

# Index theorems for holomorphic self-maps

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## Introduction

The usual index theorems for holomorphic self-maps, like for instance the classical holomorphic Lefschetz theorem (see, e.g., [GH]), assume that the fixed-points set contains only isolated points. The aim of this paper, on the contrary, is to prove index theorems for holomorphic self-maps having a positive dimensional fixed-points set.

The origin of our interest in this problem lies in holomorphic dynamics. A main tool for the complete generalization to two complex variables of the classical Leau-Fatou flower theorem for maps tangent to the identity achieved in [A2] was an index theorem for holomorphic self-maps of a complex surface fixing pointwise a smooth complex curve  $S$ . This theorem (later generalized in [BT] to the case of a singular  $S$ ) presented uncanny similarities with the Camacho-Sad index theorem for invariant leaves of a holomorphic foliation on a complex surface (see [CS]). So we started to investigate the reasons for these similarities; and this paper contains what we have found.

The main idea is that the simple fact of being pointwise fixed by a holomorphic self-map  $f$  induces a lot of structure on a (possibly singular) subvariety  $S$  of a complex manifold  $M$ . First of all, we shall introduce (in §3) a canonically defined holomorphic section  $X_f$  of the bundle  $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ , where  $N_S$  is the normal bundle of  $S$  in  $M$  (here we are assuming  $S$  smooth; however, we can also define  $X_f$  as a section of a suitable sheaf even when  $S$  is singular — see Remark 3.3 — but it turns out that only the behavior on the regular part of  $S$  is relevant for our index theorems), and  $\nu_f$  is a positive integer, the *order of contact* of  $f$  with  $S$ , measuring how close  $f$  is to being the identity in a neighborhood  $S$  (see §1). Roughly speaking, the section  $X_f$  describes the directions in which  $S$  is pushed by  $f$ ; see Proposition 8.1 for a more precise description of this phenomenon when  $S$  is a hypersurface.

The canonical section  $X_f$  can also be seen as a morphism from  $N_S^{\otimes \nu_f}$  to  $TM|_S$ ; its image  $\Xi_f$  is the *canonical distribution*. When  $\Xi_f$  is contained in  $TS$  (we shall say that  $f$  is *tangential*) and integrable (this happens for instance if  $S$  is a hypersurface), then on  $S$  we get a singular holomorphic

foliation induced by  $f$  — and this is a first concrete connection between our discrete dynamical theory and the continuous dynamics studied in foliation theory. We stress, however, that we get a well-defined foliation on  $S$  *only*, while in the continuous setting one usually assumes that  $S$  is invariant under a foliation defined in a *whole neighborhood* of  $S$ . Thus even in the tangential codimension-one case our results will not be a direct consequence of foliation theory.

As we shall momentarily discuss, to get index theorems it is important to have a section of  $TS \otimes (N_S^*)^{\otimes \nu_f}$  (as in the case when  $f$  is tangential) instead of merely a section of  $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ . In Section 3, when  $f$  is not tangential (which is a situation akin to dicriticality for foliations; see Propositions 1.4 and 8.1) we shall define other holomorphic sections  $H_{\sigma,f}$  and  $H_{\sigma,f}^1$  of  $TS \otimes (N_S^*)^{\otimes \nu_f}$  which are as good as  $X_f$  when  $S$  satisfies a geometric condition which we call *comfortably embedded* in  $M$ , meaning, roughly speaking, that  $S$  is a first-order approximation of the zero section of a vector bundle (see §2 for the precise definition, amounting to the vanishing of two sheaf cohomology classes — or, in other terms, to the triviality of two canonical extensions of  $N_S$ ).

The canonical section is not the only object we are able to associate to  $S$ . Having a section  $X$  of  $TS \otimes F^*$ , where  $F$  is any vector bundle on  $S$ , is equivalent to having an  $F^*$ -valued derivation  $X^\#$  of the sheaf of holomorphic functions  $\mathcal{O}_S$  (see §5). If  $E$  is another vector bundle on  $S$ , a *holomorphic action of  $F$  on  $E$  along  $X$*  is a  $\mathbb{C}$ -linear map  $\tilde{X}: \mathcal{E} \rightarrow \mathcal{F}^* \otimes \mathcal{E}$  (where  $\mathcal{E}$  and  $\mathcal{F}$  are the sheafs of germs of holomorphic sections of  $E$  and  $F$ ) satisfying  $\tilde{X}(gs) = X^\#(g) \otimes s + g\tilde{X}(s)$  for any  $g \in \mathcal{O}_S$  and  $s \in E$ ; this is a generalization of the notion of  $(1, 0)$ -connection on  $E$  (see Example 5.1).

In Section 5 we shall show that when  $S$  is a hypersurface and  $f$  is tangential (or  $S$  is comfortably embedded in  $M$ ) there is a natural way to define a holomorphic action of  $N_S^{\otimes \nu_f}$  on  $N_S$  along  $X_f$  (or along  $H_{\sigma,f}$  or  $H_{\sigma,f}^1$ ). And this will allow us to bring into play the general theory developed by Lehmann and Suwa (see, e.g., [Su]) on a cohomological approach to index theorems. So, exactly as Lehmann and Suwa generalized, to any dimension, the Camacho-Sad index theorem, we are able to generalize the index theorems of [A2] and [BT] in the following form (see Theorem 6.2):

**THEOREM 0.1.** *Let  $S$  be a compact, globally irreducible, possibly singular hypersurface in an  $n$ -dimensional complex manifold  $M$ . Let  $f: M \rightarrow M$ ,  $f \neq \text{id}_M$ , be a holomorphic self-map of  $M$  fixing pointwise  $S$ , and denote by  $\text{Sing}(f)$  the zero set of  $X_f$ . Assume that*

- (a)  *$f$  is tangential to  $S$ , and then set  $X = X_f$ , or that*
- (b)  *$S^0 = S \setminus (\text{Sing}(S) \cup \text{Sing}(f))$  is comfortably embedded into  $M$ , and then set  $X = H_{\sigma,f}$  if  $\nu_f > 1$ , or  $X = H_{\sigma,f}^1$  if  $\nu_f = 1$ .*

Assume moreover  $X \neq O$  (a condition always satisfied when  $f$  is tangential), and denote by  $\text{Sing}(X)$  the zero set of  $X$ . Let  $\text{Sing}(S) \cup \text{Sing}(X) = \bigcup_{\lambda} \Sigma_{\lambda}$  be the decomposition of  $\text{Sing}(S) \cup \text{Sing}(X)$  in connected components. Finally, let  $[S]$  be the line bundle on  $M$  associated to the divisor  $S$ . Then there exist complex numbers  $\text{Res}(X, S, \Sigma_{\lambda}) \in \mathbb{C}$  depending only on the local behavior of  $X$  and  $[S]$  near  $\Sigma_{\lambda}$  such that

$$\sum_{\lambda} \text{Res}(X, S, \Sigma_{\lambda}) = \int_S c_1^{n-1}([S]),$$

where  $c_1([S])$  is the first Chern class of  $[S]$ .

Furthermore, when  $\Sigma_{\lambda}$  is an isolated point  $\{x_{\lambda}\}$ , we have explicit formulas for the computation of the residues  $\text{Res}(X, S, \{x_{\lambda}\})$ ; see Theorem 6.5.

Since  $X$  is a global section of  $TS \otimes (N_S^*)^{\otimes \nu_f}$ , if  $S$  is smooth and  $X$  has only isolated zeroes it is well-known that the top Chern class  $c_{n-1}(TS \otimes (N_S^*)^{\otimes \nu_f})$  counts the zeroes of  $X$ . Our result shows that  $c_1^{n-1}(N_S)$  is related in a similar (but deeper) way to the zero set of  $X$ . See also Section 8 for examples of results one can obtain using both Chern classes together.

If the codimension of  $S$  is greater than one, and  $S$  is smooth, we can blow-up  $M$  along  $S$ ; then the exceptional divisor  $E_S$  is a hypersurface, and we can apply to it the previous theorem. In this way we get (see Theorem 7.2):

**THEOREM 0.2.** *Let  $S$  be a compact complex submanifold of codimension  $1 < m < n$  in an  $n$ -dimensional complex manifold  $M$ . Let  $f: M \rightarrow M$ ,  $f \neq \text{id}_M$ , be a holomorphic self-map of  $M$  fixing pointwise  $S$ , and assume that  $f$  is tangential, or that  $\nu_f > 1$  (or both). Let  $\bigcup_{\lambda} \Sigma_{\lambda}$  be the decomposition in connected components of the set of singular directions (see §7 for the definition) for  $f$  in  $E_S$ . Then there exist complex numbers  $\text{Res}(f, S, \Sigma_{\lambda}) \in \mathbb{C}$ , depending only on the local behavior of  $f$  and  $S$  near  $\Sigma_{\lambda}$ , such that*

$$\sum_{\lambda} \text{Res}(f, S, \Sigma_{\lambda}) = \int_S \pi_* c_1^{n-1}([E_S]),$$

where  $\pi_*$  denotes integration along the fibers of the bundle  $E_S \rightarrow S$ .

Theorems 0.1 and 0.2 are only two of the index theorems we can derive using this approach. Indeed, we are also able to obtain versions for holomorphic self-maps of two other main index theorems of foliation theory, the Baum-Bott index theorem and the Lehmann-Suwa-Khanedani (or variation) index theorem: see Theorems 6.3, 6.4, 6.6, 7.3 and 7.4. In other words, it turns out that the existence of holomorphic actions of suitable complex vector bundles defined only on  $S$  is an efficient tool to get index theorems, both in our setting and (under slightly different assumptions) in foliation theory; and this is another reason for the similarities noticed in [A2].

Finally, in Section 8 we shall present a couple of applications of our results to the discrete dynamics of holomorphic self-maps of complex surfaces, thus closing the circle and coming back to the arguments that originally inspired our work.

**1. The order of contact**

Let  $M$  be an  $n$ -dimensional complex manifold, and  $S \subset M$  an irreducible subvariety of codimension  $m$ . We shall denote by  $\mathcal{O}_M$  the sheaf of holomorphic functions on  $M$ , and by  $\mathcal{I}_S$  the subsheaf of functions vanishing on  $S$ . With a slight abuse of notations, we shall use the same symbol to denote both a germ at  $p$  and any representative defined in a neighborhood of  $p$ . We shall denote by  $TM$  the holomorphic tangent bundle of  $M$ , and by  $\mathcal{T}_M$  the sheaf of germs of local holomorphic sections of  $TM$ . Finally, we shall denote by  $\text{End}(M, S)$  the set of (germs about  $S$  of) holomorphic self-maps of  $M$  fixing  $S$  pointwise.

Let  $f \in \text{End}(M, S)$  be given,  $f \neq \text{id}_M$ , and take  $p \in S$ . For every  $h \in \mathcal{O}_{M,p}$  the germ  $h \circ f$  is well-defined, and we have  $h \circ f - h \in \mathcal{I}_{S,p}$ .

*Definition 1.1.* The  $f$ -order of vanishing at  $p$  of  $h \in \mathcal{O}_{M,p}$  is given by

$$\nu_f(h; p) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{I}_{S,p}^\mu\},$$

and the order of contact  $\nu_f(p)$  of  $f$  at  $p$  with  $S$  by

$$\nu_f(p) = \min\{\nu_f(h; p) \mid h \in \mathcal{O}_{M,p}\}.$$

We shall momentarily prove that  $\nu_f(p)$  does not depend on  $p$ .

Let  $(z^1, \dots, z^n)$  be local coordinates in a neighborhood of  $p$ . If  $h$  is any holomorphic function defined in a neighborhood of  $p$ , the definition of order of contact yields the important relation

$$(1.1) \quad h \circ f - h = \sum_{j=1}^n (f^j - z^j) \frac{\partial h}{\partial z^j} \pmod{\mathcal{I}_{S,p}^{2\nu_f(p)}},$$

where  $f^j = z^j \circ f$ .

As a consequence we have

LEMMA 1.1. (i) *Let  $(z^1, \dots, z^n)$  be any set of local coordinates at  $p \in S$ . Then*

$$\nu_f(p) = \min_{j=1, \dots, n} \{\nu_f(z^j; p)\}.$$

(ii) *For any  $h \in \mathcal{O}_{M,p}$  the function  $p \mapsto \nu_f(h; p)$  is constant in a neighborhood of  $p$ .*

(iii) *The function  $p \mapsto \nu_f(p)$  is constant.*

*Proof.* (i) Clearly,  $\nu_f(p) \leq \min_{j=1, \dots, n} \{\nu_f(z^j; p)\}$ . The opposite inequality follows from (1.1).

(ii) Let  $h \in \mathcal{O}_{M,p}$ , and choose a set  $\{\ell^1, \dots, \ell^k\}$  of generators of  $\mathcal{I}_{S,p}$ . Then we can write

$$(1.2) \quad h \circ f - h = \sum_{|I|=\nu_f(h;p)} \ell^I g_I,$$

where  $I = (i_1, \dots, i_k) \in \mathbb{N}^k$  is a  $k$ -multi-index,  $|I| = i_1 + \dots + i_k$ ,  $\ell^I = (\ell^1)^{i_1} \dots (\ell^k)^{i_k}$  and  $g_I \in \mathcal{O}_{M,p}$ . Furthermore, there is a multi-index  $I_0$  such that  $g_{I_0} \notin \mathcal{I}_{S,p}$ . By the coherence of the sheaf of ideals of  $S$ , the relation (1.2) holds for the corresponding germs at all points  $q \in S$  in a neighborhood of  $p$ . Furthermore,  $g_{I_0} \notin \mathcal{I}_{S,p}$  means that  $g_{I_0}|_S \neq 0$  in a neighborhood of  $p$ , and thus  $g_{I_0} \notin \mathcal{I}_{S,q}$  for all  $q \in S$  close enough to  $p$ . Putting these two observations together we get the assertion.

(iii) By (i) and (ii) we see that the function  $p \mapsto \nu_f(p)$  is locally constant and since  $S$  is connected, it is constant everywhere. □

We shall then denote by  $\nu_f$  the *order of contact* of  $f$  with  $S$ , computed at any point  $p \in S$ .

As we shall see, it is important to compare the order of contact of  $f$  with the  $f$ -order of vanishing of germs in  $\mathcal{I}_{S,p}$ .

*Definition 1.2.* We say that  $f$  is *tangential* at  $p$  if

$$\min\{\nu_f(h; p) \mid h \in \mathcal{I}_{S,p}\} > \nu_f.$$

LEMMA 1.2. *Let  $\{\ell^1, \dots, \ell^k\}$  be a set of generators of  $\mathcal{I}_{S,p}$ . Then*

$$\nu_f(h; p) \geq \min\{\nu_f(\ell^1; p), \dots, \nu_f(\ell^k; p), \nu_f + 1\}$$

for all  $h \in \mathcal{I}_{S,p}$ . In particular,  $f$  is tangential at  $p$  if and only if

$$\min\{\nu_f(\ell^1; p), \dots, \nu_f(\ell^k; p)\} > \nu_f.$$

*Proof.* Let us write  $h = g_1 \ell^1 + \dots + g_k \ell^k$  for suitable  $g_1, \dots, g_k \in \mathcal{O}_{M,p}$ . Then

$$h \circ f - h = \sum_{j=1}^k [(g_j \circ f)(\ell^j \circ f - \ell^j) + (g_j \circ f - g_j)\ell^j],$$

and the assertion follows. □

COROLLARY 1.3. *If  $f$  is tangential at one point  $p \in S$ , then it is tangential at all points of  $S$ .*

*Proof.* The coherence of the sheaf of ideals of  $S$  implies that if  $\{\ell^1, \dots, \ell^k\}$  are generators of  $\mathcal{I}_{S,p}$  then the corresponding germs are generators of  $\mathcal{I}_{S,q}$  for

all  $q \in S$  close enough to  $p$ . Then Lemmas 1.1.(ii) and 1.2 imply that both the set of points where  $f$  is tangential and the set of points where  $f$  is not tangential are open; hence the assertion follows because  $S$  is connected.  $\square$

Of course, we shall then say that  $f$  is *tangential* along  $S$  if it is tangential at any point of  $S$ .

*Example 1.1.* Let  $p$  be a smooth point of  $S$ , and choose local coordinates  $z = (z^1, \dots, z^n)$  defined in a neighborhood  $U$  of  $p$ , centered at  $p$  and such that  $S \cap U = \{z^1 = \dots = z^m = 0\}$ . We shall write  $z' = (z^1, \dots, z^m)$  and  $z'' = (z^{m+1}, \dots, z^n)$ , so that  $z''$  yields local coordinates on  $S$ . Take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ ; then in local coordinates the map  $f$  can be written as  $(f^1, \dots, f^n)$  with

$$f^j(z) = z^j + \sum_{h \geq 1} P_h^j(z', z''),$$

where each  $P_h^j$  is a homogeneous polynomial of degree  $h$  in the variables  $z'$ , with coefficients depending holomorphically on  $z''$ . Then Lemma 1.1 yields

$$\nu_f = \min\{h \geq 1 \mid \exists 1 \leq j \leq n : P_h^j \neq 0\}.$$

Furthermore,  $\{z^1, \dots, z^m\}$  is a set of generators of  $\mathcal{I}_{S,p}$ ; therefore by Lemma 1.2 the map  $f$  is tangential if and only if

$$\min\{h \geq 1 \mid \exists 1 \leq j \leq m : P_h^j \neq 0\} > \min\{h \geq 1 \mid \exists m + 1 \leq j \leq n : P_h^j \neq 0\}.$$

*Remark 1.1.* When  $S$  is smooth, the differential of  $f$  acts linearly on the normal bundle  $N_S$  of  $S$  in  $M$ . If  $S$  is a hypersurface,  $N_S$  is a line bundle, and the action is multiplication by a holomorphic function  $b$ ; if  $S$  is compact, this function is a constant. It is easy to check that in local coordinates chosen as in the previous example the expression of the function  $b$  is exactly  $1 + P_1^1(z)/z^1$ ; therefore we must have  $P_1^1(z) = (b_f - 1)z^1$  for a suitable constant  $b_f \in \mathbb{C}$ . In particular, if  $b_f \neq 1$  then necessarily  $\nu_f = 1$  and  $f$  is not tangential along  $S$ .

*Remark 1.2.* The number  $\mu$  introduced in [BT, (2)] is, by Lemma 1.1, our order of contact; therefore our notion of tangential is equivalent to the notion of nondegeneracy defined in [BT] when  $n = 2$  and  $m = 1$ . On the other hand, as already remarked in [BT], a nondegenerate map in the sense defined in [A2] when  $n = 2$ ,  $m = 1$  and  $S$  is smooth is tangential if and only if  $b_f = 1$  (which was the case mainly considered in that paper).

