the polydisk is given by (see, e.g., [JP, Example 3.5.6])

$$\kappa_{\Delta^n}(z;v)=\max_{j=1,\dots,n}\left\{\frac{|v_j|}{1-|z_j|^2}\right\};$$

therefore $s_x(v)\le 1$ for all $x\in\partial\Delta^n$ and $v\in\mathbb{C}^n$, $v\ne O$. It is also clear that if $v_j=0$ when $|x_j|=1$, then $s_x(v)=0\in\mathbf{s}_x(v)$. On the other hand, if there is j such that $|x_j|=1$ and $v_j\ne 0$ then $s_x(v)=1\in\mathbf{s}_x(v)$. Indeed, by (2.9) we see that if $z\in K^{\Delta^n}_O(x,M)$ we have

$$\frac{1-\|z\|}{1-|z_j|}\ge\frac12\frac{1-\|z\|}{1+\|z\|}\frac{1-|z_j|^2}{|x_j-z_j|^2}>\frac{1}{2M^2}$$

for all j such that $|x_j|=1$. Therefore if $z\in K^{\Delta^n}_O(x,M)$ is close enough to x, setting

$$c=\min\{|v_j|\mid |x_j|=1,|v_j|\ne 0\}>0,$$

we have

$$(1-\|z\|)\kappa_{\Delta^n}(z;v) \ge \frac{c}{2} \max_{\substack{|x_j|=1 \\ |v_j|\ne 0}} \left\{ \frac{1-\|z\|}{1-|z_j|} \right\} > \frac{c}{4M^2} > 0,$$

and so $s_x(v) = 1 \in \mathbf{s}_x(v)$ as claimed.

These results seem to suggest that the Kobayashi type might be the inverse of the D'Angelo type of ∂D at x along the direction v (that is, the highest order of contact of ∂D with a complex curve tangent to v at x), but we do not even try to prove such a statement here. Another open question is whether $s_x(v)$ always belongs to $\mathbf{s}_x(v)$ or not.

We can now state a very general Julia-Wolff-Carathéodory theorem:

Theorem 4.2: *Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain equipped with a bounded geometrical projection device at $x \in \partial D$. Let $f \in \mathrm{Hol}(D, \Delta)$ be β-Julia at x, that is such that*

$$\liminf_{z\to x} \left[k_D(z_0, z) - \omega\big(0, f(z)\big)\right] = \tfrac{1}{2}\log\beta < +\infty.$$

Then for every $v \in \mathbb{C}^n$ and $s \in \mathbf{s}_x(v)$ the function

$$d(z, \partial D)^{s-1} \frac{\partial f}{\partial v} \tag{4.2}$$

is K-bounded. Furthermore, it has K-limit zero at x if $s > s_x(v)$.

This statements holds, for instance, in domains locally biholomorphic to C^2 convex or to convex circular domains.

Remark 4.1: The previous statement is optimal with regard to K-boundedness, but it is only asymptotically optimal with regard to the existence of the limit, as the proof will make clear. The more interesting limit case, that is the behavior of (4.2) when $s = s_x(v) \in \mathbf{s}_x(v)$, requires deeper tools. As we shall discuss later, in several instances (4.2) will admit *restricted* K-limit but it will not admit K-limit at x when $s = s_x(v)$: see Example 4.6. So it will be necessary to use all the machinery we discussed in the previous section. Furthermore, the specific tools we shall use to deal with this case will depend more strongly on the actual shape of the domain; a fully satisfying template approach to the limit case is yet to be found.

Example 4.6: This example, due to Rudin [Ru, 8.5.8], shows that we cannot expect the function (4.2) to be K-bounded, let alone to have a restricted K-limit, if $s < s_x(v)$. Let $\psi \in \mathrm{Hol}(\Delta, \Delta)$ be given by

$$\psi(\zeta) = \exp\left(-\frac{\pi}{2} - i\log(1-\zeta)\right).$$

As $\zeta \to 1$, the function $\psi(\zeta)$ spirals around the origin without limit; moreover,

$$\psi'(\zeta) = \frac{i}{1-\zeta}\psi(\zeta).$$

Let $f \in \mathrm{Hol}(B^2, \Delta)$ be given by

$$f(z_1, z_2) = z_1 + \frac{1}{2} z_2^2 \psi(z_1).$$

Then f is 1-Julia at $x = (1, 0)$, and admits K-limit 1 at x. But

$$\frac{\partial f}{\partial z_1} = 1 + \frac{i z_2^2}{2(1-z_1)}\psi(z_1), \qquad \frac{\partial f}{\partial z_2} = z_2 \psi(z_1);$$

therefore $\partial f/\partial z_1$ has restricted K-limit 1 at x while $d(z, \partial B^2)^{s-1}\partial f/\partial z_1$ blows-up at x for all $s < 1$. Similarly, $d(z, \partial B^2)^{-1/2}\partial f/\partial z_2$ has restricted K-limit 0 at x but $d(z, \partial B^2)^{s-1}\partial f/\partial z_2$ is not K-bounded if $s < 1/2$. Notice furthermore that the K-limit of $\partial f/\partial z_1$ at x does not exist.

We can now start with the

Proof of Theorem 4.2: Take $s \in \mathbf{s}_x(v)$. We shall argue mimicking the proof of the one-dimensional Julia-Wolff-Carathéodory theorem. We shall first show that a sort of incremental ratio is K-bounded, and then we shall use a integral formula to prove K-boundedness of the partial derivative. After that, deriving the existence of the K-limit will be easy.