*Example 1.2.* A particularly interesting example (actually, the one inspiring this paper) of map  $f \in \text{End}(M, S)$  is obtained by blowing up a map tangent to the identity. Let  $f_o$  be a (germ of) holomorphic self-map of  $\mathbb{C}^n$  (or of any complex  $n$ -manifold) fixing the origin (or any other point) and *tangent to the*

identity, that is, such that  $d(f_o)_O = \text{id}$ . If  $\pi: M \rightarrow \mathbb{C}^n$  denotes the blow-up of the origin, let  $S = \pi^{-1}(O) \cong \mathbb{P}^{n-1}(\mathbb{C})$  be the exceptional divisor, and  $f \in \text{End}(M, S)$  the lifting of  $f_o$ , that is, the unique holomorphic self-map of  $M$  such that  $f_o \circ \pi = \pi \circ f$  (see, e.g., [A1] for details). If

$$f_o^j(w) = w^j + \sum_{h \geq 2} Q_h^j(w)$$

is the expansion of  $f_o^j$  in a series of homogeneous polynomials (for  $j = 1, \dots, n$ ), then in the canonical coordinates centered in  $p = [1 : 0 : \dots : 0]$  the map  $f$  is given by

$$f^j(z) = \begin{cases} z^1 + \sum_{h \geq 2} Q_h^1(1, z'')(z^1)^h & \text{for } j = 1, \\ z^j + \frac{\sum_{h \geq 2} [Q_h^j(1, z'') - z^j Q_h^1(1, z'')](z^1)^{h-1}}{1 + \sum_{h \geq 2} Q_h^1(1, z'')(z^1)^{h-1}} & \text{for } j = 2, \dots, n, \end{cases}$$

where  $z'' = (z^2, \dots, z^n)$ . Therefore  $b_f = 1$ ,

$$\nu_f(z^1; p) = \min\{h \geq 2 \mid Q_h^1(1, z'') \neq 0\},$$

and

$$\nu_f = \min\{\nu_f(z^1; p), \min\{h \geq 1 \mid \exists 2 \leq j \leq n : Q_{h+1}^j(1, z'') - z^j Q_{h+1}^1(1, z'') \neq 0\}\}.$$

Let  $\nu(f_o) \geq 2$  be the order of  $f_o$ , that is, the minimum  $h$  such that  $Q_h^j \neq 0$  for some  $1 \leq j \leq n$ . Clearly,  $\nu_f(z^1; p) \geq \nu(f_o)$  and  $\nu_f \geq \nu(f_o) - 1$ . More precisely, if there is  $2 \leq j \leq n$  such that  $Q_{\nu(f_o)}^j(1, z'') \neq z^j Q_{\nu(f_o)}^1(1, z'')$ , then  $\nu_f = \nu(f_o) - 1$  and  $f$  is tangential. If on the other hand we have  $Q_{\nu(f_o)}^j(1, z'') \equiv z^j Q_{\nu(f_o)}^1(1, z'')$  for all  $2 \leq j \leq n$ , then necessarily  $Q_{\nu(f_o)}^1(1, z'') \neq 0$ ,  $\nu_f(z^1; p) = \nu(f_o) = \nu_f$ , and  $f$  is not tangential.

Borrowing a term from continuous dynamics, we say that a map  $f_o$  tangent to the identity at the origin is *dicritical* if  $w^h Q_{\nu(f_o)}^k(w) \equiv w^k Q_{\nu(f_o)}^h(w)$  for all  $1 \leq h, k \leq n$ . Then we have proved that:

**PROPOSITION 1.4.** *Let  $f_o \in \text{End}(\mathbb{C}^n, O)$  be a (germ of) holomorphic self-map of  $\mathbb{C}^n$  tangent to the identity at the origin, and let  $f \in \text{End}(M, S)$  be its blow-up. Then  $f$  is not tangential if and only if  $f_o$  is dicritical. Furthermore,  $\nu_f = \nu(f_o) - 1$  if  $f_o$  is not dicritical, and  $\nu_f = \nu(f_o)$  if  $f_o$  is dicritical.*

In particular, most maps obtained with this procedure are tangential.

### 2. Comfortably embedded submanifolds

Up to now  $S$  was any complex subvariety of the manifold  $M$ . However, some of the proofs in the following sections do not work in this generality; so this section is devoted to describe the kind of properties we shall (sometimes) need on  $S$ .

Let  $E'$  and  $E''$  be two vector bundles on the same manifold  $S$ . We recall (see, e.g., [Ati, §1]) that an *extension* of  $E''$  by  $E'$  is an exact sequence of vector bundles

$$O \longrightarrow E' \xrightarrow{\iota} E \xrightarrow{\pi} E'' \longrightarrow O.$$

Two extensions are *equivalent* if there is an isomorphism of exact sequences which is the identity on  $E'$  and  $E''$ .

A *splitting* of an extension  $O \longrightarrow E' \xrightarrow{\iota} E \xrightarrow{\pi} E'' \longrightarrow O$  is a morphism  $\sigma: E'' \rightarrow E$  such that  $\pi \circ \sigma = \text{id}_{E''}$ . In particular,  $E = \iota(E') \oplus \sigma(E'')$ , and we shall say that the extension *splits*. We explicitly remark that an extension splits if and only if it is equivalent to the trivial extension  $O \rightarrow E' \rightarrow E' \oplus E'' \rightarrow E'' \rightarrow O$ .

Let  $S$  now be a complex submanifold of a complex manifold  $M$ . We shall denote by  $TM|_S$  the restriction to  $S$  of the tangent bundle of  $M$ , and by  $N_S = TM|_S/TS$  the normal bundle of  $S$  into  $M$ . Furthermore,  $\mathcal{T}_{M,S}$  will be the sheaf of germs of holomorphic sections of  $TM|_S$  (which is different from the restriction  $\mathcal{T}_M|_S$  to  $S$  of the sheaf of holomorphic sections of  $TM$ ), and  $\mathcal{N}_S$  the sheaf of germs of holomorphic sections of  $N_S$ .

*Definition 2.1.* Let  $S$  be a complex submanifold of codimension  $m$  in an  $n$ -dimensional complex manifold  $M$ . A chart  $(U_\alpha, z_\alpha)$  of  $M$  is *adapted* to  $S$  if either  $S \cap U_\alpha = \emptyset$  or  $S \cap U_\alpha = \{z_\alpha^1 = \dots = z_\alpha^m = 0\}$ , where  $z_\alpha = (z_\alpha^1, \dots, z_\alpha^n)$ . In particular,  $\{z_\alpha^1, \dots, z_\alpha^m\}$  is a set of generators of  $\mathcal{I}_{S,p}$  for all  $p \in S \cap U_\alpha$ . An atlas  $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$  of  $M$  is *adapted* to  $S$  if all charts in  $\mathfrak{U}$  are. If  $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$  is adapted to  $S$  we shall denote by  $\mathfrak{U}_S = \{(U''_\alpha, z''_\alpha)\}$  the atlas of  $S$  given by  $U''_\alpha = U_\alpha \cap S$  and  $z''_\alpha = (z_\alpha^{m+1}, \dots, z_\alpha^n)$ , where we are clearly considering only the indices such that  $U_\alpha \cap S \neq \emptyset$ . If  $(U_\alpha, z_\alpha)$  is a chart adapted to  $S$ , we shall denote by  $\partial_{\alpha,r}$  the projection of  $\partial/\partial z_\alpha^r|_{S \cap U_\alpha}$  in  $N_S$ , and by  $\omega_\alpha^r$  the local section of  $N_S^*$  induced by  $dz_\alpha^r|_{S \cap U_\alpha}$ ; thus  $\{\partial_{\alpha,1}, \dots, \partial_{\alpha,m}\}$  and  $\{\omega_\alpha^1, \dots, \omega_\alpha^m\}$  are local frames for  $N_S$  and  $N_S^*$  respectively over  $U_\alpha \cap S$ , dual to each other.

From now on, every chart and atlas we consider on  $M$  will be adapted to  $S$ .

*Remark 2.1.* We shall use the Einstein convention on the sum over repeated indices. Furthermore, indices like  $j, h, k$  will run from 1 to  $n$ ; indices like  $r, s, t, u, v$  will run from 1 to  $m$ ; and indices like  $p, q$  will run from  $m + 1$  to  $n$ .



*Definition 2.2.* We shall say that  $S$  splits into  $M$  if the extension  $O \rightarrow TS \rightarrow TM|_S \rightarrow N_S \rightarrow O$  splits.

*Example 2.1.* It is well-known that if  $S$  is a rational smooth curve with negative self-intersection in a surface  $M$ , then  $S$  splits into  $M$ .

**PROPOSITION 2.1.** *Let  $S$  be a complex submanifold of codimension  $m$  in an  $n$ -dimensional complex manifold  $M$ . Then  $S$  splits into  $M$  if and only if there is an atlas  $\hat{\mathfrak{U}} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$  adapted to  $S$  such that*

$$(2.1) \quad \left. \frac{\partial \hat{z}_\beta^p}{\partial \hat{z}_\alpha^r} \right|_S \equiv 0,$$

for all  $r = 1, \dots, m, p = m + 1, \dots, n$  and indices  $\alpha$  and  $\beta$ .

*Proof.* It is well known (see, e.g., [Ati, Prop. 2]) that there is a one-to-one correspondence between equivalence classes of extensions of  $N_S$  by  $TS$  and the cohomology group  $H^1(S, \text{Hom}(\mathcal{N}_S, \mathcal{T}_S))$ , and an extension splits if and only if it corresponds to the zero cohomology class.

The class corresponding to the extension  $O \rightarrow TS \rightarrow TM|_S \rightarrow N_S \rightarrow O$  is the class  $\delta(\text{id}_{N_S})$ , where  $\delta: H^0(S, \text{Hom}(\mathcal{N}_S, \mathcal{N}_S)) \rightarrow H^1(S, \text{Hom}(\mathcal{N}_S, \mathcal{T}_S))$  is the connecting homomorphism in the long exact sequence of cohomology associated to the short exact sequence obtained by applying the functor  $\text{Hom}(\mathcal{N}_S, \cdot)$  to the extension sequence. More precisely, if  $\mathfrak{U}$  is an atlas adapted to  $S$ , we get a local splitting morphism  $\sigma_\alpha: N_{U''_\alpha} \rightarrow TM|_{U''_\alpha}$  by setting  $\sigma_\alpha(\partial_{r,\alpha}) = \partial/\partial z^r_\alpha$ , and then the element of  $H^1(\mathfrak{U}_S, \text{Hom}(\mathcal{N}_S, \mathcal{T}_S))$  associated to the extension is  $\{\sigma_\beta - \sigma_\alpha\}$ . Now,

$$(\sigma_\beta - \sigma_\alpha)(\partial_{r,\alpha}) = \left. \frac{\partial z^s_\beta}{\partial z^r_\alpha} \right|_S \frac{\partial}{\partial z^s_\beta} - \frac{\partial}{\partial z^r_\alpha} = \left. \frac{\partial z^s_\beta}{\partial z^r_\alpha} \frac{\partial z^p_\alpha}{\partial z^s_\beta} \right|_S \frac{\partial}{\partial z^p_\alpha}.$$

So, if (2.1) holds, then  $S$  splits into  $M$ . Conversely, assume that  $S$  splits into  $M$ ; then we can find an atlas  $\mathfrak{U}$  adapted to  $S$  and a 0-cochain  $\{c_\alpha\} \in H^0(\mathfrak{U}_S, \mathcal{T}_S \otimes \mathcal{N}_S^*)$  such that

$$(2.2) \quad \left. \frac{\partial z^s_\beta}{\partial z^r_\alpha} \frac{\partial z^p_\alpha}{\partial z^s_\beta} \right|_S = (c_\beta)_s^q \left. \frac{\partial z^s_\beta}{\partial z^r_\alpha} \frac{\partial z^p_\alpha}{\partial z^q_\beta} \right|_S - (c_\alpha)_r^p$$

on  $U_\alpha \cap U_\beta \cap S$ . We claim that the coordinates

$$(2.3) \quad \begin{cases} \hat{z}_\alpha^r = z^r_\alpha, \\ \hat{z}_\alpha^p = z^p_\alpha + (c_\alpha)_s^p (z''_\alpha)^s z^s_\alpha \end{cases}$$

satisfy (2.1) when restricted to suitable open sets  $\hat{U}_\alpha \subseteq U_\alpha$ . Indeed, (2.2) yields

$$\begin{aligned} \frac{\partial z_\beta^p}{\partial \hat{z}_\alpha^r} &= \frac{\partial z_\beta^p}{\partial z_\alpha^s} \frac{\partial z_\alpha^s}{\partial \hat{z}_\alpha^r} + \frac{\partial z_\beta^p}{\partial z_\alpha^q} \frac{\partial z_\alpha^q}{\partial \hat{z}_\alpha^r} = \frac{\partial z_\beta^p}{\partial z_\alpha^r} - (c_\alpha)_r^q \frac{\partial z_\beta^p}{\partial z_\alpha^q} + R_1 \\ &= \frac{\partial z_\beta^p}{\partial z_\alpha^r} + (c_\beta)_s^p \frac{\partial z_\beta^s}{\partial z_\alpha^r} - (c_\alpha)_r^q \frac{\partial z_\beta^p}{\partial z_\alpha^q} + R_1 = R_1, \end{aligned}$$

where  $R_1$  denotes terms vanishing on  $S$ , and we are done. □

*Definition 2.3.* Assume that  $S$  splits into  $M$ . An atlas  $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$  adapted to  $S$  and satisfying (2.1) will be called a *splitting atlas* for  $S$ . It is easy to see that for any splitting morphism  $\sigma: N_S \rightarrow TM|_S$  there exists a splitting atlas  $\mathfrak{U}$  such that  $\sigma(\partial_{r,\alpha}) = \partial/\partial z_\alpha^r$  for all  $r = 1, \dots, m$  and indices  $\alpha$ ; we shall say that  $\mathfrak{U}$  is *adapted* to  $\sigma$ .

*Example 2.2.* A *local holomorphic retraction* of  $M$  onto  $S$  is a holomorphic retraction  $\rho: W \rightarrow S$ , where  $W$  is a neighborhood of  $S$  in  $M$ . It is clear that the existence of such a local holomorphic retraction implies that  $S$  splits into  $M$ .

*Example 2.3.* Let  $\pi: M \rightarrow S$  be a rank  $m$  holomorphic vector bundle on  $S$ . If we identify  $S$  with the zero section of the vector bundle,  $\pi$  becomes a (global) holomorphic retraction of  $M$  on  $S$ . The charts given by the trivialization of the bundle clearly give a splitting atlas. Furthermore, if  $(U_\alpha, z_\alpha)$  and  $(U_\beta, z_\beta)$  are two such charts, we have  $z''_\beta = \varphi_{\beta\alpha}(z''_\alpha)$  and  $z'_\beta = a_{\beta\alpha}(z''_\alpha)z'_\alpha$ , where  $a_{\beta\alpha}$  is an invertible matrix depending only on  $z''_\alpha$ . In particular, we have

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \equiv 0 \quad \text{and} \quad \frac{\partial^2 z_\beta^r}{\partial z_\alpha^s \partial z_\alpha^t} \equiv 0$$

for all  $r, s, t = 1, \dots, m, p = m + 1, \dots, n$  and indices  $\alpha$  and  $\beta$ .

The previous example, compared with (2.1), suggests the following

*Definition 2.4.* Let  $S$  be a codimension  $m$  complex submanifold of an  $n$ -dimensional complex manifold  $M$ . We say that  $S$  is *comfortably embedded* in  $M$  if  $S$  splits into  $M$  and there exists a splitting atlas  $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$  such that

$$(2.4) \quad \left. \frac{\partial^2 z_\beta^r}{\partial z_\alpha^s \partial z_\alpha^t} \right|_S \equiv 0$$

for all  $r, s, t = 1, \dots, m$  and indices  $\alpha$  and  $\beta$ .

An atlas satisfying the previous condition is said to be *comfortable* for  $S$ . Roughly speaking, then, a comfortably embedded submanifold is like a first-order approximation of the zero section of a vector bundle.

Let us express condition (2.4) in a different way. If  $(U_\alpha, z_\alpha)$  and  $(U_\beta, z_\beta)$  are two charts about  $p \in S$  adapted to  $S$ , we can write

$$(2.5) \quad z_\beta^r = (a_{\beta\alpha})_s^r z_\alpha^s$$

for suitable  $(a_{\beta\alpha})_s^r \in \mathcal{O}_{M,p}$ . The germs  $(a_{\beta\alpha})_s^r$  (unless  $m = 1$ ) are not uniquely determined by (2.5); indeed, all the other solutions of (2.5) are of the form  $(a_{\beta\alpha})_s^r + e_s^r$ , where the  $e_s^r$ 's are holomorphic and satisfy

$$(2.6) \quad e_s^r z_\alpha^s \equiv 0.$$

Differentiating with respect to  $z_\alpha^t$  we get

$$(2.7) \quad e_t^r + \frac{\partial e_s^r}{\partial z_\alpha^t} z_\alpha^s \equiv 0;$$

in particular,  $e_t^r|_S \equiv 0$ , and so the restriction of  $(a_{\beta\alpha})_s^r$  to  $S$  is uniquely determined — and it indeed gives the 1-cocycle of the normal bundle  $N_S$  with respect to the atlas  $\mathfrak{U}_S$ .

Differentiating (2.7) we obtain

$$(2.8) \quad \frac{\partial e_t^r}{\partial z_\alpha^s} + \frac{\partial e_s^r}{\partial z_\alpha^t} + \frac{\partial^2 e_u^r}{\partial z_\alpha^s \partial z_\alpha^t} z_\alpha^u \equiv 0;$$

in particular,

$$\left[ \frac{\partial e_t^r}{\partial z_\alpha^s} + \frac{\partial e_s^r}{\partial z_\alpha^t} \right] \Big|_S \equiv 0,$$

and so the restriction of

$$\frac{\partial(a_{\beta\alpha})_t^r}{\partial z_\alpha^s} + \frac{\partial(a_{\beta\alpha})_s^r}{\partial z_\alpha^t}$$

to  $S$  is uniquely determined for all  $r, s, t = 1, \dots, m$ .

With this notation, we have

$$\frac{\partial^2 z_\beta^r}{\partial z_\alpha^s \partial z_\alpha^t} = \frac{\partial(a_{\beta\alpha})_s^r}{\partial z_\alpha^t} + \frac{\partial(a_{\beta\alpha})_t^r}{\partial z_\alpha^s} + \frac{\partial^2(a_{\beta\alpha})_u^r}{\partial z_\alpha^s \partial z_\alpha^t} z_\alpha^u,$$

therefore (2.4) is equivalent to requiring

$$(2.9) \quad \left( \frac{\partial(a_{\beta\alpha})_t^r}{\partial z_\alpha^s} + \frac{\partial(a_{\beta\alpha})_s^r}{\partial z_\alpha^t} \right) \Big|_S \equiv 0$$

for all  $r, s, t = 1, \dots, m$ , and indices  $\alpha$  and  $\beta$ .

*Example 2.4.* It is easy to check that the exceptional divisor  $S$  in Example 1.2 is comfortably embedded into the blow-up  $M$ .

Then the main result of this section is

**THEOREM 2.2.** *Let  $S$  be a codimension  $m$  complex submanifold of an  $n$ -dimensional complex manifold  $M$ . Assume that  $S$  splits into  $M$ , and let  $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$  be a splitting atlas. Define a 1-cochain  $\{h_{\beta\alpha}\}$  of  $\mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*$  by setting*

$$(2.10) \quad \begin{aligned} h_{\beta\alpha} &= \frac{1}{2} \frac{\partial z_\alpha^r}{\partial z_\beta^u} \frac{\partial^2 z_\beta^u}{\partial z_\alpha^s \partial z_\alpha^t} \Big|_S \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t \\ &= \frac{1}{2} (a_{\alpha\beta})_u^r \left( \frac{\partial (a_{\beta\alpha})_s^u}{\partial z_\alpha^t} + \frac{\partial (a_{\beta\alpha})_t^u}{\partial z_\alpha^s} \right) \Big|_S \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t. \end{aligned}$$

Then:

- (i)  $\{h_{\beta\alpha}\}$  defines an element  $[h] \in H^1(S, \mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*)$  independent of  $\mathfrak{U}$ ;
- (ii)  $S$  is comfortably embedded in  $M$  if and only if  $[h] = 0$ .

*Proof.* (i) Let us first prove that  $\{h_{\beta\alpha}\}$  is a 1-cocycle with values in  $\mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*$ . We know that

$$(a_{\alpha\beta})_u^r (a_{\beta\alpha})_s^u = \delta_s^r + e_s^r,$$

where  $\delta_s^r$  is Kronecker's delta, and the  $e_s^r$ 's satisfy (2.6). Differentiating we get

$$\frac{\partial (a_{\alpha\beta})_u^r}{\partial z_\alpha^t} (a_{\beta\alpha})_s^u + (a_{\alpha\beta})_u^r \frac{\partial (a_{\beta\alpha})_s^u}{\partial z_\alpha^t} = \frac{\partial e_s^r}{\partial z_\alpha^t};$$

therefore (2.8) yields

$$(a_{\beta\alpha})_s^u \frac{\partial (a_{\alpha\beta})_u^r}{\partial z_\alpha^t} \Big|_S + (a_{\beta\alpha})_t^u \frac{\partial (a_{\alpha\beta})_u^r}{\partial z_\alpha^s} \Big|_S = -(a_{\alpha\beta})_u^r \left( \frac{\partial (a_{\beta\alpha})_s^u}{\partial z_\alpha^t} + \frac{\partial (a_{\beta\alpha})_t^u}{\partial z_\alpha^s} \right) \Big|_S.$$

Hence

$$\begin{aligned} h_{\alpha\beta} &= \frac{1}{2} (a_{\beta\alpha})_u^r \left( \frac{\partial (a_{\alpha\beta})_s^u}{\partial z_\beta^t} + \frac{\partial (a_{\alpha\beta})_t^u}{\partial z_\beta^s} \right) \Big|_S \partial_{\beta,r} \otimes \omega_\beta^s \otimes \omega_\beta^t \\ &= \frac{1}{2} (a_{\beta\alpha})_u^r (a_{\alpha\beta})_{r_1}^{r_1} (a_{\beta\alpha})_{s_1}^s (a_{\beta\alpha})_{t_1}^t \\ &\quad \times \left( (a_{\alpha\beta})_{t_2}^{t_2} \frac{\partial (a_{\alpha\beta})_s^u}{\partial z_\alpha^{t_2}} + (a_{\alpha\beta})_{s_2}^{s_2} \frac{\partial (a_{\alpha\beta})_t^u}{\partial z_\alpha^{s_2}} \right) \Big|_S \partial_{\alpha,r_1} \otimes \omega_\alpha^{s_1} \otimes \omega_\alpha^{t_1} \\ &= \frac{1}{2} \left( (a_{\beta\alpha})_{s_1}^s \frac{\partial (a_{\alpha\beta})_{s_1}^{r_1}}{\partial z_\alpha^{t_1}} + (a_{\beta\alpha})_{t_1}^t \frac{\partial (a_{\alpha\beta})_{t_1}^{r_1}}{\partial z_\alpha^{s_1}} \right) \Big|_S \partial_{\alpha,r_1} \otimes \omega_\alpha^{s_1} \otimes \omega_\alpha^{t_1} \\ &= -h_{\beta\alpha}, \end{aligned}$$

where in the second equality we used (2.1). Analogously one proves that  $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0$ , and thus  $\{h_{\beta\alpha}\}$  is a 1-cocycle as claimed.

Now we have to prove that the cohomology class  $[h]$  is independent of the atlas  $\mathfrak{U}$ . Let  $\hat{\mathfrak{U}} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$  be another splitting atlas; up to taking a common

refinement we can assume that  $U_\alpha = \hat{U}_\alpha$  for all  $\alpha$ . Choose  $(A_\alpha)_s^r \in \mathcal{O}(U_\alpha)$  so that  $\hat{z}_\alpha^r = (A_\alpha)_s^r z_\alpha^s$ ; as usual, the restrictions to  $S$  of  $(A_\alpha)_s^r$  and of

$$\frac{\partial(A_\alpha)_s^r}{\partial z_\alpha^t} + \frac{\partial(A_\alpha)_t^r}{\partial z_\alpha^s}$$

are uniquely defined. Set, now,

$$C_\alpha = \frac{1}{2}(A_\alpha^{-1})_u^r \left[ \frac{\partial(A_\alpha)_s^u}{\partial z_\alpha^t} + \frac{\partial(A_\alpha)_t^u}{\partial z_\alpha^s} \right] \Big|_S \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t;$$

then it is not difficult to check that

$$h_{\beta\alpha} - \hat{h}_{\beta\alpha} = C_\beta - C_\alpha,$$

where  $\{\hat{h}_{\beta\alpha}\}$  is the 1-cocycle built using  $\hat{\mathcal{U}}$ , and this means exactly that both  $\{h_{\beta\alpha}\}$  and  $\{\hat{h}_{\beta\alpha}\}$  determine the same cohomology class.

(ii) If  $S$  is comfortably embedded, using a comfortable atlas we immediately see that  $[h] = 0$ . Conversely, assume that  $[h] = 0$ ; therefore we can find a splitting atlas  $\mathcal{U}$  and a 0-cochain  $\{c_\alpha\}$  of  $\mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*$  such that  $h_{\beta\alpha} = c_\alpha - c_\beta$ . Writing

$$c_\alpha = (c_\alpha)_{st}^r \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t,$$

with  $(c_\alpha)_{ts}^r$  symmetric in the lower indices, we define  $\hat{z}_\alpha$  by setting

$$\begin{cases} \hat{z}_\alpha^r = z_\alpha^r + (c_\alpha)_{st}^r(z_\alpha^s) z_\alpha^t & \text{for } r = 1, \dots, m, \\ \hat{z}_\alpha^p = z_\alpha^p & \text{for } p = m + 1, \dots, n, \end{cases}$$

on a suitable  $\hat{U}_\alpha \subseteq U_\alpha$ . Then  $\hat{\mathcal{U}} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$  clearly is a splitting atlas; we claim that it is comfortable too. Indeed, by definition the functions

$$(\hat{a}_{\beta\alpha})_s^r = [\delta_u^r + (c_\beta)_{uv}^r (a_{\beta\alpha})_t^v z_\alpha^t] (a_{\beta\alpha})_{u_1}^u d_s^{u_1}$$

satisfy (2.5) for  $\hat{\mathcal{U}}$ , where the  $d_s^{u_1}$ 's are such that  $z_\alpha^{u_1} = d_s^{u_1} \hat{z}_\alpha^s$ . Hence

$$\begin{aligned} \left( \frac{\partial(\hat{a}_{\beta\alpha})_s^r}{\partial \hat{z}_\alpha^t} + \frac{\partial(\hat{a}_{\beta\alpha})_t^r}{\partial \hat{z}_\alpha^s} \right) \Big|_S &= 2(c_\beta)_{uv}^r (a_{\beta\alpha})_s^u (a_{\beta\alpha})_t^v \Big|_S + \left( \frac{\partial(a_{\beta\alpha})_s^r}{\partial z_\alpha^t} + \frac{\partial(a_{\beta\alpha})_t^r}{\partial z_\alpha^s} \right) \Big|_S \\ &\quad + (a_{\beta\alpha})_u^r \left( \frac{\partial d_s^u}{\partial z_\alpha^t} + \frac{\partial d_t^u}{\partial z_\alpha^s} \right) \Big|_S. \end{aligned}$$

Now, differentiating

$$z_\alpha^u = d_v^u (z_\alpha^v + (c_\alpha)_{rs}^v z_\alpha^r z_\alpha^s)$$

we get

$$\delta_t^u = \frac{\partial d_v^u}{\partial z_\alpha^t} (z_\alpha^v + (c_\alpha)_{rs}^v z_\alpha^r z_\alpha^s) + d_v^u (\delta_t^v + 2(c_\alpha)_{rt}^v z_\alpha^r)$$

and

$$0 = \left( \frac{\partial d_s^u}{\partial z_\alpha^t} + \frac{\partial d_t^u}{\partial z_\alpha^s} \right) \Big|_S + 2(c_\alpha)_{st}^u.$$

Recalling that  $h_{\beta\alpha} = c_\alpha - c_\beta$  we then see that  $\hat{\mathcal{U}}$  satisfies (2.9), and we are done.  $\square$

*Remark 2.2.* Since  $N_S \otimes N_S^* \otimes N_S^* \cong \text{Hom}(N_S, \text{Hom}(N_S, N_S))$ , the previous theorem asserts that to any submanifold  $S$  splitting into  $M$  we can canonically associate an extension

$$O \rightarrow \text{Hom}(N_S, N_S) \rightarrow E \rightarrow N_S \rightarrow O$$

of  $N_S$  by  $\text{Hom}(N_S, N_S)$ , and  $S$  is comfortably embedded in  $M$  if and only if this extension splits. See also [ABT] for more details on comfortably embedded submanifolds.

### 3. The canonical sections

Our next aim is to associate to any  $f \in \text{End}(M, S)$  different from the identity a section of a suitable vector bundle, indicating (very roughly speaking) how  $f$  would move  $S$  if it did not keep it fixed. To do so, in this section we still assume that  $S$  is a smooth complex submanifold of a complex manifold  $M$ ; however, in Remark 3.3 we shall describe the changes needed to avoid this assumption.

Given  $f \in \text{End}(M, S)$ ,  $f \not\equiv \text{id}_M$ , it is clear that  $df|_{TS} = \text{id}$ ; therefore  $df - \text{id}$  induces a map from  $N_S$  to  $TM|_S$ , and thus a holomorphic section over  $S$  of the bundle  $TM|_S \otimes N_S^*$ . If  $(U, z)$  is a chart adapted to  $S$ , we can define germs  $g_r^h$  for  $h = 1, \dots, n$  and  $r = 1, \dots, m$  by writing

$$z^h \circ f - z^h = z^1 g_1^h + \dots + z^m g_m^h.$$

It is easy to check that the germ of the section of  $TM|_S \otimes N_S^*$  defined by  $df - \text{id}$  is locally expressed by

$$g_r^h|_{U \cap S} \frac{\partial}{\partial z^h} \otimes \omega^r,$$

where we are again indicating by  $\omega^r$  the germ of section of the conormal bundle induced by the 1-form  $dz^r$  restricted to  $S$ .

A problem with this section is that it vanishes identically if (and only if)  $\nu_f > 1$ . The solution consists in expanding  $f$  at a higher order.

*Definition 3.1.* Given a chart  $(U, z)$  adapted to  $S$ , set  $f^j = z^j \circ f$ , and write

$$(3.1) \quad f^j - z^j = z^{r_1} \dots z^{r_{\nu_f}} g_{r_1 \dots r_{\nu_f}}^j,$$

where the  $g_{r_1 \dots r_{\nu_f}}^j$ 's are symmetric in  $r_1, \dots, r_{\nu_f}$  and do not all vanish restricted to  $S$ . Let us then define

$$(3.2) \quad \mathcal{X}_f = g_{r_1 \dots r_{\nu_f}}^h \frac{\partial}{\partial z^h} \otimes dz^{r_1} \otimes \dots \otimes dz^{r_{\nu_f}}.$$

This is a local section of  $TM \otimes (T^*M)^{\otimes \nu_f}$ , defined in a neighborhood of a point of  $S$ ; furthermore, when restricted to  $S$ , it induces a local section of  $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ .

*Remark 3.1.* When  $m > 1$  the  $g_{r_1 \dots r_{\nu_f}}^j$ 's are *not* uniquely determined by (3.1). Indeed, if  $e_{r_1 \dots r_{\nu_f}}^j$  are such that

$$(3.3) \quad e_{r_1 \dots r_{\nu_f}}^j z^1 \dots z^{r_{\nu_f}} \equiv 0$$

then  $g_{r_1 \dots r_{\nu_f}}^j + e_{r_1 \dots r_{\nu_f}}^j$  still satisfies (3.1). This means that the section (3.2) is not uniquely determined too; but, as we shall see, this will not be a problem. For instance, (3.3) implies that  $e_{r_1 \dots r_{\nu_f}}^j \in \mathcal{I}_S$ ; therefore  $\mathcal{X}_f|_{U \cap S}$  is always uniquely determined — though *a priori* it might depend on the chosen chart. On the other hand, when  $m = 1$  both the  $g_{r_1 \dots r_{\nu_f}}^j$ 's and  $\mathcal{X}_f$  are uniquely determined; this is one of the reasons making the codimension-one case simpler than the general case.

We have already remarked that when  $\nu_f = 1$  the section  $\mathcal{X}_f$  restricted to  $U \cap S$  coincides with the restriction of  $df - \text{id}$  to  $S$ . Therefore when  $\nu_f = 1$  the restriction of  $\mathcal{X}_f$  to  $S$  gives a globally well-defined section. Actually, this holds for any  $\nu_f \geq 1$ :

**PROPOSITION 3.1.** *Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . Then the restriction of  $\mathcal{X}_f$  to  $S$  induces a global holomorphic section  $X_f$  of the bundle  $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ .*

*Proof.* Let  $(U, z)$  and  $(\hat{U}, \hat{z})$  be two charts about  $p \in S$  adapted to  $S$ . Then we can find holomorphic functions  $a_s^r$  such that

$$(3.4) \quad \hat{z}^r = a_s^r z^s;$$

in particular,

$$(3.5) \quad \frac{\partial \hat{z}^r}{\partial z^s} = a_s^r \pmod{\mathcal{I}_S} \quad \text{and} \quad \frac{\partial \hat{z}^r}{\partial z^p} = 0 \pmod{\mathcal{I}_S}.$$

Now set  $f^j = z^j \circ f$ ,  $\hat{f}^j = \hat{z}^j \circ f$ , and define  $g_{r_1 \dots r_{\nu_f}}^j$  and  $\hat{g}_{r_1 \dots r_{\nu_f}}^j$  using (3.1) with  $(U, z)$  and  $(\hat{U}, \hat{z})$  respectively. Then (3.4) and (1.1) yield

$$\begin{aligned} a_{s_1}^{r_1} \dots a_{s_{\nu_f}}^{r_{\nu_f}} \hat{g}_{r_1 \dots r_{\nu_f}}^j z^{s_1} \dots z^{s_{\nu_f}} &= \hat{g}_{r_1 \dots r_{\nu_f}}^j \hat{z}^{r_1} \dots \hat{z}^{r_{\nu_f}} \\ &= \hat{f}^j - \hat{z}^j = (f^h - z^h) \frac{\partial \hat{z}^j}{\partial z^h} + R_{2\nu_f} \\ &= g_{s_1 \dots s_{\nu_f}}^h \frac{\partial \hat{z}^j}{\partial z^h} z^{s_1} \dots z^{s_{\nu_f}} + R_{2\nu_f}, \end{aligned}$$

where the remainder terms  $R_{2\nu_f}$  belong to  $\mathcal{I}_S^{2\nu_f}$ . Therefore we find

$$(3.6) \quad a_{s_1}^{r_1} \dots a_{s_{\nu_f}}^{r_{\nu_f}} \hat{g}_{r_1 \dots r_{\nu_f}}^j = \frac{\partial \hat{z}^j}{\partial z^h} g_{s_1 \dots s_{\nu_f}}^h \pmod{\mathcal{I}_S}.$$

Recalling (3.5) we then get

$$\begin{aligned} & \hat{g}_{r_1 \dots r_{\nu_f}}^j \frac{\partial}{\partial \hat{z}^j} \otimes d\hat{z}^{r_1} \otimes \dots \otimes d\hat{z}^{r_{\nu_f}} \\ &= \frac{\partial z^h}{\partial \hat{z}^j} \frac{\partial \hat{z}^{r_1}}{\partial z^{k_1}} \dots \frac{\partial \hat{z}^{r_{\nu_f}}}{\partial z^{k_{\nu_f}}} \hat{g}_{r_1 \dots r_{\nu_f}}^j \frac{\partial}{\partial z^h} \otimes dz^{k_1} \otimes \dots \otimes dz^{k_{\nu_f}} \\ &= a_{s_1}^{r_1} \dots a_{s_{\nu_f}}^{r_{\nu_f}} \hat{g}_{r_1 \dots r_{\nu_f}}^j \frac{\partial z^h}{\partial \hat{z}^j} \frac{\partial}{\partial z^h} \otimes dz^{s_1} \otimes \dots \otimes dz^{s_{\nu_f}} \pmod{\mathcal{I}_S} \\ &= g_{s_1 \dots s_{\nu_f}}^h \frac{\partial}{\partial z^h} \otimes dz^{s_1} \otimes \dots \otimes dz^{s_{\nu_f}} \pmod{\mathcal{I}_S}, \end{aligned}$$

and we are done. □

*Remark 3.2.* For later use, we explicitly notice that when  $m = 1$  the germs  $a_s^r$  are uniquely determined, and (3.6) becomes

$$(3.7) \quad (a_1^1)^{\nu_f} \hat{g}_{1 \dots 1}^j = \frac{\partial \hat{z}^j}{\partial z^h} g_{1 \dots 1}^h \pmod{\mathcal{I}_S^{\nu_f}}.$$

*Definition 3.2.* Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . The *canonical section*  $X_f \in H^0(S, \mathcal{T}_{M,S} \otimes (\mathcal{N}_S^*)^{\otimes \nu_f})$  associated to  $f$  is defined by setting

$$(3.8) \quad X_f = g_{s_1 \dots s_{\nu_f}}^h |_S \frac{\partial}{\partial z^h} \otimes \omega^{s_1} \otimes \dots \otimes \omega^{s_{\nu_f}}$$

in any chart adapted to  $S$ . Since  $(\mathcal{N}_S^*)^{\otimes \nu_f} = (\mathcal{N}_S^{\otimes \nu_f})^*$ , we can also think of  $X_f$  as a holomorphic section of  $\text{Hom}(\mathcal{N}_S^{\otimes \nu_f}, TM|_S)$ , and introduce the *canonical distribution*  $\Xi_f = X_f(\mathcal{N}_S^{\otimes \nu_f}) \subseteq TM|_S$ .

In particular we can now justify the term “tangential” previously introduced:

**COROLLARY 3.2.** *Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . Then  $f$  is tangential if and only if the canonical distribution is tangent to  $S$ , that is if and only if  $\Xi_f \subseteq TS$ .*

*Proof.* This follows from Lemma 1.2. □

*Example 3.1.* By the notation introduced in Example 1.2, if  $f$  is obtained by blowing up a map  $f_o$  tangent to the identity, then the canonical coordinates centered in  $p = [1 : 0 : \dots : 0]$  are adapted to  $S$ . In particular, if  $f_o$  is non-dicritical (that is, if  $f$  is tangential) then in a neighborhood of  $p$ ,

$$X_f = [Q_{\nu(f_o)}^q(1, z'') - z^q Q_{\nu(f_o)}^1(1, z'')] \frac{\partial}{\partial z^q} \otimes (\omega^1)^{\otimes (\nu(f_o)-1)}.$$

*Remark 3.3.* To be more precise,  $x_f$  is a section of the subsheaf  $\mathcal{T}_{M,S} \otimes \text{Sym}^{\nu_f}(\mathcal{N}_S^*)$ , where  $\text{Sym}^{\nu_f}(\mathcal{N}_S^*)$  is the symmetric  $\nu_f$ -fold tensor product of  $\mathcal{N}_S^*$ .



Now, the sheaf  $\mathcal{N}_S^*$  is isomorphic to  $\mathcal{I}_S/\mathcal{I}_S^2$ , and it is known that  $\text{Sym}^{\nu_f} \mathcal{I}_S/\mathcal{I}_S^2$  is isomorphic to  $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1}$ . This allows us to define  $X_f$  as a global section of the coherent sheaf  $\mathcal{T}_{M,S} \otimes \text{Sym}^{\nu_f}(\mathcal{I}_S/\mathcal{I}_S^2)$  even when  $S$  is singular. Indeed, if  $(U, z)$  is a local chart adapted to  $S$ , for  $j = 1, \dots, n$  the functions  $f^j - z^j$  determine local sections  $[f^j - z^j]$  of  $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1}$ . But, since for any other chart  $(\hat{U}, \hat{z})$ ,

$$\hat{f}^j - \hat{z}^j = (f^h - z^h) \frac{\partial \hat{z}^j}{\partial z^h} + R_{2\nu_f},$$

then  $(\partial/\partial z^j) \otimes [f^j - z^j]$  is a well-defined global section of  $\mathcal{T}_{M,S} \otimes (\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f}$  which coincides with  $X_f$  when  $S$  is smooth.

*Remark 3.4.* When  $f$  is tangential and  $\Xi_f$  is involutive as a sub-distribution of  $TS$  — for instance when  $m = 1$  — we thus get a holomorphic singular foliation on  $S$  canonically associated to  $f$ . As already remarked in [Br], this possibly is the reason explaining the similarities discovered in [A2] between the local dynamics of holomorphic maps tangent to the identity and the dynamics of singular holomorphic foliations.