Let us first show that an incremental ratio is K-bounded. Take $z \in K^D_{z_0}(x,M)$ and set

$$\tfrac{1}{2}\log R(z) = \log M - k_D(z_0,z)$$

so that $z \in E^D_{z_0}\big(x,R(z)\big)$. Then $f(z) \in E\big(\tau,\beta R(z)\big)$, which implies as usual that

$$|\tau - f(z)| \le 2\beta R(z).$$

On the other hand,

$$\tfrac{1}{2}\log R(z) \le \log M - \omega\big(0,\tilde p_x(z)\big) \le \tfrac{1}{2}\log\big[M^2|1-\tilde p_x(z)|\big],$$

and so

$$\left|\frac{\tau - f(z)}{1-\tilde p_x(z)}\right| \le 2\beta M^2. \tag{4.3}$$

Now the integral formula. Take $z \in K^D_{z_0}(x,M)$. Since D is complete hyperbolic (and hence taut), we can find a holomorphic map $\psi\colon\Delta\to D$ with $\psi(0)=z$ and $\psi'(0)=v/\kappa_D(z;v)$. Then for any $r\in(0,1)$ we can write

$$\begin{aligned} d(z,\partial D)^{s-1}\frac{\partial f}{\partial v}(z) &= d(z,\partial D)^{s-1}\kappa_D(z;v)(f\circ\psi)'(0) = d(z,\partial D)^{s-1}\frac{\kappa_D(z;v)}{2\pi i}\int\limits_{|\zeta|=r}\frac{f\big(\psi(\zeta)\big)}{\zeta^2}\,d\zeta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{f\big(\psi(re^{i\theta})\big)-\tau}{\tilde p_x\big(\psi(re^{i\theta})\big)-1}\,\frac{\tilde p_x\big(\psi(re^{i\theta})\big)-1}{\tilde p_x(z)-1}\left(\frac{\tilde p_x(z)-1}{d(z,\partial D)}\right)^s\frac{d(z,\partial D)^s\kappa_D(z;v)}{re^{i\theta}}\,d\theta. \end{aligned} \tag{4.4}$$

We must then prove that the four factors in the integrand are bounded as z varies in $K^D_{z_0}(x,M)$.

The last factor is K-bounded by the choice of s. The third factor is K-bounded because the projection device is bounded. To prove that the other two factors are K-bounded we shall need the following two lemmas:

Lemma 4.3: *Let $D\subset\mathbb{C}^n$ be a domain equipped with a geometrical projection device at x. Then $\tilde p_x\big(K^D_{z_0}(x,M)\big)\subseteq K(1,M)$ for all $M>1$.*

Proof: Take $z\in K^D_{z_0}(x,M)$. Then

$$\begin{aligned} \omega\big(\tilde p_x(z),\zeta\big)-\omega(0,\zeta)+\omega\big(0,\tilde p_x(\zeta)\big) &\le k_D\big(z,\varphi_x(\zeta)\big)-\omega(0,\zeta)+k_D(z_0,z)\\ &= k_D\big(z,\varphi_x(\zeta)\big)-k_D\big(z_0,\varphi_x(\zeta)\big)+k_D(z_0,z)+k_D\big(z_0,\varphi_x(\zeta)\big)-\omega(0,\zeta)\\ &\le k_D\big(z,\varphi_x(\zeta)\big)-k_D\big(z_0,\varphi_x(\zeta)\big)+k_D(z_0,z), \end{aligned}$$

and taking the lim sup as $\zeta\to1$ we get the assertion. □

Lemma 4.4: *Let $D\subset\mathbb{C}^n$ be a complete hyperbolic domain, $z_0\in D$, and $x\in\partial D$. If $M_1>M>1$ set*

$$r=\frac{M_1-M}{M_1+M}. \tag{4.5}$$

Then $\psi(\overline{\Delta_r})\subset K^D_{z_0}(x,M_1)$ for all holomorphic maps $\psi\colon\Delta\to D$ such that $\psi(0)\in K^D_{z_0}(x,M)$.

Proof: Take $\zeta\in\overline{\Delta_r}$. Then

$$\begin{aligned} &\limsup_{w\to x}\big[k_D\big(\psi(\zeta),w\big)-k_D(z_0,w)\big]+k_D\big(z_0,\psi(\zeta)\big)\\ &\qquad\le 2k_D\big(\psi(\zeta),\psi(0)\big)+\limsup_{w\to x}\big[k_D\big(\psi(0),w\big)-k_D(z_0,w)\big]+k_D\big(z_0,\psi(0)\big)\\ &\qquad< 2\omega(0,\zeta)+\log M\le\log\left(M\frac{1+r}{1-r}\right)=\log M_1, \end{aligned}$$

that is $\psi(\zeta)\in K^D_{z_0}(x,M_1)$. □

Now we can deal with the first two factors in (4.4). Choose $M_1 > M$ and r as in (4.5). Then $\psi(re^{i\theta}) \in K^D_{z_0}(x, M_1)$ for all $\theta \in [0, 2\pi]$, and (4.3) yields

$$\left| \frac{f\bigl(\psi(re^{i\theta})\bigr) - \tau}{\tilde{p}_x\bigl(\psi(re^{i\theta})\bigr) - 1} \right| \le 2\beta M_1^2,$$

that is the first factor is bounded too. Finally, Lemmas 4.3 and 4.4 imply that

$$\left| \frac{1 - \tilde{p}_x\bigl(\psi(\zeta)\bigr)}{1 - \tilde{p}_x(z)} \right| \le M_1 \frac{1 - \bigl|\tilde{p}_x\bigl(\psi(\zeta)\bigr)\bigr|}{1 - |\tilde{p}_x(z)|} \le \frac{M_1}{2} \frac{1 + |\tilde{p}_x(z)|}{1 - |\tilde{p}_x(z)|} \frac{1 - \bigl|\tilde{p}_x\bigl(\psi(\zeta)\bigr)\bigr|}{1 + \bigl|\tilde{p}_x\bigl(\psi(\zeta)\bigr)\bigr|}$$

for all $\zeta \in \overline{\Delta_r}$. Therefore

$$\begin{aligned} \tfrac{1}{2} \log \left| \frac{1 - \tilde{p}_x\bigl(\psi(\zeta)\bigr)}{1 - \tilde{p}_x(z)} \right| &\le \tfrac{1}{2} \log \frac{M_1}{2} + \omega\bigl(0, \tilde{p}_x(z)\bigr) - \omega\bigl(0, \tilde{p}_x\bigl(\psi(\zeta)\bigr)\bigr) \\ &\le \tfrac{1}{2} \log \frac{M_1}{2} + \omega\bigl(\tilde{p}_x\bigl(\psi(0)\bigr), \tilde{p}_x\bigl(\psi(\zeta)\bigr)\bigr) \le \tfrac{1}{2} \log \frac{M_1}{2} + \omega(0, r), \end{aligned}$$

and the second factor is bounded too.