*Definition 3.3.* A point  $p \in S$  is *singular* for  $f$  if there exists  $v \in (N_S)_p$ ,  $v \neq O$ , such that  $X_f(v \otimes \dots \otimes v) = O$ . We shall denote by  $\text{Sing}(f)$  the set of singular points of  $f$ .

In Section 7 it will become clear why we choose this definition for singular points. In Section 8 we shall describe a dynamical interpretation of  $X_f$  at nonsingular points in the codimension-one case; see Proposition 8.1.

*Remark 3.5.* If  $S$  is a hypersurface, the normal bundle is a line bundle. Therefore  $\Xi_f$  is a 1-dimensional distribution, and the singular points of  $f$  are the points where  $\Xi_f$  vanishes. Recalling (3.8), we then see that  $p \in \text{Sing}(f)$  if and only if  $g_{1\dots 1}^1(p) = \dots = g_{1\dots 1}^n(p) = 0$  for any adapted chart, and thus both the strictly fixed points of [A2] and the singular points of [BT], [Br] are singular in our case as well.

As we shall see later on, our index theorems will need a section of  $TS \otimes (N_S^*)^{\otimes \nu_f}$ ; so it will be natural to assume  $f$  tangential. When  $f$  is not tangential but  $S$  splits in  $M$  we can work too.

Let  $O \rightarrow TS \xrightarrow{\iota} TM|_S \xrightarrow{\pi} N_S \rightarrow O$  be the usual extension. Then we can associate to any splitting morphism  $\sigma: N_S \rightarrow TM|_S$  a morphism  $\sigma': TM|_S \rightarrow TS$  such that  $\sigma' \circ \iota = \text{id}_{TS}$ , by  $\sigma' = \iota^{-1} \circ (\sigma \circ \pi - \text{id}_{TM|_S})$ . Conversely, if there is a morphism  $\sigma': TM|_S \rightarrow TS$  such that  $\sigma' \circ \iota = \text{id}_{TS}$ , we get a splitting morphism by setting  $\sigma = (\pi|_{\text{Ker } \sigma'})^{-1}$ . Then

*Definition 3.4.* Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , and assume that  $S$  splits in  $M$ . Choose a splitting morphism  $\sigma: N_S \rightarrow TM|_S$  and let  $\sigma': TM|_S \rightarrow TS$

be the induced morphism. We set

$$H_{\sigma,f} = (\sigma' \otimes \text{id}) \circ X_f \in H^0(S, \mathcal{T}_S \otimes (N_S^*)^{\otimes \nu_f}).$$

Since the differential of  $f$  induces a morphism from  $N_S$  into itself, we have a dual morphism  $(df)^*: N_S^* \rightarrow N_S^*$ . Then if  $\nu_f = 1$  we also set

$$H_{\sigma,f}^1 = (\text{id} \otimes (df)^*) \circ H_{\sigma,f} \in H^0(S, \mathcal{T}_S \otimes N_S^*).$$

*Remark 3.6.* We defined  $H_{\sigma,f}^1$  only for  $\nu_f = 1$  because when  $\nu_f > 1$  one has  $(df)^* = \text{id}$ . On the other hand, when  $\nu_f = 1$  one has  $(df)^* = \text{id}$  if and only if  $f$  is tangential. Finally, we have  $X_f \equiv H_{\sigma,f}$  if and only if  $f$  is tangential, and  $H_{\sigma,f} \equiv O$  if and only if  $\Xi_f \subseteq \text{Im } \sigma = \text{Ker } \sigma'$ .

Finally, if  $(U, z)$  is a chart in an atlas adapted to the splitting  $\sigma$ , locally we have

$$H_{\sigma,f} = g_{s_1 \dots s_{\nu_f}}^p |_S \frac{\partial}{\partial z^p} \otimes \omega^{s_1} \otimes \dots \otimes \omega^{s_{\nu_f}},$$

and, if  $\nu_f = 1$ ,

$$H_{\sigma,f}^1 = (\delta_r^s + g_r^s) g_s^p |_S \frac{\partial}{\partial z^p} \otimes \omega^r.$$

#### 4. Local extensions

As we have already remarked, while  $X_f$  is well-defined, its extension  $\mathcal{X}_f$  in general is not. However, we shall now derive formulas showing how to control the ambiguities in the definition of  $\mathcal{X}_f$ , at least in the cases that interest us most.

In this section we assume  $m = 1$ , i.e., that  $S$  has codimension one in  $M$ . To simplify notation we shall write  $g^j$  for  $g_{1 \dots 1}^j$  and  $a$  for  $a_1^1$ . We shall also use the following notation:

- $T_1$  will denote any sum of terms of the form  $g \frac{\partial}{\partial z^p} \otimes dz^{h_1} \otimes \dots \otimes dz^{h_{\nu_f}}$  with  $g \in \mathcal{I}_S$ ;
- $R_k$  will denote any local section with coefficients in  $\mathcal{I}_S^k$ .

For instance, if  $(U, z)$  and  $(\hat{U}, \hat{z})$  are two charts adapted to  $S$ ,

$$(4.1) \quad \begin{aligned} \frac{\partial}{\partial \hat{z}^h} \otimes (d\hat{z}^1)^{\otimes \nu_f} &= a^{\nu_f} \frac{\partial z^k}{\partial \hat{z}^h} \frac{\partial}{\partial z^k} \otimes (dz^1)^{\otimes \nu_f} \\ &+ \frac{\partial z^1}{\partial \hat{z}^h} a^{\nu_f-1} z^1 \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{j_\ell}} \frac{\partial}{\partial z^1} \otimes dz^1 \otimes \dots \\ &\dots \otimes dz^{j_\ell} \otimes \dots \otimes dz^1 + T_1 + R_2, \end{aligned}$$

where

$$T_1 = \frac{\partial z^p}{\partial z^h} a^{\nu_f-1} z^1 \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{j_\ell}} \frac{\partial}{\partial z^p} \otimes dz^1 \otimes \dots \otimes dz^{j_\ell} \otimes \dots \otimes dz^1.$$

Assume now that  $f$  is tangential, and let  $(U, z)$  be a chart adapted to  $S$ . We know that  $f^1 - z^1 \in \mathcal{I}_S^{\nu_f+1}$ , and thus we can write

$$f^1 - z^1 = h^1(z^1)^{\nu_f+1},$$

where  $h^1$  is uniquely determined. Now, if  $(\hat{U}, \hat{z})$  is another chart adapted to  $S$  then

$$\begin{aligned} a^{\nu_f+1} \hat{h}^1(z^1)^{\nu_f+1} &= \hat{f}^1 - \hat{z}^1 = (a \circ f) f^1 - a z^1 \\ &= a(f^1 - z^1) + (a \circ f - a) z^1 + (a \circ f - a)(f^1 - z^1) \\ &= a(f^1 - z^1) + \frac{\partial a}{\partial z^p} (f^p - z^p) z^1 + R_{\nu_f+2} \\ &= \left[ ah^1 + \frac{\partial a}{\partial z^p} g^p \right] (z^1)^{\nu_f+1} + R_{\nu_f+2}. \end{aligned}$$

Therefore

$$(4.2) \quad a^{\nu_f+1} \hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} g^p + R_1.$$

Since  $g^1 = h^1 z^1$  we then get

$$(4.3) \quad a^{\nu_f} \hat{g}^1 = ag^1 + \frac{\partial a}{\partial z^p} g^p z^1 + R_2,$$

which generalizes (3.6) when  $f$  is tangential and  $m = 1$ .

Putting (4.3), (3.6) and (4.1) into (3.2) we then get

LEMMA 4.1. *Let  $f \in \text{End}(M, S)$ ,  $f \not\equiv \text{id}_M$ . Assume that  $f$  is tangential, and that  $S$  has codimension 1. Let  $(\hat{U}, \hat{z})$  and  $(U, z)$  be two charts about  $p \in S$  adapted to  $S$ , and let  $\hat{\mathcal{X}}_f, \mathcal{X}_f$  be given by (3.2) in the respective coordinates. Then*

$$\hat{\mathcal{X}}_f = \mathcal{X}_f + T_1 + R_2.$$

When  $S$  is comfortably embedded in  $M$  and of codimension one we shall also need nice local extensions of  $H_{\sigma,f}$  and  $H_{\sigma,f}^1$ , and to know how they behave under change of (comfortable) coordinates.

Definition 4.1. Let  $S$  be comfortably embedded in  $M$  and of codimension 1, and take  $f \in \text{End}(M, S)$ ,  $f \not\equiv \text{id}_M$ . Let  $(U, z)$  be a chart in a comfortable atlas, and set  $b^1(z) = g^1(O, z'')$ ; notice that  $f$  is tangential if and only if  $b^1 \equiv O$ . Write  $g^1 = b^1 + h^1 z^1$  for a well-defined holomorphic function  $h^1$ ; then set

$$(4.4) \quad \mathcal{H}_{\sigma,f} = h^1 z^1 \frac{\partial}{\partial z^1} \otimes (dz^1)^{\otimes \nu_f} + g^p \frac{\partial}{\partial z^p} \otimes (dz^1)^{\otimes \nu_f},$$

and if  $\nu_f = 1$  set

$$(4.5) \quad \mathcal{H}_{\sigma,f}^1 = h^1 z^1 \frac{\partial}{\partial z^1} \otimes dz^1 + g^p(1 + b^1) \frac{\partial}{\partial z^p} \otimes dz^1.$$

Notice that  $\mathcal{H}_{\sigma,f}$  (respectively,  $\mathcal{H}_{\sigma,f}^1$ ) restricted to  $S$  yields  $H_{\sigma,f}$  (respectively,  $H_{\sigma,f}^1$ ).

PROPOSITION 4.2. *Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . Assume that  $S$  is comfortably embedded in  $M$ , and of codimension one. Fix a comfortable atlas  $\mathfrak{U}$ , and let  $(U, z)$ ,  $(\hat{U}, \hat{z})$  be two charts in  $\mathfrak{U}$  about  $p \in S$ . Then if  $\nu_f = 1$ ,*

$$(4.6) \quad \hat{\mathcal{H}}_{\sigma,f}^1 = \mathcal{H}_{\sigma,f}^1 + T_1 + R_2,$$

while if  $\nu_f > 1$ ,

$$(4.7) \quad \hat{\mathcal{H}}_{\sigma,f} = \mathcal{H}_{\sigma,f} + T_1 + R_2,$$

where  $T_1 = T_1^o + T_1^1$  with

$$T_1^o = \frac{1}{a} g^q z^1 \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{p_\ell}} \frac{\partial}{\partial z^q} \otimes dz^1 \otimes \dots \otimes dz^{p_\ell} \otimes \dots \otimes dz^1,$$

$$T_1^1 = -ag^1 \frac{\partial z^q}{\partial \hat{z}^1} \frac{\partial}{\partial z^q} \otimes (dz^1)^{\otimes \nu_f}.$$

*Proof.* First of all, from (3.7),  $a^{\nu_f} \hat{b}^1 = ab^1 \pmod{\mathcal{I}_S}$ . But since we are using a comfortable atlas we get

$$\frac{\partial(a^{\nu_f} \hat{b}^1 - ab^1)}{\partial z^1} = (\nu_f a^{\nu_f-1} \hat{b}^1 - b^1) \frac{\partial a}{\partial z^1} + R_1 = R_1,$$

and thus

$$(4.8) \quad a^{\nu_f} \hat{b}^1 = ab^1 \pmod{\mathcal{I}_S^2}.$$

If  $\nu_f > 1$  then by (3.7) and (4.8),

$$a^{\nu_f} \hat{h}^1 \hat{z}^1 = (ah^1 + \frac{\partial a}{\partial z^p} g^p) z^1 \pmod{\mathcal{I}_S^2},$$

which implies

$$(4.9) \quad a^{\nu_f+1} \hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} g^p \pmod{\mathcal{I}_S}.$$

If  $\nu_f = 1$ , using (2.4) we can write

$$\begin{aligned} \hat{b}^1 \hat{z}^1 + \hat{h}^1 (\hat{z}^1)^2 &= \hat{f}^1 - \hat{z}^1 \\ &= \frac{\partial \hat{z}^1}{\partial z^j} (f^j - z^j) + \frac{1}{2} \frac{\partial^2 \hat{z}^1}{\partial z^h \partial z^k} (f^h - z^h)(f^k - z^k) + R_3 \\ &= ab^1 z^1 + \left[ ah^1 + \frac{\partial a}{\partial z^p} g^p (1 + b^1) \right] (z^1)^2 + R_3, \end{aligned}$$

and by (4.8),

$$(4.10) \quad a^2 \hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} g^p (1 + b^1) \pmod{\mathcal{I}_S}.$$

So if we compute  $\hat{\mathcal{H}}_{\sigma,f}$  for  $\nu_f > 1$  (respectively,  $\hat{\mathcal{H}}_{\sigma,f}^1$  for  $\nu_f = 1$ ) using (3.7), (4.1) and (4.9) (respectively, (3.7), (4.1), (4.8) and (4.10)), we get the assertions.  $\square$

### 5. Holomorphic actions

The index theorems to be discussed depend on actions of vector bundles. This concept was introduced by Baum and Bott in [BB], and later generalized in [CL], [LS], [LS2] and [Su]. Let us recall here the relevant definitions.

Let  $S$  again be a submanifold of codimension  $m$  in an  $n$ -dimensional complex manifold  $M$ , and let  $\pi_F: F \rightarrow S$  be a holomorphic vector bundle on  $S$ . We shall denote by  $\mathcal{F}$  the sheaf of germs of holomorphic sections of  $F$ , by  $\mathcal{T}_S$  the sheaf of germs of holomorphic sections of  $TS$ , and by  $\Omega_S^1$  (respectively,  $\Omega_M^1$ ) the sheaf of holomorphic 1-forms on  $S$  (respectively, on  $M$ ).

A section  $X$  of  $\mathcal{T}_S \otimes \mathcal{F}^*$  (or, equivalently, a holomorphic section of  $TS \otimes F^*$ ) can be interpreted as a morphism  $X: \mathcal{F} \rightarrow \mathcal{T}_S$ . Therefore it induces a derivation  $X^\#: \mathcal{O}_S \rightarrow \mathcal{F}^*$  by setting

$$(5.1) \quad X^\#(g)(u) = X(u)(g)$$

for any  $p \in S$ ,  $g \in \mathcal{O}_{S,p}$  and  $u \in \mathcal{F}_p$ . If  $\{f_1^*, \dots, f_k^*\}$  is a local frame for  $F^*$  about  $p$ , and  $X$  is locally given by  $X = \sum_j v_j \otimes f_j^*$ , then

$$(5.2) \quad X^\#(g) = \sum_j v_j(g) f_j^*.$$

Notice that if  $X^*: \Omega_S^1 \rightarrow \mathcal{F}^*$  denotes the dual morphism of  $X: \mathcal{F} \rightarrow \mathcal{T}_S$ , by definition we have

$$X^*(\omega)(u) = \omega(X(u))$$

for any  $p \in S$ ,  $\omega \in (\Omega_S^1)_p$  and  $u \in \mathcal{F}_p$ , and so

$$X^\#(g) = X^*(dg).$$

*Definition 5.1.* Let  $\pi_E: E \rightarrow S$  be another holomorphic vector bundle on  $S$ , and denote by  $\mathcal{E}$  the sheaf of germs of holomorphic sections of  $E$ . Let  $X$  be a section of  $\mathcal{T}_S \otimes \mathcal{F}^*$ . A *holomorphic action of  $F$  on  $E$  along  $X$*  (or an  *$X$ -connection on  $E$* ) is a  $\mathbb{C}$ -linear map  $\tilde{X}: \mathcal{E} \rightarrow \mathcal{F}^* \otimes \mathcal{E}$  such that

$$(5.3) \quad \tilde{X}(gs) = X^\#(g) \otimes s + g\tilde{X}(s)$$

for any  $g \in \mathcal{O}_S$  and  $s \in \mathcal{E}$ .

*Example 5.1.* If  $F = TS$ , and the section  $X$  is the identity  $\text{id}: TS \rightarrow TS$ , then  $X^\#(g) = dg$ , and a holomorphic action of  $TS$  on  $E$  along  $X$  is just a  $(1,0)$ -connection on  $E$ .

*Definition 5.2.* A point  $p \in S$  is a *singularity* of a holomorphic section  $X$  of  $\mathcal{T}_S \otimes \mathcal{F}^*$  if the induced map  $X_p: F_p \rightarrow T_p S$  is not injective. The set of singular points of  $X$  will be denoted by  $\text{Sing}(X)$ , and we shall set  $S^0 = S \setminus \text{Sing}(X)$  and  $\Xi_X = X(F|_{S^0}) \subseteq TS^0$ . Notice that  $\Xi_X$  is a holomorphic subbundle of  $TS^0$ .

The canonical section previously introduced suggests the following definition:

*Definition 5.3.* A *Camacho-Sad action* on  $S$  is a holomorphic action of  $N_S^{\otimes \nu}$  on  $N_S$  along a section  $X$  of  $\mathcal{T}_S \otimes (N_S^{\otimes \nu})^*$ , for a suitable  $\nu \geq 1$ .

*Remark 5.1.* The rationale behind the name is the following: as we shall see, the index theorem in [A2] is induced by a holomorphic action of  $N_S^{\otimes \nu_f}$  on  $N_S$  along  $X_f$  when  $f$  is tangential, and this index theorem was inspired by the Camacho-Sad index theorem [CS].

Let us describe a way to get Camacho-Sad actions. Let  $\pi: TM|_S \rightarrow N_S$  be the canonical projection; we shall use the same symbol for all other projections naturally induced by it. Let  $X$  be any global section of  $TS \otimes (N_S^{\otimes \nu})^*$ . Then we might try to define an action  $\tilde{X}: \mathcal{N}_S \rightarrow (\mathcal{N}_S^{\otimes \nu})^* \otimes \mathcal{N}_S = \text{Hom}(\mathcal{N}_S^{\otimes \nu}, \mathcal{N}_S)$  by setting

$$(5.4) \quad \tilde{X}(s)(u) = \pi([\mathcal{X}(\tilde{u}), \tilde{s}]|_S)$$

for any  $s \in \mathcal{N}_S$  and  $u \in \mathcal{N}_S^{\otimes \nu}$ , where:  $\tilde{s}$  is any element in  $\mathcal{T}_M|_S$  such that  $\pi(\tilde{s}|_S) = s$ ;  $\tilde{u}$  is any element in  $\mathcal{T}_M|_S^{\otimes \nu_f}$  such that  $\pi(\tilde{u}|_S) = u$ ; and  $\mathcal{X}$  is a *suitably chosen* local section of  $\mathcal{T}_M \otimes (\Omega_M^1)^{\otimes \nu}$  that restricted to  $S$  induces  $X$ .