We are left to show that (4.2) has K-limit 0 when $s > s_x(v)$. But indeed choose $s_x(v) < s_1 < s$; then we can write

$$d(z, \partial D)^{s-1} \frac{\partial f}{\partial v}(z) = d(z, \partial D)^{s - s_1} \left[d(z, \partial D)^{s_1 - 1} \frac{\partial f}{\partial v}(z) \right],$$

and thus it converges to 0 as $z \to x$ inside $K^D_{z_0}(x, M)$. □

Let us now discuss what happens when $s = s_x(v) \in \mathbf{s}_x(v)$. As anticipated before, we shall need to apply Lindelöf principles and the material of the previous section. We shall still prove some general results, but the deeper theorems will work for specific classes of domains only.

We begin dealing with directions transversal to the boundary. If f being β-Julia at $x \in \partial D$ would imply $f \circ \varphi_x$ being β-Julia at 1, one could try to apply the classical Julia-Wolff-Carathéodory theorem to $f \circ \varphi_x$. It turns out that this approach is viable when the projection device preserves horospheres:

Lemma 4.5: *Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain equipped with a projection device at $x \in \partial D$ preserving horospheres. Then for any $f \in \mathrm{Hol}(D, \Delta)$ which is β-Julia at x, the composition $f \circ \varphi_x$ is β-Julia at 1. In particular, $(f \circ \varphi_x)'$ has non-tangential limit $\beta\tau$ at 1, where $\tau \in \partial\Delta$ is the K-limit of f at x.*

Proof: Since f is β-Julia and the projection device preserves horospheres we have

$$f \circ \varphi_x\bigl(E(1, R)\bigr) \subseteq E(\tau, \beta R)$$

for all $R > 0$. Then Lemma 1.4 yields

$$\begin{aligned} \liminf_{z \to x} \bigl[k_D(z_0, z) - \omega\bigl(0, f(z)\bigr)\bigr] = \tfrac{1}{2} \log \beta &\ge \liminf_{\zeta \to 1} \bigl[\omega(0, \zeta) - \omega\bigl(0, f\bigl(\varphi_x(\zeta)\bigr)\bigr)\bigr] \\ &\ge \liminf_{\zeta \to 1} \bigl[k_D\bigl(z_0, \varphi_x(\zeta)\bigr) - \omega\bigl(0, f\bigl(\varphi_x(\zeta)\bigr)\bigr)\bigr] \\ &\ge \liminf_{z \to x} \bigl[k_D(z_0, z) - \omega\bigl(0, f(z)\bigr)\bigr], \end{aligned}$$

and we are done. □

This is enough to deal with transversal directions, under the mild hypotheses that $s_x(v) \le 1$ for all $v \in \mathbb{C}^n$ (which might be possibly true in all complete hyperbolic domains) and that the radial limit of φ_x' at 1 exists (this happens in convex domains of finite type or in convex circular domains, for instance). Indeed we have

Corollary 4.6: *Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain equipped with a good bounded geometrical projection device at $x \in \partial D$ preserving horospheres. Assume moreover that $1 \in \mathbf{s}_x(v)$ for all $v \in \mathbb{C}^n$, $v \neq O$, and that the radial limit $\nu_x = \varphi_x'(1)$ of φ_x' at 1 exists. Finally, let $f \in \mathrm{Hol}(D, \Delta)$ be β-Julia at x, and denote by $\tau \in \partial\Delta$ its K-limit at x. Then:*

(i) *$s_x(\nu_x) = 1$ and $\partial f/\partial \nu_x$ has restricted K-limit $\beta\tau \neq 0$ at x;*
(ii) *if moreover $s_x(v_T) < 1$ for all $v_T \neq O$ orthogonal to ν_x, then for all v_N not orthogonal to ν_x the function $\partial f/\partial v_N$ has non-zero restricted K-limit at x, and $s_x(v_N) = 1$.*

Proof: The previous lemma implies that $(f \circ \varphi_x)'$ has radial limit $\beta\tau \neq 0$ at 1. Now write

$$\frac{\partial f}{\partial \nu_x}\bigl(\varphi_x(t)\bigr) = df_{\varphi_x(t)}(\nu_x) = (f \circ \varphi_x)'(t) + df_{\varphi_x(t)}\bigl(\nu_x - \varphi_x'(t)\bigr). \tag{4.6}$$

Since $1 \in \mathbf{s}_x(v)$ for all $v \in \mathbb{C}^n$, Theorem 4.2 implies that the norm of df_z is K-bounded at x; therefore (4.6) yields

$$\lim_{t \to 1^-} \frac{\partial f}{\partial \nu_x}\bigl(\varphi_x(t)\bigr) = \beta\tau,$$

and (i) follows from Theorems 3.2 and 4.2, because $\beta\tau \neq 0$. If v_N is not orthogonal to ν_x, we can write $v_N = \lambda\nu_x + v_T$ with $\lambda \neq 0$ and v_T orthogonal to ν_x. Therefore

$$\frac{\partial f}{\partial v_N} = \lambda \frac{\partial f}{\partial \nu_x} + \frac{\partial f}{\partial v_T},$$

and (ii) follows from (i) and Theorem 4.2. □

We recall that for this statement to hold it is necessary to use restricted K-limits and not K-limits: see Example 4.6.

Remark 4.2: If D is strongly convex or convex of finite type then ([Le], [AT2]) ν_x is a complex multiple of $\mathbf{n}_x$. Therefore in these cases "orthogonal to ν_x" means "complex tangential to ∂D at x", and "not orthogonal to ν_x" means "transversal to ∂D at x". But the previous corollary does not a priori require any smoothness on ∂D.

Remark 4.3: As it stands, Corollary 4.6 applies for instance to projection devices that are the localization of the canonical projection device in strongly convex domains, or of the canonical projection device in strictly linearly convex domains of finite type. However, once this statement holds for a projection device, one might derive similar statements for not necessarily geometrical projection devices. For instance, in [A4] it is shown how to derive a similar result for the localization of the euclidean projection device in strongly pseudoconvex domains knowing Corollary 4.6 for the (geometrical, good and bounded) localization of the canonical projection device.

If D is a convex Reinhardt domain, we can use a completely different approach, not depending on the degree of smoothness of the boundary. We already noticed that in this case the canonical projection device is bounded and preserves horospheres; if moreover it is good (as, for instance, in the polydisk) we have the following

Theorem 4.7: *Let $D \subset\subset \mathbb{C}^n$ be a convex Reinhardt domain, equipped with the canonical projection device at $x \in \partial D$ with $\varphi_x(\zeta) = \zeta x$. Assume that this projection device is good. Let $f \in \mathrm{Hol}(D, \Delta)$ be β-Julia at x, and let $v \in \mathbb{C}^n$ be such that $s_x(v) = 1 \in \mathbf{s}_x(v)$. Then $\partial f/\partial v$ has restricted K-limit at x.*

Proof: By Theorems 3.2 and 4.2 it suffices to prove that $\partial f/\partial v(tx)$ converges as $t \to 1^-$.

Let M be the set of all n-uple of natural numbers $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $k_1, \ldots, k_n$ relatively prime and $|k| = k_1 + \cdots + k_n > 0$. For $z \in \mathbb{C}^n$ and $k \in \mathbb{N}^n$ we write $z^k = z_1^{k_1} \cdots z_n^{k_n}$. For each $k \in M$ we choose a point $y(k) \in \partial D$ such that

$$\max\{|x^k| \mid x \in \partial D\} = |y(k)^k|,$$

and we set

$$\Sigma_k = \{x \in \partial D \mid |x_j| = |y_j(k)| \text{ for } j = 1, \ldots, n\}.$$

For instance, if $D = \Delta^n$ we can take $y(k) = (1, \dots, 1)$ and $\Sigma_k = (\partial\Delta)^n$ for all $k \in M$.

Let $\tau \in \partial\Delta$ be the K-limit of f at x. Since the function $(\tau + f)/(\tau - f)$ has positive real part, the generalized Herglotz representation formula proved in [KK] yields

$$\frac{\tau + f(z)}{\tau - f(z)} = \sum_{k\in M} \left[\int\limits_{\Sigma_k} \frac{w^k + z^k}{w^k - z^k}\, d\mu_k(w) + C_k \right], \tag{4.7}$$

for suitable $C_k \in \mathbb{C}$ and positive Borel measures μ_k on Σ_k; the sum is absolutely converging.

Let $X_k = \{w \in \Sigma_k \mid w^k = x^k\}$, and set $\beta_k = \mu_k(X_k) \ge 0$ and $\mu_k^o = \mu_k - \mu_k|_{X_k}$, where $\mu_k|_{X_k}$ is the restriction of μ_k to X_k (i.e., $\mu_k|_{X_k}(E) = \mu_k(E \cap X_k)$ for every Borel subset E).

Using these notations (4.7) becomes

$$\frac{\tau + f(z)}{\tau - f(z)} = \sum_{k\in M} \left[\beta_k \frac{x^k + z^k}{x^k - z^k} + \int\limits_{\Sigma_k} \frac{w^k + z^k}{w^k - z^k}\, d\mu_k^o(w) + C_k \right]. \tag{4.8}$$

In particular, if $z = tx = \varphi_x(t)$ we get

$$\frac{\tau + f(tx)}{\tau - f(tx)} = \sum_{k\in M} \left[\beta_k \frac{1 + t^{|k|}}{1 - t^{|k|}} + \int\limits_{\Sigma_k} \frac{w^k + t^{|k|}x^k}{w^k - t^{|k|}x^k}\, d\mu_k^o(w) + C_k \right]. \tag{4.9}$$

Let us multiply both sides by $(1-t)$, and then take the limit as $t \to 1^-$. The left-hand side, by Lemma 4.5 and Proposition 1.6, tends to $2\tau/\beta$. For the right-hand side, first of all we have

$$\frac{1 + t^{|k|}}{1 - t^{|k|}}(1-t) = \frac{1 + t^{|k|}}{1 + \cdots + t^{|k|-1}} \to \frac{2}{|k|}.$$

Next, if $|x^k| < |y(k)^k|$ it is clear that

$$(1-t) \int\limits_{\Sigma_k} \frac{w^k + t^{|k|}x^k}{w^k - t^{|k|}x^k}\, d\mu_k^o(w) \to 0. \tag{4.10}$$

Otherwise, since $\mu_k^o(X_k) = 0$, for every $\varepsilon > 0$ there exists an open neighborhood A_ε of X_k in Σ_k such that $\mu_k^o(A_\varepsilon) < \varepsilon$. Then

$$\begin{aligned}
(1-t) &\left| \int\limits_{\Sigma_k} \frac{w^k + t^{|k|}x^k}{w^k - t^{|k|}x^k}\, d\mu_k^o(w) \right| \\
&= (1-t) \left| \int\limits_{A_\varepsilon} \frac{w^k + t^{|k|}x^k}{w^k - t^{|k|}x^k}\, d\mu_k^o(w) + \int\limits_{\Sigma_k \setminus A_\varepsilon} \frac{w^k + t^{|k|}x^k}{w^k - t^{|k|}x^k}\, d\mu_k^o(w) \right| \\
&\le 2\,\frac{1-t}{1-t^{|k|}}\varepsilon + (1-t) \left| \int\limits_{\Sigma_k \setminus A_\varepsilon} \frac{w^k + t^{|k|}x^k}{w^k - t^{|k|}x^k}\, d\mu_k^o(w) \right| \to \frac{2}{|k|}\varepsilon.
\end{aligned}$$

Since this happens for all $\varepsilon > 0$, it follows that (4.10) holds in this case too. Summing up, we have found

$$\frac{\tau}{\beta} = \sum_{k\in M} \frac{\beta_k}{|k|}; \tag{4.11}$$

in particular, the series on the right-hand side is converging.