Surprisingly enough, we can make this definition work in the cases interesting to us:

**THEOREM 5.1.** *Let  $f \in \text{End}(M, S)$ ,  $f \not\equiv \text{id}_M$ , be given. Assume that  $S$  has codimension one in  $M$  and that*

- (a)  *$f$  is tangential to  $S$ , or that*
- (b)  *$S$  is comfortably embedded into  $M$ .*

*Then we can use (5.4) to define a Camacho-Sad action on  $S$  along  $X_f$  in case (a), along  $H_{\sigma, f}$  in case (b) when  $\nu_f > 1$ , and along  $H_{\sigma, f}^1$  in case (b) when  $\nu_f = 1$ .*

*Proof.* We shall denote by  $X$  the section  $X_f$ ,  $H_{\sigma, f}$  or  $H_{\sigma, f}^1$  depending on the case we are considering. Let  $\mathcal{U}$  be an atlas adapted to  $S$ , comfortable and adapted to the splitting morphism  $\sigma$  in case (b), and let  $\mathcal{X}$  be the local

extension of  $X$  defined in a chart belonging to  $\mathfrak{U}$  by Definition 3.1 (respectively, Definition 4.1). We first prove that the right-hand side of (5.4) does not depend on the chart chosen. Take  $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$  to be local charts about  $p \in S$ . Using Lemma 4.1 and Proposition 4.2 we get

$$[\hat{\mathcal{X}}(\tilde{u}), \tilde{s}] = [(\mathcal{X} + T_1 + R_2)(\tilde{u}), \tilde{s}] = [\mathcal{X}(\tilde{u}) + T_1 + R_2, \tilde{s}] = [\mathcal{X}(\tilde{u}), \tilde{s}] + T_0 + R_1,$$

where  $T_0$  represents a local section of  $TM$  that restricted to  $S$  is tangent to it. Thus

$$\pi([\hat{\mathcal{X}}(\tilde{u}), \tilde{s}]|_S) = \pi([\mathcal{X}(\tilde{u}), \tilde{s}]|_S),$$

as desired.

We must now show that the right-hand side of (5.4) does not depend on the extensions of  $s$  and  $u$  chosen. If  $\tilde{s}'$  and  $\tilde{u}'$  are other extensions of  $s$  and  $u$  respectively, we have  $(\tilde{s}' - \tilde{s})|_S = T_0$ , while  $(\tilde{u}' - \tilde{u})|_S$  is a sum of terms of the form  $V_1 \otimes \dots \otimes V_{\nu_f}$  with at least one  $V_\ell$  tangent to  $S$ . Therefore  $\mathcal{X}(\tilde{u}' - \tilde{u})|_S = O$  and

$$\begin{aligned} [\mathcal{X}(\tilde{u}'), \tilde{s}']|_S &= [\mathcal{X}(\tilde{u}), \tilde{s}]|_S + [\mathcal{X}(\tilde{u}), \tilde{s}' - \tilde{s}]|_S + [\mathcal{X}(\tilde{u}' - \tilde{u}), \tilde{s}]|_S \\ &\quad + [\mathcal{X}(\tilde{u}' - \tilde{u}), \tilde{s}' - \tilde{s}]|_S = [\mathcal{X}(\tilde{u}), \tilde{s}]|_S + T_0, \end{aligned}$$

so that  $\pi([\mathcal{X}(\tilde{u}'), \tilde{s}']|_S) = \pi([\mathcal{X}_f(\tilde{u}), \tilde{s}]|_S)$ , as wanted.

We are left to show that  $\tilde{X}$  is actually an action. Take  $g \in \mathcal{O}_S$ , and let  $\tilde{g} \in \mathcal{O}_M|_S$  be any extension. First of all,

$$\tilde{X}(s)(gu) = \pi([\mathcal{X}(\tilde{g}\tilde{u}), \tilde{s}]|_S) = g\tilde{X}(s)(u) - \tilde{s}(\tilde{g})|_S \pi(X(u)) = g\tilde{X}(s)(u),$$

and so  $\tilde{X}(s)$  is a morphism. Finally, (5.1) yields

$$\mathcal{X}(\tilde{u})(\tilde{g})|_S = X^\#(g)(u),$$

and so

$$\tilde{X}(gs)(u) = \pi([\mathcal{X}(\tilde{u}), \tilde{g}\tilde{s}]|_S) = g\tilde{X}(s)(u) + \mathcal{X}(\tilde{u})(\tilde{g})|_S s = g\tilde{X}(s)(u) + X^\#(g)(u)s,$$

and we are done. □

*Remark 5.2.* If  $\nu_f = 1$  and  $f$  is not tangential then (5.4) with  $\mathcal{X} = \mathcal{H}_{\sigma,f}$  does not define an action. This is the reason why we introduced the new section  $H_{\sigma,f}^1$  and its extension  $\mathcal{H}_{\sigma,f}^1$ .

Later it will be useful to have an expression of  $\tilde{X}_f, \tilde{H}_{\sigma,f}$  and  $\tilde{H}_{\sigma,f}^1$  in local coordinates. Let then  $(U, z)$  be a local chart belonging to a (comfortable, if necessary) atlas adapted to  $S$ , so that  $\{\partial_1\}$  is a local frame for  $N_S$ , and  $\{(\omega^1)^{\otimes \nu_f} \otimes \partial_1\}$  is a local frame for  $(N_S^{\otimes \nu_f})^* \otimes N_S$ . There is a holomorphic function  $M_f$  such that

$$\tilde{X}_f(\partial_1)(\partial_1^{\otimes \nu_f}) = M_f \partial_1.$$

Now, recalling (3.2), we obtain

$$\begin{aligned} \tilde{X}_f(\partial_1)(\partial_1^{\otimes \nu_f}) &= \pi \left( \left[ \mathcal{X}_f \left( \left( \frac{\partial}{\partial z^1} \right)^{\otimes \nu_f} \right), \frac{\partial}{\partial z^1} \right] \Big|_S \right) \\ &= \pi \left( \left[ g^j \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^1} \right] \Big|_S \right) = - \frac{\partial g^1}{\partial z^1} \Big|_S \partial_1, \end{aligned}$$

and so

$$(5.5) \quad M_f = - \frac{\partial g^1}{\partial z^1} \Big|_S.$$

In particular, recalling that  $f$  is tangential we can write  $g^1 = z^1 h^1$ , and hence (5.5) yields

$$(5.6) \quad M_f = -h^1|_S.$$

Similarly, if we write  $\tilde{H}_{\sigma,f}(\partial_1)(\partial_1^{\otimes \nu_f}) = M_{\sigma,f}\partial_1$  and  $\tilde{H}_{\sigma,f}^1(\partial_1)(\partial_1) = M_{\sigma,f}^1\partial_1$ , we obtain

$$(5.7) \quad M_{\sigma,f} = M_{\sigma,f}^1 = -h^1|_S,$$

where  $h^1$  is defined by  $f^1 - z^1 = b^1(z^1)^{\nu_f} + h^1(z^1)^{\nu_f+1}$ .

Following ideas originally due to Baum and Bott (see [BB]), we can also introduce a holomorphic action on the virtual bundle  $TS - N_S^{\otimes \nu_f}$ . But let us first define what we mean by a holomorphic action on such a bundle.

*Definition 5.4.* Let  $S^0$  be an open dense subset of a complex manifold  $S$ ,  $F$  a vector bundle on  $S$ ,  $X \in H^0(S, \mathcal{T}_S \otimes \mathcal{F}^*)$ ,  $W$  a vector bundle over  $S^0$  and  $\tilde{W}$  any extension of  $W$  over  $S$  in  $K$ -theory. Then we say that  $F$  acts holomorphically on  $\tilde{W}$  along  $X$  if  $F|_{S^0}$  acts holomorphically on  $W$  along  $X|_{S^0}$ .

Let  $S$  be a codimension-one submanifold of  $M$  and take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , as usual. If  $f$  is tangential set  $X = X_f$ . If not, assume that  $S$  is comfortably embedded in  $M$  and set  $X = H_{\sigma,f}$  or  $X = H_{\sigma,f}^1$  according to the value of  $\nu_f$ ; in this case, we shall also assume that  $X \neq O$ . Set  $S^0 = S \setminus \text{Sing}(X)$ , and let  $\mathcal{Q}_f = \mathcal{T}_S/X(\mathcal{N}_S^{\otimes \nu_f})$ . The sheaf  $\mathcal{Q}_f$  is a coherent analytic sheaf which is locally free over  $S^0$ . The associated vector bundle (over  $S^0$ ) is denoted by  $Q_f$  and it is called the *normal bundle of  $f$* . Then the virtual bundle  $TS - N_S^{\otimes \nu_f}$ , represented by the sheaf  $\mathcal{Q}_f$ , is an extension (in the sense of  $K$ -theory) of  $Q_f$ .

*Definition 5.5.* A *Baum-Bott action* on  $S$  is a holomorphic action of  $N_S^{\otimes \nu}$  on the virtual bundle  $TS - N_S^{\otimes \nu}$  along a section  $X$  of  $\mathcal{T}_S \otimes N_S^{\otimes \nu}$ , for a suitable  $\nu \geq 1$ .



**THEOREM 5.2.** *Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given. Assume that  $S$  has codimension one in  $M$ , and that either  $f$  is tangential to  $S$  (and then set  $X = X_f$ ) or  $S$  is comfortably embedded into  $M$  (and then set  $X = H_{\sigma, f}$  or  $X = H_{\sigma, f}^1$  according to the value of  $\nu_f$ ). Assume moreover that  $X \neq 0$ . Then there exists a Baum-Bott action  $\tilde{B}: \mathcal{Q}_f \rightarrow (\mathcal{N}_S^{\otimes \nu_f})^* \otimes \mathcal{Q}_f$  of  $N_S^{\otimes \nu_f}$  on  $TS - N_S^{\otimes \nu_f}$  along  $X$  defined by*

$$(5.8) \quad \tilde{B}(s)(u) = \pi_f([X(u), \tilde{s}])$$

where  $\pi_f: \mathcal{T}_S \rightarrow \mathcal{Q}_f$  is the natural projection, and  $\tilde{s} \in \mathcal{T}_S$  is any section such that  $\pi_f(\tilde{s}) = s$ .

*Proof.* If  $\hat{s} \in \mathcal{T}_S$  is another section such that  $\pi_f(\hat{s}) = s$  we have  $\hat{s} - \tilde{s} \in X(\mathcal{N}_S^{\otimes \nu_f})$ ; hence  $\pi_f([X(u), \hat{s} - \tilde{s}]) = 0$ , and (5.8) does not depend on the choice of  $\tilde{s}$ . Finally, one can easily check that  $\tilde{B}$  is a holomorphic action on  $S^0$ .  $\square$

*Remark 5.3.* Since  $S$  has codimension one,  $X: N_S^{\otimes \nu_f} \rightarrow TS$  yields a (possibly singular) holomorphic foliation on  $S$ , and the previous action coincides with the one used in [BB] for the case of foliations.

We can also define a third holomorphic action, on the virtual bundle  $TM|_S - N_S^{\otimes \nu_f}$ . Assume that  $f$  is tangential, and let  $S^0 = S \setminus \text{Sing}(X_f)$ , as before. Then the sheaf  $\mathcal{W}_f = \mathcal{T}_{M, S}/X_f(\mathcal{N}_S^{\otimes \nu_f})$  is a coherent analytic sheaf, locally free over  $S^0$ ; let  $W_f = TM|_{S^0}/X_f(N_S^{\otimes \nu_f}|_{S^0})$  be the associated vector bundle over  $S^0$ . Then the virtual bundle  $TM|_S - N_S^{\otimes \nu_f}$ , represented by the sheaf  $\mathcal{W}_f$ , is an extension (in the sense of  $K$ -theory) of  $W_f$ .

*Definition 5.6.* A *Lehmann-Suwa action* on  $S$  is a holomorphic action of  $N_S^{\otimes \nu}$  on  $TM|_S - N_S^{\otimes \nu}$  along a section  $X$  of  $\mathcal{T}_S \otimes N_S^{\otimes \nu}$ , for a suitable  $\nu \geq 1$ .

Again, the name is chosen to honor the ones who first discovered the analogous action for holomorphic foliations in any dimension; see [LS], [LS2] (and [KS] for dimension two).

To present an example of such an action we first need a definition.

*Definition 5.7.* Let  $S$  be a codimension-one, comfortably embedded submanifold of  $M$ , and choose a comfortable atlas  $\mathfrak{U}$  adapted to a splitting morphism  $\sigma: N_S \rightarrow TM|_S$ . If  $v \in (\mathcal{N}_S^{\otimes \nu})_p$  and  $(U, \varphi) \in \mathfrak{U}$  is a chart about  $p \in S$ , we can write  $v = \lambda(z'')\partial_1^{\otimes \nu}$  for a suitable  $\lambda \in \mathcal{O}(U \cap S)$ . Then the *local extension of  $v$  along the fibers of  $\sigma$*  is the local section  $\tilde{v} = \lambda(z'')(\partial/\partial z^1)^{\otimes \nu} \in (\mathcal{T}_M|_S^{\otimes \nu})_p$ .

If  $(\hat{U}, \hat{z})$  is another chart in  $\mathfrak{U}$  about  $p$ , and  $v \in (\mathcal{N}_S^{\otimes \nu})_p$ , we can also write  $v = \hat{\lambda}\hat{\partial}_1^{\otimes \nu}$ , and we clearly have  $\hat{\lambda} = (a|_S)^\nu \lambda$ . But since  $S$  is comfortably

embedded in  $M$  we also have

$$\left. \frac{\partial(\hat{\lambda} - a^\nu \lambda)}{\partial z^1} \right|_S \equiv 0,$$

and thus

$$a^\nu \lambda = \hat{\lambda} + R_2.$$

Therefore if  $\hat{v}$  denotes the local extension of  $v$  along the fibers of  $\sigma$  in the chart  $(\hat{U}, \hat{\varphi})$  we have

$$(5.9) \quad \hat{v} = \hat{\lambda} \left( \frac{\partial}{\partial \hat{z}^1} \right)^{\otimes \nu} = a^\nu \lambda \frac{\partial z^{h_1}}{\partial \hat{z}^1} \cdots \frac{\partial z^{h_\nu}}{\partial \hat{z}^1} \frac{\partial}{\partial z^{h_1}} \otimes \cdots \otimes \frac{\partial}{\partial z^{h_\nu}} + R_2 = \tilde{v} + T_1 + R_2,$$

where

$$T_1 = a\lambda \sum_{\ell=1}^{\nu} \frac{\partial z^{p_\ell}}{\partial \hat{z}^1} \frac{\partial}{\partial z^1} \otimes \cdots \otimes \frac{\partial}{\partial z^{p_\ell}} \otimes \cdots \otimes \frac{\partial}{\partial z^1}.$$

Hence:

**THEOREM 5.3.** *Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given. Assume that  $S$  is of codimension one and comfortably embedded in  $M$ , and that  $f$  is tangential with  $\nu_f > 1$ . Let  $\rho_f: \mathcal{T}_{M,S} \rightarrow \mathcal{W}_f$  be the natural projection. Then a Lehmann-Suwa action  $\tilde{V}: \mathcal{W}_f \rightarrow (\mathcal{N}_S^{\otimes \nu_f})^* \otimes \mathcal{W}_f$  of  $\mathcal{N}_S^{\otimes \nu_f}$  on  $TM|_S - \mathcal{N}_S^{\otimes \nu_f}$  may be defined along  $X_f$  by setting*

$$(5.10) \quad \tilde{V}(s)(v) = \rho_f([\mathcal{X}_f(\tilde{v}), \tilde{s}]|_S),$$

for  $s \in \mathcal{W}_f$  and  $v \in \mathcal{N}_S^{\otimes \nu}$ , where  $\tilde{s}$  is any element in  $\mathcal{T}_M|_S$  such that  $\rho_f(\tilde{s}|_S) = s$ , and  $\tilde{v} \in \mathcal{T}_M|_S^{\otimes \nu_f}$  is an extension of  $v$  constant along the fibers of a splitting morphism  $\sigma$ .

*Proof.* Since  $\mathcal{X}_f(\tilde{v})|_S \in \mathcal{T}_S$  then clearly (5.10) does not depend on the extension  $\tilde{s}$  chosen. Using (5.9) and (4.7), since  $f$  tangential implies  $\mathcal{X}_f = \mathcal{H}_{\sigma,f}$  and  $T_1^1 = R_2$ , we have

$$[\hat{\mathcal{X}}_f(\hat{v}), \tilde{s}] = [(\mathcal{X}_f + T_1^o + R_2)(\tilde{v} + T_1 + R_2), \tilde{s}] = [\mathcal{X}_f(\tilde{v}), \tilde{s}] + R_1,$$

and therefore (5.10) does not depend on the comfortable coordinates chosen to define it. Finally, arguing as in Theorem 5.1 we can show that  $\tilde{V}$  actually is a holomorphic action, and we are done. □

### 6. Index theorems for hypersurfaces

Let  $S$  be a compact, globally irreducible, possibly singular hypersurface in a complex manifold  $M$ , and set  $S' = S \setminus \text{Sing}(S)$ . Given the following data:

- (a) a line bundle  $F$  over  $S'$ ;

- (b) a holomorphic section  $X$  of  $TS' \otimes F^*$ ;
- (c) a vector bundle  $E$  defined on  $M$ ;
- (d) a holomorphic action  $\tilde{X}$  of  $F|_{S'}$  on  $E|_{S'}$  along  $X$ ;

we can recover a partial connection (in the sense of Bott) on  $E$  restricted to  $S^0 = S' \setminus \text{Sing}(X)$  as follows: since, by definition of  $S^0$ , the dual map  $X^*: \Xi_X^* \rightarrow F^*|_{S^0}$  is an isomorphism, we can define a partial connection (in the sense of Bott [Bo])  $D: \Xi_X \times H^0(S^0, E|_{S^0}) \rightarrow H^0(S^0, E|_{S^0})$  by setting

$$D_v(s) = (X^* \otimes \text{id})^{-1}(\tilde{X}(s))(v)$$

for  $p \in S^0$ ,  $v \in (\Xi_X)_p$  and  $s \in H^0(S^0, E|_{S^0})$ . Furthermore, we can always extend this partial connection  $D$  to a  $(1, 0)$ -connection on  $E|_{S^0}$ , for instance by using a partition of unity (see, e.g., [BB]). Any such connection (which is a  $\Xi_X$ -connection in the terminology of [Bo], [Su]) will be said to be *induced* by the holomorphic action  $\tilde{X}$ .

We can then apply the general theory developed by Lehmann and Suwa for foliations (see in particular Theorem 1' and Proposition 4 of [LS], as well as [Su, Th. VI.4.8]) to get the following:

**THEOREM 6.1.** *Let  $S$  be a compact, globally irreducible, possibly singular hypersurface in an  $n$ -dimensional complex manifold  $M$ , and set  $S' = S \setminus \text{Sing}(S)$ . Let  $F$  be a line bundle over  $S'$  admitting an extension to  $M$ , and  $X$  a holomorphic section of  $TS' \otimes F^*$ . Set  $S^0 = S' \setminus \text{Sing}(X)$ , and let  $\text{Sing}(S) \cup \text{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$  be the decomposition of  $\text{Sing}(S) \cup \text{Sing}(X)$  in connected components. Finally, let  $E$  be a vector bundle defined on  $M$ . Then for any holomorphic action  $\tilde{X}$  of  $F|_{S'}$  on  $E|_{S'}$  along  $X$  and any homogeneous symmetric polynomial  $\varphi$  of degree  $n - 1$ , there are complex numbers  $\text{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) \in \mathbb{C}$ , depending only on the local behavior of  $\tilde{X}$  and  $E$  near  $\Sigma_\lambda$ , such that*

$$\sum_\lambda \text{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) = \int_S \varphi(E),$$

where  $\varphi(E)$  is the evaluation of  $\varphi$  on the Chern classes of  $E$ .

Recalling the results of the previous section, we then get the following index theorem for holomorphic self-maps:

**THEOREM 6.2.** *Let  $S$  be a compact, globally irreducible, possibly singular hypersurface in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given. Assume that*

- (a)  $f$  is tangential to  $S$ , and  $X = X_f$ , or that
- (b)  $S^0 = S \setminus (\text{Sing}(S) \cup \text{Sing}(f))$  is comfortably embedded into  $M$ , and  $X = H_{\sigma, f}$  if  $\nu_f > 1$ , or  $X = H_{\sigma, f}^1$  if  $\nu_f = 1$ .