Without loss of generality, we can assume that $v = \partial/\partial z_1$. Then differentiating (4.8) with respect to z_1 we get

$$\frac{\partial f}{\partial z_1}(z) = \overline{\tau}\big(\tau - f(z)\big)^2 \sum_{k\in M} k_1 \frac{z^k}{z_1}\left[\beta_k \frac{x^k}{(x^k - z^k)^2} + \int\limits_{\Sigma_k} \frac{w^k}{(w^k - z^k)^2}\, d\mu_k^o(w)\right]. \tag{4.12}$$

Then we have

$$\begin{aligned}\frac{\partial f}{\partial z_1}(tx) = \overline{\tau}\left(\frac{\tau - f(tx)}{1-t}\right)^2 \sum_{k\in M} k_1 t^{|k|-1}\overline{x_1}\Bigg[\beta_k\left(\frac{1-t}{1-t^{|k|}}\right)^2 \\ + x^k(1-t)^2 \int\limits_{\Sigma_k} \frac{w^k}{(w^k - t^{|k|}x^k)^2}\, d\mu_k^o(w)\Bigg].\end{aligned}$$

The same argument used before shows that

$$(1-t)^2 \int\limits_{\Sigma_k} \frac{w^k}{(w^k - t^{|k|}x^k)^2}\, d\mu_k^o(w) \to 0$$

as $t \to 1^-$. Therefore

$$\lim_{t\to 1^-} \frac{\partial f}{\partial z_1}(tx) = \beta^2 \tau \overline{x_1} \sum_{k\in M} \beta_k \frac{k_1}{|k|^2}, \tag{4.13}$$

where the series is converging because $\beta_k k_1/|k|^2 \le \beta_k/|k|$, and we are done. □

We end this survey by presenting two results for the case of complex tangential directions.

In the polydisk we have seen that either $s_x(v) = 1$ or $s_x(v) = 0$ for any $v \in \mathbb{C}^n$, and in the latter case $\kappa_D(z; v)$ is bounded for z close to x. The former case is dealt with in the previous theorem; but even in the latter case (which is the embodiment of "complex tangential" for the polydisk) we can prove that $\partial f/\partial v$ behaves as we expect:

Proposition 4.8: *Let $f \in \mathrm{Hol}(\Delta^n, \Delta)$ be β-Julia at $x \in \partial\Delta^n$. Take $v \in \mathbb{C}^n$ such that $v_j = 0$ when $|x_j| = 1$, so that $s_x(v) = 0 \in \mathbf{s}_x(v)$. Then*

$$d(z, \partial D)^{-1} \frac{\partial f}{\partial v}(z)$$

has restricted K-limit 0 at x.

Proof: Since the canonical projection device is bounded, it suffices to prove that the holomorphic function

$$\big(1 - \tilde{p}_x(z)\big)^{-1} \frac{\partial f}{\partial v}(z) \tag{4.14}$$

has restricted K-limit 0; we recall that, in this case, $\tilde{p}_x\colon \Delta^n \to \Delta$ is given by (see [A5])

$$\tilde{p}_x(z) = \frac{1}{d_x}(z, \check{x}),$$

where d_x is the number of components of x of modulus 1, and

$$\hat{x}_j = \begin{cases} x_j & \text{if } |x_j| = 1, \\ 0 & \text{if } |x_j| < 1. \end{cases}$$

Therefore it is enough to prove that (4.14) is K-bounded and that it tends to 0 when restricted to the radial curve $t \mapsto tx$.

We argue as in the proof of Theorem 4.2. For $j = 1, \ldots, n$ and $z \in \Delta^n$ set

$$w_j(z) = \frac{v_j}{(1-|z_j|^2)\kappa_{\Delta^n}(z;v)}$$

and define $\psi_z \in \mathrm{Hol}(\Delta, \Delta^n)$ by

$$\psi_z(\zeta) = \left(\frac{\zeta w_1 + z_1}{1 + \overline{z_1} w_1 \zeta}, \ldots, \frac{\zeta w_n + z_n}{1 + \overline{z_n} w_n \zeta} \right).$$

Then we have $\psi_z(0) = z$ and $\psi_z'(0) = v/\kappa_{\Delta^n}(z;v)$, so that we can write

$$\begin{aligned} \big(1 - \tilde{p}_x(z)\big)^{-1} \frac{\partial f}{\partial v}(z) &= \big(1 - \tilde{p}_x(z)\big)^{-1} \kappa_{\Delta^n}(z;v)(f \circ \psi_z)'(0) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f\big(\psi_z(re^{i\theta})\big) - \tau}{\tilde{p}_x\big(\psi_z(re^{i\theta})\big) - 1} \, \frac{\tilde{p}_x\big(\psi_z(re^{i\theta})\big) - 1}{\tilde{p}_x(z) - 1} \, \frac{\kappa_{\Delta^n}(z;v)}{re^{i\theta}} \, d\theta, \end{aligned} \tag{4.15}$$

for any $r \in (0,1)$. The proof of Theorem 4.2 then shows that (4.14) is K-bounded as well as all the factors in the integrand.

Now let $z = \varphi_x(t) = tx$. Then Lemma 4.5, Theorem 3.2 and Proposition 1.6 imply that the first factor in the integrand converges to $\beta\tau$ as $t \to 1^-$, where $\tau \in \partial\Delta$ is the K-limit of f at x. The choice of v implies that

$$\lim_{t \to 1^-} \kappa_{\Delta^n}(tx;v) = \max_{|x_j|<1} \left\{ \frac{|v_j|}{1 - |x_j|^2} \right\} = c < +\infty.$$

Finally, again the choice of v implies $\tilde{p}_x\big(\psi_{tx}(\zeta)\big) = t = \tilde{p}_x(tx)$ for all $t > 0$, and thus

$$\lim_{t \to 1^-} \big(1 - \tilde{p}_x(tx)\big)^{-1} \frac{\partial f}{\partial v}(tx) = \frac{c\beta\tau}{2r\pi} \int_0^{2\pi} \frac{d\theta}{e^{i\theta}} = 0,$$

and the assertion follows from Theorem 3.2. □

We remark that this result is more precise than [A5, Proposition 4.8].

We end with a final result in the case of complex tangential directions, whose proof has a fairly different flavor. We shall state the result for strongly convex domains only; but the same argument (due to Rudin [Ru, Proposition 8.5.7]) works in convex domains of finite type too (see [AT2]).

Proposition 4.9: *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain, equipped with the canonical projection device at $x \in \partial D$. Let $v \in \mathbb{C}^n$ be complex tangential to ∂D at x, so that $s_x(v) = 1/2 \in \mathbf{s}_x(v)$. Let $f \in \mathrm{Hol}(D, \Delta)$ be β-Julia at x. Then*

$$d(z, \partial D)^{-1/2} \frac{\partial f}{\partial v}(z)$$

has restricted K-limit 0 at x.

Proof: As usual, it suffices to show that

$$\lim_{t \to 1} \frac{1}{(1-t)^{1/2}} \frac{\partial f}{\partial v}\big(\varphi_x(t)\big) = 0. \tag{4.16}$$

We need some preparation. Consider the map $\Phi \colon \Delta \times \mathbb{C} \to \mathbb{C}^n$ given by

$$\Phi(\zeta, \eta) = \varphi_x(\zeta) + \eta v.$$

Clearly, $\Phi^{-1}(D) \cap (\mathbb{C} \times \{0\}) = \Delta$ and $\Phi^{-1}(D) \cap (\{\zeta\} \times \mathbb{C})$ is convex for all $\zeta \in \Delta$. Furthermore, since D is strongly convex, v is complex tangential to ∂D at x and $t \mapsto \varphi_x(t)$ is transversal, there is an euclidean ball $B \subset \Phi^{-1}(D)$ of center $(t_0, 0)$ and radius $1 - t_0$ for a suitable $t_0 \in (0,1)$.

Now define $\tilde{h}\colon B \to \Delta$ by

$$\tilde{h}(\zeta, \eta) = f\bigl(\Phi(\zeta, \eta)\bigr).$$

We remark that $\tilde{h}(\zeta, 0) = f\bigl(\varphi_x(\zeta)\bigr)$ and $\partial\tilde{h}(\zeta, 0)/\partial\zeta = \partial f\bigl(\varphi_x(\zeta)\bigr)/\partial v$. Hence we can write

$$\tilde{h}(\zeta, \eta) = f\bigl(\varphi_x(\zeta)\bigr) + \eta\,\frac{\partial f}{\partial v}\bigl(\varphi_x(\zeta)\bigr) + o(|\eta|).$$

Set

$$h(\zeta, \eta) = f\bigl(\varphi_x(\zeta)\bigr) + \tfrac{1}{2}\eta\,\frac{\partial f}{\partial v}\bigl(\varphi_x(\zeta)\bigr) = f\bigl(\varphi_x(\zeta)\bigr) + \eta(1-\zeta)^{1/2}g(\zeta),$$

where $g(\zeta) = \frac{1}{2}(1-\zeta)^{-1/2}\partial f\bigl(\varphi_x(\zeta)\bigr)/\partial v$. Since h is the arithmetic mean of the first two partial sums of the power series expansion of $\tilde{h}$, it sends B into Δ. Furthermore, (4.16) is equivalent to $g(t) \to 0$ as $t \to 1$.

Choose $\varepsilon > 0$ and set $c = \beta^2/\varepsilon^2(1-t_0)$. We wish to estimate

$$\limsup_{t\to 1}\bigl|g\bigl(t + ic(1-t)\bigr)\bigr|.$$

Set $\zeta_t = t + ic(1-t)$; it is easy to check that $(\zeta_t, 0) \in B$ if $(1-t) \le 2(1-t_0)/(1+c^2)$. Moreover

$$(1-t_0)^2 - |\zeta_t - t_0|^2 > (1-t_0)(1-t)$$

if $(1-t) < (1-t_0)/(1+c^2)$; hence if t is sufficiently close to 1 we can find $\eta_t \in \mathbb{C}$ such that

$$(1-t_0)^2 - |\zeta_t - t_0|^2 > |\eta_t|^2 > (1-t_0)(1-t) \tag{4.17}$$

and

$$\eta_t(1-\zeta_t)^{1/2}g(\zeta_t) \in \mathbb{R}. \tag{4.18}$$

In particular, $(\zeta_t, \eta_t) \in B$ if $(1-t) < (1-t_0)/(1+c^2)$. By definition,

$$|1-\zeta_t| = (1-t)\sqrt{1+c^2} \ge c\,(1-t);$$

hence (4.17) yields

$$|\eta_t(1-\zeta_t)^{1/2}g(\zeta_t)| \ge (1-t_0)^{1/2}c^{1/2}(1-t)|g(\zeta_t)|. \tag{4.19}$$