Assume moreover  $X \not\equiv O$ . Let  $\text{Sing}(S) \cup \text{Sing}(X) = \bigcup_{\lambda} \Sigma_{\lambda}$  be the decomposition of  $\text{Sing}(S) \cup \text{Sing}(X)$  in connected components. Finally, let  $[S]$  be the line bundle on  $M$  associated to the divisor  $S$ . Then there exist complex numbers  $\text{Res}(X, S, \Sigma_{\lambda}) \in \mathbb{C}$ , depending only on the local behavior of  $X$  and  $[S]$  near  $\Sigma_{\lambda}$ , such that

$$\sum_{\lambda} \text{Res}(X, S, \Sigma_{\lambda}) = \int_S c_1^{n-1}([S]).$$

*Proof.* By Theorem 5.1 we have a Camacho-Sad action on  $S$  along  $X$  on  $N_{S^0}$ . Since  $[S]$  is an extension to  $M$  of  $N_{S^0}$ , we can apply Theorem 6.1.  $\square$

*Remark 6.1.* If  $M$  has dimension two, and  $S$  has at least one singularity or  $X_f$  has at least one zero, then  $S' \setminus \text{Sing}(f)$  is *always* comfortably embedded in  $M$ . Indeed, it is an open Riemann surface; so  $H^1(S' \setminus \text{Sing}(f), \mathcal{F}) = O$  for any coherent analytic sheaf  $\mathcal{F}$ , and the result follows from Proposition 2.1 and Theorem 2.2.

In a similar way, applying [Su, Th. IV.5.6], Theorem 5.3, and recalling that  $\varphi(H - L) = \varphi(H \otimes L^*)$  for any vector bundle  $H$ , line bundle  $L$  and homogeneous symmetric polynomial  $\varphi$ , we get

**THEOREM 6.3.** *Let  $S$  be a compact, globally irreducible, possibly singular hypersurface in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \not\equiv \text{id}_M$ , be given. Assume that  $S' = S \setminus \text{Sing}(S)$  is comfortably embedded into  $M$ , and that  $f$  is tangential to  $S$  with  $\nu_f > 1$ . Let  $\text{Sing}(S) \cup \text{Sing}(X_f) = \bigcup_{\lambda} \Sigma_{\lambda}$  be the decomposition of  $\text{Sing}(S) \cup \text{Sing}(X_f)$  in connected components. Finally, let  $[S]$  be the line bundle on  $M$  associated to the divisor  $S$ . Then for any homogeneous symmetric polynomial  $\varphi$  of degree  $n - 1$  there exist complex numbers  $\text{Res}_{\varphi}(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_{\lambda}) \in \mathbb{C}$ , depending only on the local behavior of  $X_f$  and  $TM|_S - [S]^{\otimes \nu_f}$  near  $\Sigma_{\lambda}$ , such that*

$$\sum_{\lambda} \text{Res}_{\varphi}(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_{\lambda}) = \int_S \varphi(TM|_S \otimes ([S]^*)^{\otimes \nu_f}).$$

Finally, applying the Baum-Bott index theorem (see [Su, Th. III.7.6]) and Theorem 5.2 we get

**THEOREM 6.4.** *Let  $S$  be a compact, globally irreducible, smooth complex hypersurface in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \not\equiv \text{id}_M$ , be given. Assume that*

- (a)  $f$  is tangential to  $S$ , and  $X = X_f$ , or that
- (b)  $S^0 = S \setminus \text{Sing}(f)$  is comfortably embedded into  $M$ , and  $X = H_{\sigma, f}$  if  $\nu_f > 1$ , or  $X = H_{\sigma, f}^1$  if  $\nu_f = 1$ .

Assume moreover  $X \neq O$ . Let  $\text{Sing}(X) = \bigcup_{\lambda} \Sigma_{\lambda}$  be the decomposition of  $\text{Sing}(S) \cup \text{Sing}(X)$  in connected components. Finally, let  $[S]$  be the line bundle on  $M$  associated to the divisor  $S$ . Then for any homogeneous symmetric polynomial  $\varphi$  of degree  $n - 1$  there exist complex numbers  $\text{Res}_{\varphi}(X, TS - [S]^{\otimes \nu_f}, \Sigma_{\lambda}) \in \mathbb{C}$ , depending only on the local behavior of  $X$  and  $TS - [S]^{\otimes \nu_f}$  near  $\Sigma_{\lambda}$ , such that

$$\sum_{\lambda} \text{Res}_{\varphi}(X, TS - [S]^{\otimes \nu_f}, \Sigma_{\lambda}) = \int_S \varphi(TS \otimes ([S]^*)^{\otimes \nu_f}).$$

Thus, we have recovered three main index theorems of foliation theory in the setting of holomorphic self-maps fixing pointwise a hypersurface.

Clearly, these index theorems are as useful as the formulas for the computation of the residues are explicit; the rest of this section is devoted to deriving such formulas in many important cases.

Let us first describe the general way these residues are defined in Lehmann-Suwa theory. Assume the hypotheses of Theorem 6.1. Let  $\tilde{U}_0$  be a tubular neighborhood of  $S^0$  in  $M$ , and denote by  $\rho: \tilde{U}_0 \rightarrow S^0$  the associated retraction. Given any connection  $D$  on  $E|_{S^0}$  induced by the holomorphic action  $\tilde{X}$  of  $F$  along  $X$ , set  $D^0 = \rho^*(D)$ . Next, choose an open set  $\tilde{U}_{\lambda} \subset M$  such that  $\tilde{U}_{\lambda} \cap (\text{Sing}(S) \cup \text{Sing}(X)) = \Sigma_{\lambda}$ , and a compact real  $2n$ -dimensional manifold  $\tilde{R}_{\lambda} \subset \tilde{U}_{\lambda}$  with  $C^{\infty}$  boundary containing  $\Sigma_{\lambda}$  in its interior and such that  $\partial\tilde{R}_{\lambda}$  intersects  $S$  transversally. Let  $D^{\lambda}$  be any connection on  $E|_{\tilde{U}_{\lambda}}$ , and denote by  $B(\varphi(D^0), \varphi(D^{\lambda}))$  the Bott difference form of  $\varphi(D^0)$  and  $\varphi(D^{\lambda})$  on  $\tilde{U}_0 \cap \tilde{U}_{\lambda}$ . Then (see [LS] and [Su, Chap. IV])

$$(6.1) \quad \text{Res}_{\varphi}(\tilde{X}, E, \Sigma_{\lambda}) = \int_{R_{\lambda}} \varphi(D^{\lambda}) - \int_{\partial R_{\lambda}} B(\varphi(D^0), \varphi(D^{\lambda})),$$

where  $R_{\lambda} = \tilde{R}_{\lambda} \cap S$ . A similar formula holds for virtual vector bundles too; see again [Su, Chap. IV].

*Remark 6.2.* When  $\Sigma_{\lambda} = \{x_{\lambda}\}$  is an isolated singularity of  $S$ , the second integral in (6.1) is taken on the link of  $x_{\lambda}$  in  $S$ . In particular if  $S$  is not irreducible at  $x_k$  then the residue is the sum of several terms, one for each irreducible component of  $S$  at  $x_k$ .

We now specialize (6.1) to our situation. Let us begin with the Camacho-Sad action: we shall compute the residues for connected components  $\Sigma_{\lambda}$  reduced to an isolated point  $x_{\lambda}$ . Let again  $[S]$  be the line bundle associated to the divisor  $S$ , and choose an open set  $\tilde{U}_{\lambda} \subset M$  containing  $x_{\lambda}$  so that  $\tilde{U}_{\lambda} \cap (\text{Sing}(S) \cup \text{Sing}(X)) = \{x_{\lambda}\}$  and  $[S]$  is trivial on  $\tilde{U}_{\lambda}$ ; take as  $D^{\lambda}$  the trivial connection for  $[S]$  on  $W$  with respect to some frame. In particular, then,  $\varphi(D^{\lambda}) = O$  on  $R_{\lambda}$ . By (6.1) the residue is then obtained simply by integrating

$B(\varphi(D^0), \varphi(D^\lambda))$  over  $\partial R_\lambda$ . Notice furthermore that since  $[S]$  is a line bundle there is only one nontrivial  $\varphi$  to consider: the  $(n - 1)^{\text{th}}$  power of the linear symmetric function, so that  $\varphi(D) = c_1^{n-1}([S])$ .

Let  $\eta^j$  be a connection one-form of  $D^j$ , for  $j = 0, \lambda$ ; with respect to a suitable frame for  $[S]$  we can assume that  $\eta^\lambda \equiv O$ . Let

$$\tilde{\eta} := t\eta^0 + (1 - t)\eta^\lambda = t\eta^0,$$

and let  $\tilde{K} := d\tilde{\eta} + \tilde{\eta} \wedge \tilde{\eta} = d\tilde{\eta}$ . From the very definition of the Bott difference form, it follows that

$$B(\varphi(D^0), \varphi(D^\lambda)) = \left(\frac{1}{2\pi i}\right)^{n-1} \int_0^1 \tilde{K}^{n-1}.$$

A straightforward computation shows that

$$\tilde{K}^{n-1} = (n - 1)t^{n-2} dt \wedge \eta^0 \wedge \overbrace{d\eta^0 \wedge \dots \wedge d\eta^0}^{n-2} + N,$$

where  $N$  is a term not containing  $dt$ . Therefore

$$(6.2) \quad B(\varphi(D^0), \varphi(D^\lambda)) = \left(\frac{1}{2\pi i}\right)^{n-1} \eta^0 \wedge \overbrace{d\eta^0 \wedge \dots \wedge d\eta^0}^{n-2}.$$

Assume now that  $x_\lambda \in \text{Sing}(X)$  and  $S$  is smooth at  $x_\lambda$ . Up to shrinking  $\tilde{U}_\lambda$  we may assume that  $\tilde{U}_\lambda$  is the domain of a chart  $z$  adapted to  $S$  (and belonging to a comfortable atlas if necessary), so that  $\{\partial_1\}$  is a local frame for  $N_{S^0}$ , and  $\{dz^2, \dots, dz^n\}$  is a local frame for  $T^*S^0$ . Then any connection  $D$  induced by the Camacho-Sad action is locally represented by the  $(1,0)$ -form  $\eta^0$  such that  $D(\partial_1) = \eta^0 \otimes \partial_1$ . To compute  $\eta^0$ , we first of all notice that  $X = g^p \frac{\partial}{\partial z^p} \otimes (\omega^1)^{\otimes \nu_f}$ , if  $X = X_f$  or  $X = H_{\sigma,f}$ , and  $X = (1 + b^1)g^p \frac{\partial}{\partial z^p} \otimes \omega^1$  if  $X = H_{\sigma,f}^1$ . Then, when  $X$  is  $X_f$  or  $H_{\sigma,f}$ ,

$$(X^*)^{-1}((\omega^1)^{\otimes \nu_f}) = \frac{1}{g^p} dz^p,$$

while when  $X = H_{\sigma,f}^1$ ,

$$(X^*)^{-1}((\omega^1)^{\otimes \nu_f}) = \frac{1}{(1 + b^1)g^p} dz^p.$$

Therefore, recalling formulas (5.6) and (5.7), we can choose  $D$  so that when  $X$  is  $X_f$  or  $H_{\sigma,f}$ ,

$$(6.3) \quad \eta^0 = (X^* \otimes \text{id})^{-1}(\tilde{X}(\partial_1)) = - \left. \frac{h^1}{g^p} \right|_S dz^p,$$

while when  $X = H_{\sigma,f}^1$ ,

$$(6.4) \quad \eta^0 = (X^* \otimes \text{id})^{-1}(\tilde{H}_{\sigma,f}^1(\partial_1)) = - \left. \frac{h^1}{(1 + b^1)g^p} \right|_S dz^p.$$

*Remark 6.3.* When  $n = 2$  and  $X = X_f$  we recover the connection form obtained in [Br]. The form  $\eta$  introduced in [A2], which is the opposite of  $\eta^0$ , is the connection form of the dual connection on  $N_{S^0}^*$ , by [A2, (1.7)]. Since the definition of Chern class implicitly used in [A2] is the opposite of the one used in [Br] everything is coherent. Finally, when  $n = 2$  and  $X = H_{\sigma,f}^1$  we have obtained the correct multiple of the form  $\eta$  introduced in [A2] when  $S$  was the smooth zero section of a line bundle (notice that  $1 + b^1$  is constant because  $S$  is compact, and that the form  $\eta$  of [A2] must be divided by  $b = 1 + b^1$  to get a connection form).

Now we can take  $R_\varepsilon = \{|g^p(x)| \leq \varepsilon \mid p = 2, \dots, n\}$  for a suitable  $\varepsilon > 0$  small enough. In particular, if we set  $\Gamma = \{|g^p(x)| = \varepsilon \mid p = 2, \dots, n\} \cap S$ , oriented so that  $d\theta^2 \wedge \dots \wedge d\theta^n > 0$  where  $\theta^p = \arg(g^p)$ , then arguing as in [L, §5] or [LS, §4] (see also [Su, pp.105–107]) we obtain

$$(6.5) \quad \text{Res}(X, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{g^2 \dots g^n} dz^2 \wedge \dots \wedge dz^n,$$

when  $X = X_f$  or  $H_{\sigma,f}$ , while when  $X = H_{\sigma,f}^1$  we have

$$(6.6) \quad \text{Res}(H_{\sigma,f}^1, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{(1 + b^1)^{n-1} g^2 \dots g^n} dz^2 \wedge \dots \wedge dz^n.$$

*Remark 6.4.* For  $n = 2$ , formulas (6.5) and (6.6) give the indices defined in [A2]. Thus, if  $S$  is smooth, Theorem 6.2 implies the index theorem of [A2], because  $c_1([S]) = c_1(N_S)$ . In an analogous way, Lehmann and Suwa (see [L], [LS], [LS2]) proved that the Camacho-Sad index theorem also is a consequence of Theorem 6.1.

When  $x_\lambda$  is an isolated singular point of  $S$  the computation of the residue is more complicated, because one cannot apply directly the results in [LS] as before, for in general there is no natural extension of  $\Xi_X$  and the Camacho-Sad action to  $\text{Sing}(S)$ . However we are able to compute explicitly the index in this case too when  $n = 2$ , and when  $n > 2$  and  $f$  is tangential with  $\nu_f > 1$ .

If  $n = 2$  we can choose local coordinates  $\{(w^1, w^2)\}$  in  $\tilde{U}_\lambda$  so that  $S \cap \tilde{U}_\lambda = \{l(w^1, w^2) = 0\}$  for some holomorphic function  $l$ , and  $dl \wedge dw^2 \neq 0$  on  $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$ . In particular  $(l, w^2)$  are local coordinates adapted to  $S^0$  near  $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$  and  $\frac{\partial}{\partial l}$  can be chosen as a local frame for  $N_{S^0}$  on  $\partial R_1$ .

*Remark 6.5.* When  $S^0$  is comfortably embedded in  $M$  the chart  $(l, w^2)$  should belong to a comfortable atlas. Studying the proofs of Propositions 2.1 and of Theorem 2.2 one sees that this is possible up to replacing  $l$  by a function of the form  $\hat{l} = (1 + c(w^2)l)l$ , where  $c$  is a holomorphic function defined on  $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$ . Since to compute the residues we only need the behavior of  $l$  and

$w^2$  near  $\partial R_1$ , it is easy to check that using  $\hat{l}$  or  $l$  in the following computations yields the same results. So for the sake of simplicity we shall not distinguish between  $l$  and  $\hat{l}$  in the sequel.

Up to shrinking  $\tilde{U}_\lambda$ , we can again assume that  $[S]$  is trivial on  $\tilde{U}_\lambda$ . The function  $l$  is a local generator of  $\mathcal{I}_S$  on  $\tilde{U}_\lambda$ . Then the dual of  $[l] \in \mathcal{I}_S/\mathcal{I}_S^2$ , denoted by  $s$ , is a holomorphic frame of  $[S]$  on  $\tilde{U}_\lambda$  which extends the holomorphic frame  $\frac{\partial}{\partial l}$  of  $N_{S'}$  (see [Su, p.86]). In particular  $s|_{\partial R_1} = \frac{\partial}{\partial l}$ . We then choose on  $[S]|_{\tilde{U}_\lambda}$  the trivial connection with respect to  $s$ , so that  $\eta^\lambda = O$ . We are left with the computation of the form  $\eta^0$  near  $\partial R^1$ . But if  $X = X_f$  or  $X = H_{\sigma,f}$  we can apply (6.3) to get

$$\eta^0|_{\partial R_1} = - \left. \frac{(l \circ f - l) - b^1 l^{\nu_f}}{l \cdot (w^2 \circ f - w^2)} \right|_{\partial R_1} dw^2,$$

where

$$b^1 = \left. \frac{l \circ f - l}{l^{\nu_f}} \right|_S$$

is identically zero when  $f$  is tangential. On the other hand, when  $X = H_{\sigma,f}^1$ , applying (6.4) we get

$$\eta^0|_{\partial R_1} = - \left. \frac{(l \circ f - l) - b^1 l}{(l + (l \circ f - l))(w^2 \circ f - w^2)} \right|_{\partial R_1} dw^2.$$

Hence the residue is

$$(6.7) \quad \text{Res}(X, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left. \frac{(l \circ f - l) - b^1 l^{\nu_f}}{l \cdot (w^2 \circ f - w^2)} \right|_S dw^2,$$

when  $X = X_f$  or  $X = H_{\sigma,f}$ , while when  $X = H_{\sigma,f}^1$ ,

$$(6.8) \quad \text{Res}(H_{\sigma,f}^1, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left. \frac{(l \circ f - l) - b^1 l}{(l + (l \circ f - l))(w^2 \circ f - w^2)} \right|_S dw^2.$$

*Remark 6.6.* When  $f$  is tangential we have  $b^1 \equiv 0$ ; therefore the formula (6.7) gives the index defined in [BT], and Theorem 6.2 implies the index theorem of [BT].