Now, $\zeta_t \in K(1, 2\sqrt{1+c^2})$ if $(1-t) < (1-t_0)/(1+c^2)$; hence, by Lemma 4.5 and Proposition 1.6,

$$\frac{1 - f\bigl(\varphi_x(\zeta_t)\bigr)}{1-\zeta_t} = \beta + o(1)$$

as $t \to 1$, where without loss of generality we have assumed that the K-limit of f at x is 1. Therefore

$$f\bigl(\varphi_x(\zeta_t)\bigr) = 1 - \bigl(\beta + o(1)\bigr)(1-ic)(1-t). \tag{4.20}$$

Putting together (4.18), (4.19) and (4.20) we get

$$1 \ge \mathrm{Re}[h(\zeta, \eta)] \ge 1 - \bigl(\beta + o(1)\bigr)(1-t) + (1-t_0)^{1/2}c^{1/2}(1-t)|g(\zeta_t)|,$$

that is

$$|g(\zeta_t)| \le \frac{\beta + o(1)}{(1-t_0)^{1/2}c^{1/2}}.$$

Therefore

$$\limsup_{t\to 1}\bigl|g\bigl(t + ic(1-t)\bigr)\bigr| \le \frac{\beta}{(1-t_0)^{1/2}c^{1/2}} = \varepsilon.$$

Clearly the same estimate holds for $\zeta'_t = t - ic(1-t)$. Since $|g(\zeta)|$ is bounded in the angular region bounded by these two lines, it follows that

$$\limsup_{t\to 1}|g(t)| \le \varepsilon.$$

Since ε is arbitrary, the assertion follows. $\square$

References

[A1] M. Abate: *Common fixed points of commuting holomorphic maps*. Math. Ann. **283** (1989), 645–655.

[A2] M. Abate: *The Lindelöf principle and the angular derivative in strongly convex domains*. J. Analyse Math. **54** (1990), 189–228.

[A3] M. Abate: **Iteration theory of holomorphic maps on taut manifolds.** Mediterranean Press, Cosenza, 1989.

[A4] M. Abate: *Angular derivatives in strongly pseudoconvex domains*. Proc. Symp. Pure Math. **52,** Part 2, (1991), 23–40.

[A5] M. Abate: *The Julia-Wolff-Carathéodory theorem in polydisks*. J. Anal. Math. **74** (1998), 275–306.

[AT1] M. Abate, R. Tauraso: *The Julia-Wolff-Carathéodory theorem(s)*. Contemp. Math. **222** (1999), 161–172.

[AT2] M. Abate, R. Tauraso: *The Lindelöf principle and angular derivatives in convex domains of finite type*. To appear in J. Austr. Math. Soc. **73** (2002).

[Ah] L. Ahlfors: **Conformal invariants.** McGraw-Hill, New York, 1973.

[BGS] W. Balmann, M. Gromov, V. Schroeder: **Manifolds of nonpositive curvature.** Birkhäuser, Basel, 1985.

[Ba] G. Bassanelli: *On horospheres and holomorphic endomorphisms of the Siegel disc*. Rend. Sem. Mat. Univ. Padova **70** (1983), 147–165.

[Bu] R.B. Burckel: **An introduction to classical complex analysis.** Academic Press, New York, 1979.

[C1] C. Carathéodory: *Über die Winkelderivierten von beschränkten analytischen Funktionen*. Sitz. Preuss. Akad. Wiss., Berlin (1929), 39–54.

[C2] C. Carathéodory: **Theory of functions of a complex variable, II.** Chelsea, New York, 1960.

[Ca] D.W. Catlin: *Estimates of invariant metrics on pseudoconvex domains of dimension two*. Math. Z. **200** (1989), 429–466.

[CK] J.A. Cima, S.G. Krantz: *The Lindelöf principle and normal functions of several complex variables*. Duke Math. J. **50** (1983), 303–328.

[Č] E.M. Čirka: *The Lindelöf and Fatou theorems in $\mathbb{C}^n$*. Math. USSR, Sb. **21** (1973), 619–641.

[D1] P.V. Dovbush: *Existence of admissible limits of functions of several complex variables*. Sib. Math. J. **28** (1987), 83–92.

[D2] P.V. Dovbush: *On the Lindelöf theorem in $\mathbb{C}^n$*. Ukrain. Math. Zh. **40** (1988), 796–799.

[DZ] Ju.N. Drožžinov, B.I. Zav'jalov: *On a multi-dimensional analogue of a theorem of Lindelöf*. Sov. Math. Dokl. **25** (1982), 51–52.

[EHRS] M. Elin, L.A. Harris, S. Reich, D. Shoikhet: *Evolution equations and geometric function theory in J^*-algebras*. J. Nonlinear Conv. Anal. **3** (2002), 81–121.

[F] K. Fan: *The angular derivative of an operator-valued analytic function*. Pacific J. Math. **121** (1986), 67–72.

[Fa] P. Fatou: *Séries trigonométriques et séries de Taylor*. Acta Math. **30** (1906), 335–400.

[FR] F. Forstneric, J.-P. Rosay: *Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings*. Math. Ann. **279** (1987), 239–252.

[G] I. Graham: *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary*. Trans. Am. Math. Soc. **207** (1975), 219–240.

[H1] K.T. Hahn: *Asymptotic behavior of normal mappings of several complex variables*. Canad. J. Math. **36** (1984), 718–746.

[H2] K.T. Hahn: *Nontangential limit theorems for normal mappings*. Pacific J. Math. **135** (1988), 57–64.

[He] M. Hervé: *Quelques propriétés des applications analytiques d'une boule à m dimensions dans elle-même*. J. Math. Pures Appl. **42** (1963), 117–147.

[J] H.L. Jackson: *Some remarks on angular derivatives and Julia's lemma*. Canad. Math. Bull. **9** (1966), 233–241.

[Ja] F. Jafari: *Angular derivatives in polydiscs*. Indian J. Math. **35** (1993), 197–212.

[JP] M. Jarnicki, P. Pflug: **Invariant distances and metrics in complex analysis.** Walter de Gruyter, Berlin, 1993.

[Ju1] G. Julia: *Extension nouvelle d'un lemme de Schwarz*. Acta Math. **42** (1920), 349–355.

[Ju2] G. Julia: **Principles géométriques d'analyse, I.** Gounod, Paris, 1930.

[K] Yu.V. Khurumov: *On Lindelöf's theorem in $\mathbb{C}^n$*. Soviet Math. Dokl. **28** (1983), 806–809.

[Kob1] S. Kobayashi: *Invariant distances on complex manifolds and holomorphic mappings*. J. Math. Soc. Japan **19** (1967), 460–480.

[Kob2] S. Kobayashi: **Hyperbolic complex spaces.** Springer, Berlin, 1998.

[Kom] Y. Komatu: *On angular derivative*. Kōdai Math. Sem. Rep. **13** (1961), 167–179.

[Kor] A. Korányi: *Harmonic functions on hermitian hyperbolic spaces*. Trans. Am. Math. Soc. **135** (1969), 507–516.

[KS] A. Korányi, E.M. Stein: *Fatou's theorem for generalized halfplanes*. Ann. Sc. Norm. Sup. Pisa **22** (1968), 107–112.

[KK] S. Kosbergenov, A.M. Kytmanov: *Generalizations of the Schwarz and Riesz-Herglotz formulas in Reinhardt domains*. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. n. 10 (1984), 60–63.

[Kr1] S.G. Krantz: *Invariant metrics and the boundary behavior of holomorphic functions on domains in $\mathbb{C}^n$*. J. Geom. Anal. **1** (1991), 71–97.

[Kr2] S.G. Krantz: *The boundary behavior of the Kobayashi metric*. Rocky Mountain J. Math. **22** (1992), 227–233.

[LV] E. Landau, G. Valiron: *A deduction from Schwarz's lemma*. J. London Math. Soc. **4** (1929), 162–163.

[Le] L. Lempert: *La métrique de Kobayashi et la représentation des domaines sur la boule*. Bull. Soc. Math. France **109** (1981), 427–474.

[Li1] E. Lindelöf: *Mémoire sur certaines inégalités dans la théorie des fonctions monogénes, et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel*. Acta Soc. Sci. Fennicae **35** (1909), 3–5.

[Li2] E. Lindelöf: *Sur un principe générale de l'analyse et ses applications à la theorie de la représentation conforme*. Acta Soc. Sci. Fennicae **46** (1915), 1–35.

[MM] M. Mackey, P. Mellon: *Angular derivatives on bounded symmetric domains*. Preprint, 2001.

[M] P. Mellon: *Holomorphic invariance on bounded symmetric domains*. J. Reine Angew. Math. **523** (2000), 199–223.

[Me] P.R. Mercer: *Another look at Julia's lemma*. Compl. Var. Th. Appl. **43** (2000), 129–138.

[Mi] A. Minialoff: *Sur une propriété des transformations dans l'espace de deux variables complexes*. C.R. Acad. Sci. Paris **200** (1935), 711–713.

[N] R. Nevanlinna: *Remarques sur le lemme de Schwarz*. C.R. Acad. Sci. Paris **188** (1929), 1027–1029.

[Po] C. Pommerenke: **Univalent functions.** Vandenhoeck & Ruprecht, Göttingen, 1975.

[R] A. Renaud: *Quelques propriétés des applications analytiques d'une boule de dimension infinie dans une autre*. Bull. Sci. Math. **97** (1973), 129–159.

[Ro] H.L. Royden: *Remarks on the Kobayashi metric*. In **Several Complex Variables, II,** Lect. Notes Math. 189, Springer, Berlin, 1971, pp. 125–137.

[RW] H.L. Royden, P.-M. Wong: *Carathéodory and Kobayashi metric on convex domains*. Preprint, 1983.

[Ru] W. Rudin: **Function theory in the unit ball of $\mathbb{C}^n$.** Berlin: Springer, 1980.

[SW] A. Szałowska, K. Włodarczyk: *Angular derivatives of holomorphic maps in infinite dimensions.* J. Math. Anal. Appl. **204** (1996), 1–28.

[St] E.M. Stein: **Boundary behavior of holomorphic functions of several complex variables.** Princeton University Press, Princeton, 1972.

[T] M. Tsuji: **Potential theory in modern function theory.** Maruzen, Tokyo, 1959.

[V1] E. Vesentini: *Variations on a theme of Carathéodory*. Ann. Sc. Norm. Sup. Pisa **6** (1979), 39–68.

[V2] E. Vesentini: *Complex geodesics*. Comp. Math. **44** (1981), 375–394.

[W] S. Wachs: *Sur quelques propriétés des transformations pseudo-conformes avec un point frontière invariant*. Bull. Soc. Math. Fr. **68** (1940), 177–198.

[Wł1] K. Włodarczyk: *Julia's lemma and Wolff's theorem for J^*-algebras*. Proc. Amer. Math. Soc. **99** (1987), 472–476.

[Wł2] K. Włodarczyk: *The Julia-Carathéodory theorem for distance decreasing maps on infinite-dimensional hyperbolic spaces*. Atti Accad. Naz. Lincei **4** (1993), 171–179.

[Wł3] K. Włodarczyk: *Angular limits and derivatives for holomorphic maps of infinite dimensional bounded homogeneous domains*. Atti Accad. Naz. Lincei **5** (1994), 43–53.

[Wł4] K. Włodarczyk: *The existence of angular derivatives of holomorphic maps of Siegel domains in a generalization of C^*-algebras*. Atti Accad. Naz. Lincei **5** (1994), 309–328.

[Wo] J. Wolff: *Sur une généralisation d'un théorème de Schwarz*. C.R. Acad. Sci. Paris **183** (1926), 500–502.

[Wu] H. Wu: *Normal families of holomorphic mappings*. Acta Math. **119** (1967), 193–233.

[Z] J.M. Zhu: *Angular derivatives of holomorphic maps in Hilbert spaces*. J. Math. (Wuhan) **19** (1999), 304–308.