When  $n > 2$ ,  $f$  is tangential and  $\nu_f > 1$ , we can define a local vector field  $\tilde{v}_f$  which generates the Camacho-Sad action  $\tilde{X}_f$  and compute explicitly the residue even at a singular point  $x_\lambda$  of  $S$ . To see this, assume  $(w^1, \dots, w^n)$  are local coordinates in  $\tilde{U}_\lambda$  so that  $S \cap \tilde{U}_\lambda = \{l(w^1, \dots, w^n) = 0\}$  for some holomorphic function  $l$ . Define the vector field  $\tilde{v}_f$  on  $\tilde{U}_\lambda$  by

$$(6.9) \quad \tilde{v}_f = \frac{w^1 \circ f - w^1}{l^{\nu_f}} \frac{\partial}{\partial w^1} + \dots + \frac{w^n \circ f - w^n}{l^{\nu_f}} \frac{\partial}{\partial w^n}.$$



We claim that the “holomorphic action”  $\theta_{\tilde{v}_f}$  in the sense of Bott [Bo] of  $\tilde{v}_f$  on  $N_{S'}$ , as defined in [LS, p.177] coincides with our Camacho-Sad action, and thus we can apply [LS, Th. 1] to compute the residue. To prove this we consider  $W_1 = \{x \in \tilde{U}_\lambda \mid \frac{\partial l}{\partial w^1}(x) \neq 0\}$ . On this open set we make the following change of coordinates:

$$\begin{cases} z^1 = l(w^1, \dots, w^n), \\ z^p = w^p \end{cases} \quad \text{for } p = 2, \dots, n.$$

The new coordinates  $(z^1, \dots, z^n)$  are adapted to  $S$  on  $W_1$ . If  $f^j = z^j + g^j(z^1)^{\nu_f}$  as usual, we have

$$(6.10) \quad w^p \circ f - w^p = g^p(z^1)^{\nu_f},$$

and

$$(6.11) \quad w^1 \circ f - w^1 = \frac{\partial w^1}{\partial z^j} g^j(z^1)^{\nu_f} + R_{2\nu_f} = \left( \frac{\partial l}{\partial w^1} \right)^{-1} \left[ g^1 - \frac{\partial l}{\partial w^p} g^p \right] (z^1)^{\nu_f} + R_{2\nu_f}.$$

Therefore, from (6.10) and (6.11), taking into account that  $\nu_f > 1$ , we get

$$(6.12) \quad \begin{aligned} \tilde{v}_f &= \left( \frac{w^1 \circ f - w^1}{(z^1)^{\nu_f}} \frac{\partial l}{\partial w^1} + \frac{w^p \circ f - w^p}{(z^1)^{\nu_f}} \frac{\partial l}{\partial w^p} \right) \frac{\partial}{\partial z^1} \\ &\quad + \frac{w^q \circ f - w^q}{(z^1)^{\nu_f}} \frac{\partial}{\partial z^q} = \mathcal{X}_f(\partial_1^{\otimes \nu_f}) + R_2, \end{aligned}$$

which gives the claim on  $W_1$ . Since the same holds on each  $W_j = \{x \in \tilde{U}_\lambda \mid \frac{\partial l}{\partial w^j}(x) \neq 0\}$ ,  $j = 1, \dots, n$ , and  $(\tilde{U}_\lambda \cap S) \setminus \{x_\lambda\} = \bigcup_j W_j$ , it follows that the Bott holomorphic action induced by  $\tilde{v}_f$  is the same as the Camacho-Sad action given by  $\tilde{X}_f$ . Thus, if we choose — as we can — the coordinates  $(w^1, \dots, w^n)$  as in [LS, Th. 2], that is so that  $\{l, (w^p \circ f - w^p)/l^{\nu_f}\}$  form a regular sequence at  $x_\lambda$ , the residue is expressed by the formula after [LS, Th. 2]. Taking into account that, since  $f$  is tangential and by (6.13), the function  $l$  divides  $dl(\tilde{v}_f)$ , we get

$$(6.13) \quad \text{Res}(X_f, S, \{x_\lambda\}) = \left( \frac{-i}{2\pi i} \right)^{n-1} \int_\Gamma \frac{\left[ \sum_{j=1}^n \frac{\partial l}{\partial w^j} (w^j \circ f - w^j) \right]^{n-1}}{l^{n-1} \prod_{p=2}^n (w^p \circ f - w^p)} dw^2 \wedge \dots \wedge dw^n,$$

where this time

$$\Gamma = \left\{ w \in \tilde{U}_\lambda \mid \left| \frac{w^p \circ f - w^p}{l^{\nu_f}}(w) \right| = \epsilon, l(w) = 0 \right\},$$

for a suitable  $0 < \epsilon \ll 1$ , and  $\Gamma$  is oriented as usual (in particular  $\Gamma = (-1)^{\lfloor \frac{n-1}{2} \rfloor} R_{u_0}$  where  $R_{u_0}$  is the set defined in [LS, Th. 2]).

Note that for  $n = 2$  we recover, when  $\nu_f > 1$ , formula (6.7). On the other hand, if  $x_\lambda$  is nonsingular for  $S$ , then the previous argument with  $l = w^1$  works for  $\nu_f = 1$  as well, and we get formula (6.5).

Summing up, we have proved the following:

**THEOREM 6.5.** *Let  $S$  be a compact, globally irreducible, possibly singular hypersurface in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given. Assume that*

- (a)  $f$  is tangential to  $S$ , and  $X = X_f$ , or that
- (b)  $S^0 = S \setminus (\text{Sing}(S) \cup \text{Sing}(f))$  is comfortably embedded into  $M$ , and  $X = H_{\sigma,f}$  if  $\nu_f > 1$ , or  $X = H_{\sigma,f}^1$  if  $\nu_f = 1$ .

Assume  $X \not\equiv O$ . Let  $x_\lambda \in S$  be an isolated point of  $\text{Sing}(S) \cup \text{Sing}(X)$ . Then the number  $\text{Res}(X, S, \{x_\lambda\}) \in \mathbb{C}$  introduced in Theorem 6.2 is given

- (i) if  $x_\lambda \in \text{Sing}(X) \cap (S \setminus \text{Sing}(S))$ , and  $f$  is tangential or  $S^0$  is comfortably embedded in  $M$  and  $\nu_f > 1$ , by

$$\text{Res}(X, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{(h^1)^{n-1}}{g^2 \cdots g^n} dz^2 \wedge \cdots \wedge dz^n;$$

- (ii) if  $x_\lambda \in \text{Sing}(X) \cap (S \setminus \text{Sing}(S))$ ,  $S^0$  is comfortably embedded in  $M$  and  $\nu_f = 1$ , by

$$\text{Res}(H_{\sigma,f}^1, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{(h^1)^{n-1}}{(1+b^1)^{n-1} g^2 \cdots g^n} dz^2 \wedge \cdots \wedge dz^n;$$

- (iii) if  $n = 2$ ,  $x_\lambda \in \text{Sing}(S)$ , and  $f$  is tangential or  $S^0$  is comfortably embedded in  $M$  and  $\nu_f > 1$ , by

$$\text{Res}(X, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left. \frac{(l \circ f - l) - b^1 l^{\nu_f}}{l \cdot (w^2 \circ f - w^2)} \right|_S dw^2;$$

- (iv) if  $n = 2$ ,  $x_\lambda \in \text{Sing}(S)$ ,  $S^0$  is comfortably embedded in  $M$  and  $\nu_f = 1$ , by

$$\text{Res}(H_{\sigma,f}^1, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left. \frac{(l \circ f - l) - b^1 l}{(l + (l \circ f - l))(w^2 \circ f - w^2)} \right|_S dw^2;$$

- (v) if  $n > 2$ ,  $x_\lambda \in \text{Sing}(S)$ ,  $f$  is tangential and  $\nu_f > 1$ , by

$$\text{Res}(X_f, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi i}\right)^{n-1} \int_{\Gamma} \frac{\left[\sum_{j=1}^n \frac{\partial l}{\partial w^j} (w^j \circ f - w^j)\right]^{n-1}}{l^{n-1} \prod_{p=2}^n (w^p \circ f - w^p)} dw^2 \wedge \cdots \wedge dw^n.$$

Our next aim is to compute the residue for the Lehmann-Suwa action, at least for an isolated smooth point  $x_\lambda \in \text{Sing}(X_f)$ . Let  $(W, w)$  be a local chart about  $x_\lambda$  belonging to a comfortable atlas. Set  $l = w^1$  and define  $\tilde{v}_f$  as in (6.9). By (6.13) the Lehmann-Suwa action  $\tilde{V}$  is given by the holomorphic action (in

the sense of Bott) of  $\tilde{v}_f$  on  $TM|_S - [S]^{\otimes \nu_f}$ . Therefore we can apply [L], [LS] (see also [Su, Ths. IV.5.3, IV.5.6], and [Su, Remark IV.5.7]) to obtain

$$\text{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\}) = \text{Res}_\varphi(X_f, TM|_S, \{x_\lambda\}),$$

where  $\text{Res}_\varphi(X_f, TM|_S, \{x_\lambda\})$  is the residue for the local Lie derivative action of  $\tilde{v}_f$  on  $TM|_S$  given by

$$\tilde{V}_l(s)(\tilde{v}_f) = [\tilde{v}_f, \tilde{s}]|_S,$$

where  $s$  is a section of  $TM|_S$  and  $\tilde{s}$  is a local extension of  $s$  constant along the fibers of  $\sigma$ .

We can write an expression of  $\tilde{V}_l$  in local coordinates. Let  $(U, z)$  be a local chart belonging to a comfortable atlas. Then  $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\}$  is a local frame for  $TM$ , and  $\{(\omega^1)^{\otimes \nu_f} \otimes \frac{\partial}{\partial z^1}|_S, \dots, (\omega^1)^{\otimes \nu_f} \otimes \frac{\partial}{\partial z^n}|_S\}$  is a local frame for  $(N_S^{\otimes \nu_f})^* \otimes TM|_S$ . Thus there exist holomorphic functions  $V_j^k$  (for  $j, k = 1, \dots, n$ ) so that

$$\tilde{V}_l\left(\frac{\partial}{\partial z^j}\right)(\partial_1^{\otimes \nu_f}) = V_j^k \frac{\partial}{\partial z^k}.$$

Now, from (4.4) we get

$$\begin{aligned} \tilde{V}_l\left(\frac{\partial}{\partial z^j}\right)(\partial_1^{\otimes \nu_f}) &= \left[ \mathcal{X}_f \left( \left(\frac{\partial}{\partial z^1}\right)^{\otimes \nu_f} \right), \frac{\partial}{\partial z^j} \right] \Big|_S \\ &= \left[ h^1 z^1 \frac{\partial}{\partial z^1} + g^p \frac{\partial}{\partial z^p}, \frac{\partial}{\partial z^j} \right] \Big|_S = -h^1|_S \delta_j^1 \frac{\partial}{\partial z^1} - \frac{\partial g^p}{\partial z^j} \Big|_S \frac{\partial}{\partial z^p}, \end{aligned}$$

and hence

$$(6.14) \quad V_1^1 = -h^1|_S, \quad V_p^1 \equiv 0, \quad V_j^p = -\frac{\partial g^p}{\partial z^j} \Big|_S.$$

Therefore [Su, Th. IV.5.3] yields

**THEOREM 6.6.** *Let  $S$  be a compact, globally irreducible, possibly singular hypersurface in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given. Assume that  $S' = S \setminus \text{Sing}(S)$  is comfortably embedded into  $M$ , and that  $f$  is tangential to  $S$  with  $\nu_f > 1$ . Let  $x_\lambda \in \text{Sing}(X_f)$  be an isolated smooth point of  $\text{Sing}(S) \cup \text{Sing}(X_f)$ . Then for any homogeneous symmetric polynomial  $\varphi$  of degree  $n - 1$  the complex number*

$$\text{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\})$$

introduced by Theorem 6.3 is given by

$$(6.15) \quad \text{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\}) = \int_\Gamma \frac{\varphi(V) dz^2 \wedge \dots \wedge dz^n}{g^2 \dots g^n},$$

where  $V = (V_j^k)$  is the matrix given by (6.14) and  $\Gamma$  is as in (6.5).

*Remark 6.7.* We adopt here the convention that if  $V$  is an  $n \times n$  matrix then  $c_j(V)$  is the  $j^{\text{th}}$  symmetric function of the eigenvalues  $V$  multiplied by  $(i/2\pi)^j$ , and  $\varphi(V) = \varphi(c_1(V), \dots, c_{n-1}(V))$ .

Finally, if  $x_\lambda$  is an isolated point in  $\text{Sing}(X)$ , the complex numbers  $\text{Res}_\varphi(X, TS - [S]^{\otimes \nu_f}, \{x_\lambda\})$  appearing in Theorem 6.4 can be computed exactly as in the foliation case using a Grothendieck residue with a formula very similar to (6.15); see [BB], [Su, Th. III.5.5].

### 7. Index theorems in higher codimension

Let  $S \subset M$  be a complex submanifold of codimension  $1 < m < n$  in a complex  $n$ -manifold  $M$ . A way to get index theorems for holomorphic self-maps of  $M$  fixing pointwise  $S$  is to blow-up  $S$  and then apply the index theorems for hypersurfaces; this is what we plan to do in this section.

We shall denote by  $\pi: M_S \rightarrow M$  the blow-up of  $M$  along  $S$ , and by  $E_S = \pi^{-1}(S)$  the exceptional divisor, which is a hypersurface in  $M_S$  isomorphic to the projectivized normal bundle  $\mathbb{P}(N_S)$ .

*Remark 7.1.* If  $S$  is singular, the blow-up  $M_S$  is in general singular too. So this approach works only for smooth submanifolds.

If  $(U, z)$  is a chart adapted to  $S$  centered in  $p \in S$ , in  $M_S$  we have  $m$  charts  $(\tilde{U}_r, w_r)$  centered in  $[\partial_1], \dots, [\partial_m]$  respectively, where if  $v \in N_{S,p}$ ,  $v \neq O$ , we denote by  $[v]$  its projection in  $\mathbb{P}(N_S)$ . The coordinates  $z^j$  and  $w_r^h$  are related by

$$z^j(w_r) = \begin{cases} w_r^j & \text{if } j = r, m + 1, \dots, n, \\ w_r^r w_r^j & \text{if } j = 1, \dots, r - 1, r + 1, \dots, m. \end{cases}$$

*Remark 7.2.* We have  $\tilde{U}_r \cap E_S = \{w_r^r = 0\}$ , and thus  $(\tilde{U}_r, w_r)$  is adapted to  $E_S$  up to a permutation of the coordinates.

Now take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , and assume that  $df$  acts as the identity on  $N_S$  (this is automatic if  $\nu_f > 1$ , while if  $\nu_f = 1$  it happens if and only if  $f$  is tangential). Then we can lift  $f$  to a unique map  $\tilde{f} \in \text{End}(M_S, E_S)$ ,  $\tilde{f} \neq \text{id}_{M_S}$ , such that  $f \circ \pi = \pi \circ \tilde{f}$  (see, e.g., [A1] for details). If  $(U, z)$  is a chart adapted to  $S$  and we set  $f^j = z^j \circ f$  and  $\tilde{f}_r^j = w_r^j \circ \tilde{f}$ ,

$$(7.1) \quad \tilde{f}_r^j(w_r) = \begin{cases} f^j(z(w_r)) & \text{if } j = r, m + 1, \dots, n, \\ \frac{f^j(z(w_r))}{f^r(z(w_r))} & \text{if } j = 1, \dots, r - 1, r + 1, \dots, m. \end{cases}$$

If  $f$  is tangential we can find holomorphic functions  $h_{r_1 \dots r_{\nu_f+1}}^r$  symmetric in the lower indices such that

$$(7.2) \quad f^r - z^r = h_{r_1 \dots r_{\nu_f+1}}^r z^{r_1} \dots z^{r_{\nu_f+1}} + R_{\nu_f+2};$$

as usual, only the restriction to  $S$  of each  $h_{r_1 \dots r_{\nu_f+1}}^r$  is uniquely defined. Set then

$$Y = h_{r_1 \dots r_{\nu_f+1}}^r|_S \partial_r \otimes \omega^{r_1} \otimes \dots \otimes \omega^{r_{\nu_f+1}};$$

it is a local section of  $N_S \otimes (N_S^*)^{\otimes(\nu_f+1)}$ .

On the other hand, if  $f$  is not tangential set  $B = (\pi \otimes \text{id})_* \circ X_f$ , where  $\pi: TM|_S \rightarrow N_S$  is the canonical projection. In this way we get a global section of  $N_S \otimes (N_S^*)^{\otimes \nu_f}$ , not identically zero if and only if  $f$  is not tangential, and given in local adapted coordinates by

$$B = g_{r_1 \dots r_{\nu_f}}^r|_S \partial_r \otimes \omega^{r_1} \otimes \dots \otimes \omega^{r_{\nu_f}}.$$

*Definition 7.1.* Take  $p \in S$ . If  $f$  is tangential, a non-zero vector  $v \in (N_S)_p$  is a *singular direction* for  $f$  at  $p$  if  $X_f(v \otimes \dots \otimes v) = O$  and  $Y(v \otimes \dots \otimes v) \wedge v = O$ . If  $f$  is not tangential,  $v$  is a *singular direction* if  $B(v \otimes \dots \otimes v) \wedge v = O$ .

*Remark 7.3.* The condition  $Y(v \otimes \dots \otimes v) \wedge v = O$  is equivalent to requiring  $Y(v \otimes \dots \otimes v) = \lambda v$  for some  $\lambda \in \mathbb{C}$ .

Of course, in the tangential case we must check that this definition is well-posed, because the morphism  $Y$  depends on the local coordinates chosen to define it. First of all, if  $(U, z)$  is a chart adapted to  $S$  and centered at  $p$  then  $X_f(v \otimes \dots \otimes v) = O$  when  $f$  is tangential means

$$(7.3) \quad g_{r_1 \dots r_{\nu_f}}^p(O) v^{r_1} \dots v^{r_{\nu_f}} \frac{\partial}{\partial z^p} = O,$$

where  $v = v^r \partial_r$ . Now let  $(\hat{U}, \hat{z})$  be another chart adapted to  $S$  centered in  $p$ . Then we can find holomorphic functions  $a_s^r$  and  $\hat{a}_s^r$  such that  $\hat{z}^r = a_s^r z^s$  and  $z^r = \hat{a}_s^r \hat{z}^s$ . Arguing as in the proof of (4.2) we get

$$a_{s_1}^{r_1} \dots a_{s_{\nu_f+1}}^{r_{\nu_f+1}} \hat{h}_{r_1 \dots r_{\nu_f+1}}^r = a_s^r h_{s_1 \dots s_{\nu_f+1}}^s + \sum_{\ell=1}^{\nu_f+1} \frac{\partial a_{s_\ell}^r}{\partial z^p} g_{s_1 \dots \hat{s}_\ell \dots s_{\nu_f+1}}^p + R_1,$$

where the index with the hat is missing from the list. Therefore

$$\hat{Y} = Y + \hat{a}_r^s \sum_{\ell=1}^{\nu_f+1} \frac{\partial a_{s_\ell}^r}{\partial z^p} g_{s_1 \dots \hat{s}_\ell \dots s_{\nu_f+1}}^p \Big|_S \partial_s \otimes \omega^{s_1} \otimes \dots \otimes \omega^{s_{\nu_f+1}};$$

in particular if  $X_f(v \otimes \dots \otimes v) = O$  equation (7.3) yields

$$\hat{Y}(v \otimes \dots \otimes v) = Y(v \otimes \dots \otimes v),$$

and the notion of singular direction when  $f$  is tangential is well-defined.

PROPOSITION 7.1. *Let  $S \subset M$  be a complex submanifold of codimension  $1 < m < n$  of a complex  $n$ -manifold  $M$ , and take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , such that  $df$  acts as the identity on  $N_S$  (that is  $f$  is tangential, or  $\nu_f > 1$ , or both). Denote by  $\pi: M_S \rightarrow M$  the blow-up of  $M$  along  $S$  with exceptional divisor  $E_S$ , and let  $\tilde{f} \in \text{End}(M_S, E_S)$  be the lifted map. Then:*

- (i) *if  $S$  is comfortably embedded in  $M$  then  $E_S$  is comfortably embedded in  $M_S$ , and the choice of a splitting morphism for  $S$  induces a splitting morphism for  $E_S$ ;*
- (ii)  *$d\tilde{f}$  acts as the identity on  $N_{E_S}$ ;*
- (iii)  *$\tilde{f}$  is always tangential; furthermore  $\nu_{\tilde{f}} = \nu_f$  if  $f$  is tangential,  $\nu_{\tilde{f}} = \nu_f - 1$  otherwise;*
- (iv) *a direction  $[v] \in E_S$  is a singular point for  $\tilde{f}$  if and only if it is a singular direction for  $f$ .*

*Proof.* (i) Let  $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$  be a comfortable atlas adapted to  $S$ ; we claim that  $\tilde{\mathfrak{U}} = \{(\tilde{U}_{\alpha,r}, w_{\alpha,r})\}$  is a comfortable atlas adapted to  $E_S$  (and in particular determines a splitting morphism for  $E_S$ ). Let us first prove that it is a splitting atlas, that is that

$$\left. \frac{\partial w_{\beta,s}^j}{\partial w_{\alpha,r}^r} \right|_{E_S} \equiv 0$$

for every  $r, s, j \neq s$  and indices  $\alpha$  and  $\beta$ . We have

$$z_\beta^j = z_\beta^j|_S + \left. \frac{\partial z_\beta^j}{\partial z_\alpha^s} \right|_S z_\alpha^s + \frac{1}{2} \left. \frac{\partial^2 z_\beta^j}{\partial z_\alpha^u \partial z_\alpha^v} \right|_S z_\alpha^u z_\alpha^v + R_3.$$

Since  $w_{\alpha,r}^r = z_\alpha^r$ , if  $j = p > m$  we immediately get

$$\left. \frac{\partial w_{\beta,s}^p}{\partial w_{\alpha,r}^r} \right|_{E_S} = \left. \frac{\partial z_\beta^p}{\partial z_\alpha^r} \right|_S \equiv 0,$$

because  $\mathfrak{U}$  is a splitting atlas. If  $j = t \leq m$ ,

$$\begin{aligned} (7.4) \quad z_\beta^t &= \left. \frac{\partial z_\beta^t}{\partial z_\alpha^s} \right|_S z_\alpha^s + \frac{1}{2} \left. \frac{\partial^2 z_\beta^t}{\partial z_\alpha^u \partial z_\alpha^v} \right|_S z_\alpha^u z_\alpha^v + R_3 \\ &= \left[ \left. \frac{\partial z_\beta^t}{\partial z_\alpha^r} \right|_S + \sum_{u \neq r} \left. \frac{\partial z_\beta^t}{\partial z_\alpha^u} \right|_S w_{\alpha,r}^u \right] w_{\alpha,r}^r + O((w_{\alpha,r}^r)^3), \end{aligned}$$

because  $\mathfrak{U}$  is a comfortable atlas. Therefore if  $t \neq s$ ,

$$w_{\beta,s}^t = \frac{z_\beta^t}{z_\beta^s} = \frac{\left. \frac{\partial z_\beta^t}{\partial z_\alpha^r} \right|_S + \sum_{u \neq r} \left. \frac{\partial z_\beta^t}{\partial z_\alpha^u} \right|_S w_{\alpha,r}^u + O((w_{\alpha,r}^r)^2)}{\left. \frac{\partial z_\beta^s}{\partial z_\alpha^r} \right|_S + \sum_{u \neq r} \left. \frac{\partial z_\beta^s}{\partial z_\alpha^u} \right|_S w_{\alpha,r}^u + O((w_{\alpha,r}^r)^2)},$$

and so

$$\frac{\partial w_{\beta,s}^t}{\partial w_{\alpha,r}^r} = O(w_{\alpha,r}^r),$$

as required.

Finally, since  $w_{\beta,s}^s = z_{\beta}^s$ , equation (7.4) yields

$$\frac{\partial^2 w_{\beta,s}^s}{\partial (w_{\alpha,r}^r)^2} = O(w_{\alpha,r}^r),$$

and  $\tilde{\mathcal{U}}$  is a comfortable atlas, as claimed.

(ii) Since  $df$  acts as the identity on  $N_S$ , in local coordinates we can write

$$f^j(z) = z^j + g_{r_1 \dots r_{\nu_f+1}}^j z^{r_1} \dots z^{r_{\nu_f}} + R_{\nu_f+1},$$

with  $g_{r_1}^s|_S \equiv 0$  if  $\nu_f = 1$ . Then (7.1) yields

$$(7.5) \quad \tilde{f}_r^j(w_r) = w_r^j + (w_r^r)^{\nu_f} g_{r_1 \dots r_{\nu_f}}^j(z(w_r)) w_r^{\hat{r}_1} \dots w_r^{\hat{r}_{\nu_f}} + O((w_r^r)^{\nu_f+1})$$

if  $j = r, m + 1, \dots, n$ , and

$$(7.6) \quad \tilde{f}_r^j(w_r) = w_r^j + (w_r^r)^{\nu_f-1} [g_{r_1 \dots r_{\nu_f}}^j(z(w_r)) - w_r^j g_{r_1 \dots r_{\nu_f}}^r(z(w_r))] w_r^{\hat{r}_1} \dots w_r^{\hat{r}_{\nu_f}} + O((w_r^r)^{\nu_f})$$

if  $j = 1, \dots, r - 1, r + 1, \dots, m$ , where  $w_r^{\hat{s}} = w_r^s$  if  $s \neq r$ , and  $w_r^{\hat{r}} = 1$ . In particular,  $d\tilde{f}$  acts as the identity on  $N_{E_S}$ .

(iii) We have

$$g_{r_1 \dots r_{\nu_f}}^j|_{E_S}(z(w_r)) = g_{r_1 \dots r_{\nu_f}}^j|_S(O, w_r'');$$

therefore if  $f$  is tangential then  $w_r^r$  divides all  $g_{r_1 \dots r_{\nu_f}}^s(z(w_r))$ , while it does not divide some  $g_{r_1 \dots r_{\nu_f}}^p(z(w_r))$ . In particular, then,  $\tilde{f}$  is tangential and  $\nu_{\tilde{f}} = \nu_f$ , by (7.5) and (7.6). On the other hand, if  $f$  is not tangential  $w_r^r$  does not divide some  $g_{r_1 \dots r_{\nu_f}}^s(z(w_r))$ ; therefore

$$\begin{aligned} & [g_{r_1 \dots r_{\nu_f}}^s(z(w_r)) - w_r^s g_{r_1 \dots r_{\nu_f}}^r(z(w_r))] |_{E_S} \\ & = g_{r_1 \dots r_{\nu_f}}^s(O, w_r'') - w_r^s g_{r_1 \dots r_{\nu_f}}^r(O, w_r'') \neq 0, \end{aligned}$$

and thus  $\nu_{\tilde{f}} = \nu_f - 1$  and  $\tilde{f}$  is again tangential.

(iv) Take  $v \in (N_S)_p, v \neq O$ , and a chart  $(U, z)$  adapted to  $S$  centered in  $p$ . Then  $v = v^s \partial_s$ , with  $v^r \neq 0$  for some  $r$ . Hence  $[v] \in \tilde{U}_r$  has coordinates

$$w_r^j([v]) = \begin{cases} 0 & \text{if } j = r, m + 1, \dots, n, \\ v^j/v^r & \text{if } j = 1, \dots, r - 1, r + 1, \dots, m. \end{cases}$$

If  $f$  is not tangential, then  $[v]$  is a singular point for  $\tilde{f}$  if and only if

$$[v^r g_{r_1 \dots r_{\nu_f}}^s(O) - v^s g_{r_1 \dots r_{\nu_f}}^r(O)] v^{r_1} \dots v^{r_{\nu_f}} = 0$$

for all  $s$ , and thus if and only if  $B(v \otimes \dots \otimes v) \wedge v = O$ , as claimed.

If  $f$  is tangential, writing  $f^s - z^s$  as in (7.2) we get

$$\begin{aligned} \tilde{f}_r^s(w_r) &= w_r^s + (w_r^r)^{\nu_f} [h_{r_1 \dots r_{\nu_f+1}}^s(z(w_r)) - w_r^s h_{r_1 \dots r_{\nu_f+1}}^r(z(w_r))] w_r^{\hat{r}_1} \cdots w_r^{\hat{r}_{\nu_f+1}} \\ &\quad + O((w_r^r)^{\nu_f+1}) \end{aligned}$$

for  $s \neq r$ , and then it is clear that  $[v]$  is a singular point for  $\tilde{f}$  if and only if  $v$  is a singular direction for  $f$ . □

We therefore get index theorems in any codimension:

**THEOREM 7.2.** *Let  $S$  be a compact complex submanifold of codimension  $1 < m < n$  in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given, and assume that  $df$  acts as the identity on  $N_S$ . Let  $\bigcup_\lambda \Sigma_\lambda$  be the decomposition in connected components of the set of singular directions for  $f$  in  $\mathbb{P}(N_S)$ . Then there exist complex numbers  $\text{Res}(f, S, \Sigma_\lambda) \in \mathbb{C}$ , depending only on the local behavior of  $f$  and  $S$  near  $\Sigma_\lambda$ , such that*

$$\sum_\lambda \text{Res}(f, S, \Sigma_\lambda) = \int_{E_S} c_1^{n-1}([E_S]) = \int_S \pi_* c_1^{n-1}([E_S]),$$

where  $E_S$  is the exceptional divisor in the blow-up  $\pi: M_S \rightarrow M$  of  $M$  along  $S$ , and  $\pi_*$  denotes the integration along the fibers of the bundle  $\pi|_{E_S}: E_S \rightarrow S$ .

*Proof.* This follows immediately from Theorem 6.2, Proposition 7.1, and the projection formula (see, e.g., [Su, Prop. II.4.5]). □

*Remark 7.4.* The restriction to  $E_S$  of the cohomology class  $c_1([E_S])$  is the Chern class  $\zeta = c_1(T)$  of the tautological bundle  $T$  on the bundle  $\pi|_{E_S}: E_S \rightarrow S$  and it satisfies the relation

$$\begin{aligned} \zeta^{n-m} - \pi|_{E_S}^* c_1(N_S) \zeta^{n-m-1} + \pi|_{E_S}^* c_2(N_S) \zeta^{n-m-2} + \cdots \\ \cdots + (-1)^{n-m} \pi|_{E_S}^* c_{n-m}(N_S) = 0 \end{aligned}$$

in the cohomology ring of the bundle (see, e.g., [GH, pp. 606–608]). This formula can sometimes be used to compute  $\zeta$  in terms of the Chern classes of  $N_S$  and  $TM$  in specific examples.

**THEOREM 7.3.** *Let  $S$  be a compact complex submanifold of codimension  $1 < m < n$  in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given, and set  $\nu = \nu_f$  if  $f$  is tangential, and  $\nu = \nu_f - 1$  otherwise. Assume that  $S$  is comfortably embedded into  $M$ , and that  $\nu > 1$ . Let  $\bigcup_\lambda \Sigma_\lambda$  be the decomposition in connected components of the set of singular directions for  $f$  in  $\mathbb{P}(N_S)$ . Finally, let  $\pi: M_S \rightarrow M$  be the blow-up of  $M$  along  $S$ , with exceptional divisor  $E_S$ . Then for any homogeneous symmetric polynomial  $\varphi$  of degree  $n - 1$  there exist complex numbers  $\text{Res}_\varphi(f, TM_S|_{E_S} - N_{E_S}^{\otimes \nu}, \Sigma_\lambda) \in \mathbb{C}$ ,*



depending only on the local behavior of  $f$  and  $TM_S|_{E_S} - N_{E_S}^{\otimes \nu}$  near  $\Sigma_\lambda$ , such that

$$\sum_{\lambda} \text{Res}_{\varphi}(f, TM_S|_{E_S} - N_{E_S}^{\otimes \nu}, \Sigma_{\lambda}) = \int_S \pi_* \varphi(TM_S|_{E_S} \otimes (N_{E_S}^*)^{\otimes \nu}),$$

where  $\pi_*$  denotes the integration along the fibers of the bundle  $E_S \rightarrow S$ .

*Proof.* This follows immediately from Theorem 6.3, Proposition 7.1, and the projection formula. □

**THEOREM 7.4.** *Let  $S$  be a compact complex submanifold of codimension  $1 < m < n$  in an  $n$ -dimensional complex manifold  $M$ . Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ , be given, and assume that  $df$  acts as the identity on  $N_S$ . Set  $\nu = \nu_f$  if  $f$  is tangential, and  $\nu = \nu_f - 1$  otherwise. Let  $\bigcup_{\lambda} \Sigma_{\lambda}$  be the decomposition in connected components of the set of singular directions for  $f$  in  $\mathbb{P}(N_S)$ . Finally, let  $\pi: M_S \rightarrow M$  be the blow-up of  $M$  along  $S$ , with exceptional divisor  $E_S$ . Then for any homogeneous symmetric polynomial  $\varphi$  of degree  $n - 1$  there exist complex numbers  $\text{Res}_{\varphi}(f, TE_S - N_{E_S}^{\otimes \nu}, \Sigma_{\lambda}) \in \mathbb{C}$ , depending only on the local behavior of  $f$  and  $TE_S - N_{E_S}^{\otimes \nu}$  near  $\Sigma_{\lambda}$ , such that*

$$\sum_{\lambda} \text{Res}_{\varphi}(f, TE_S - N_{E_S}^{\otimes \nu}, \Sigma_{\lambda}) = \int_S \pi_* \varphi(TE_S \otimes (N_{E_S}^*)^{\otimes \nu}),$$

where  $\pi_*$  denotes the integration along the fibers of the bundle  $E_S \rightarrow S$ .

*Proof.* This follows immediately from Theorem 6.4, Proposition 7.1, and the projection formula. □

### 8. Applications to dynamics

We conclude this paper with applications to the study of the dynamics of endomorphisms of complex manifolds, first recalling a definition from [A2]:

*Definition 8.1.* Let  $f \in \text{End}(M, p)$  be a germ at  $p \in M$  of a holomorphic self-map of a complex manifold  $M$  fixing  $p$ . A *parabolic curve* for  $f$  at  $p$  is a injective holomorphic map  $\varphi: \Delta \rightarrow M$  satisfying the following properties:

- (i)  $\Delta$  is a simply connected domain in  $\mathbb{C}$  with  $0 \in \partial\Delta$ ;
- (ii)  $\varphi$  is continuous at the origin, and  $\varphi(0) = p$ ;
- (iii)  $\varphi(\Delta)$  is invariant under  $f$ , and  $(f|_{\varphi(\Delta)})^n \rightarrow p$  as  $n \rightarrow \infty$ .

Furthermore, we say that  $\varphi$  is *tangent to a direction*  $v \in T_pM$  at  $p$  if for one (and hence any) chart  $(U, z)$  centered at  $p$  the direction of  $z(\varphi(\zeta))$  converges to the direction  $dz_p(v)$  as  $\zeta \rightarrow 0$ .

Now we have the promised dynamical interpretation of  $X_f$  at nonsingular points:

PROPOSITION 8.1. *Assume that  $S$  has codimension one in  $M$ , and take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . Let  $p \in S$  be a regular point of  $X_f$ , that is  $X_f(p) \neq O$ . Then*

- (i) *If  $f$  is tangential then no infinite orbit of  $f$  can stay arbitrarily close to  $p$ . More precisely, there is a neighborhood  $U$  of  $p$  such that for every  $q \in U$  there is  $n_0 \in \mathbb{N}$  such that  $f^{n_0}(q) \notin U$  or  $f^{n_0}(q) \in S$ .*
- (ii) *If  $\Xi_f(p)$  is transversal to  $T_pS$  (so in particular  $f$  is non-tangential) and  $\nu_f > 1$  then there exists at least one parabolic curve for  $f$  at  $p$  tangent to  $\Xi_f(p)$ .*
- (iii) *If  $\Xi_f(p)$  is transversal to  $T_pS$ ,  $\nu_f = 1$ , and  $|b(p)| \neq 0, 1$  or  $b(p) = \exp(2\pi i\theta)$  where  $\theta$  satisfies the Bryuno condition (and  $b$  is the function defined in Remark 1.1) then there exists an  $f$ -invariant one-dimensional holomorphic disk  $\Delta$  passing through  $p$  tangent to  $\Xi_f(p)$  such that  $f|_\Delta$  is holomorphically conjugated to the multiplication by  $b(p)$ .*

*Proof.* In local adapted coordinates centered at  $p \in S$  we can write

$$f^j(z) = z^j + (z^1)^{\nu_f} g^j(z),$$

so that

$$\Xi_f(p) = \text{Span} \left( g^1(O) \frac{\partial}{\partial z^1} \Big|_p + \dots + g^n(O) \frac{\partial}{\partial z^n} \Big|_p \right).$$

In case (i), we have  $g^1 = z^1 h^1$  for a suitable holomorphic function  $h^1$ , and  $g^{p_0}(O) \neq 0$  for some  $2 \leq p_0 \leq n$ , because  $p$  is not singular. Therefore we can apply [AT, Prop. 2.1] (see also [A2, Prop. 2.1]), and the assertion follows.

Now,  $\Xi_f(p)$  is transversal to  $T_pS$  if and only if  $g^1(O) \neq 0$ . In case (ii) we can then write

$$f^j(z) = z^j + g^j(O)(z^1)^{\nu_f} + O(\|z\|^{\nu_f+1})$$

with  $g^1(O) \neq 0$ . Then  $\Xi_f(p)$  is a non-degenerate characteristic direction of  $f$  at  $p$  in the sense of Hakim, and thus by [H1, 2] there exist at least  $\nu_f - 1$  parabolic curves for  $f$  at  $p$  tangent to  $\Xi_f(p)$ .

If  $\nu_f = 1$ , it is easy to see that  $b^1(p) = 1 + g^1(O)$ , and  $b^1(p) \neq 1$  because  $\Xi_f(p)$  is transversal to  $T_pS$ . Therefore we can write

$$f^j(z) = \begin{cases} b^1(p)z^1 + O(\|z\|^2) & \text{if } j = 1, \\ z^j + g^j(O)z^1 + O(\|z\|^2) & \text{if } 2 \leq j \leq n, \end{cases}$$

and the assertion in case (iii) follows immediately from [Pö] (see also [N]).  $\square$

In other words,  $X_f$  essentially dictates the dynamical behavior of  $f$  away from the singularities — or, from another point of view, we can say that the interesting dynamics is concentrated near the singularities of  $X_f$ .

*Remark 8.1.* If  $\Xi_f(p)$  is transversal to  $T_pS$ ,  $\nu_f = 1$  and  $b(p) = 0$  or  $b(p) = \exp(2\pi i\theta)$  with  $\theta$  irrational not satisfying the Bryuno condition, there might still be an  $f$ -invariant one-dimensional holomorphic disk passing through  $p$  and tangent to  $\Xi_f(p)$ . On the other hand, if  $b(p) = \exp(2\pi i\theta)$  is a  $k^{\text{th}}$  root of unity, necessarily different from one, several things might happen. For instance, if  $b(p) = -1$ , up to a linear change of coordinates we can write

$$f^j(z) = \begin{cases} z^1 + z^1(-2 + (z^1)^{\mu_1}\hat{g}^1(z)) & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}\hat{g}^j(z) & \text{if } j = 2, \dots, n, \end{cases}$$

for suitable  $\mu_1, \dots, \mu_n \in \mathbb{N}$  and holomorphic functions  $\hat{g}^j$  not divisible by  $z^1$  and such that  $\hat{g}^j(O) = 0$  if  $\mu_j = 0$ . Then if  $\mu_1 = 0$ ,

$$\begin{aligned} &(f \circ f)^j(z) \\ &= \begin{cases} z^1 - z^1[\hat{g}^1(z) + \hat{g}^1(f(z)) - \hat{g}^1(z)\hat{g}(f(z))] & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}[\hat{g}^j(z) - (-1 + \hat{g}^1(z))^{\mu_j+1}\hat{g}^j(f(z))] & \text{if } j = 2, \dots, n. \end{cases} \end{aligned}$$

So  $\nu_{f \circ f} = 1$ ,  $f \circ f$  is non-tangential but  $p$  is singular for  $f \circ f$ . On the other hand, if  $\mu_1 = 1$ ,

$$\begin{aligned} &(f \circ f)^j(z) \\ &= \begin{cases} z^1 - (z^1)^2[\hat{g}^1(z) - \hat{g}^1(f(z)) + O(z^1)] & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}[\hat{g}^j(z) + (-1)^{\mu_j}\hat{g}^j(f(z)) + O(z^1)] & \text{if } j = 2, \dots, n. \end{cases} \end{aligned}$$

Now if, for instance,  $\mu_2 = 0$  we get  $\nu_{f \circ f} = 1$ , but  $f \circ f$  is tangential and  $p$  is singular for  $f \circ f$ . But if  $\mu_2 = 2$  and  $\mu_j \geq 2$  for  $j \geq 3$  we get  $\nu_{f \circ f} = 3$  and  $p$  can be either singular or nonsingular for  $f \circ f$ .

*Remark 8.2.* If  $\nu_f = 1$ ,  $\Xi_f(p) \subset T_pS$  and  $S$  is compact, necessarily  $f$  is tangential, because  $b \equiv 1$  and then  $g^1(0, z'') \equiv 0$ . If  $S$  is not compact we might have an isolated point of tangency, and in that case we might have parabolic curves at  $p$  not tangent to  $\Xi_f(p)$ . For instance, the methods of [A1] show that this happens for the map

$$f^j(z) = \begin{cases} z^1 + z^1(az^2 + bz^3 + h_1(z'') + z^1h_2(z)) & \text{if } j = 1, \\ z^2 + z^1(c + h_3(z)) & \text{if } j = 2, \\ z^3 + z^1g^3(z) & \text{if } j = 3, \end{cases}$$

when  $a, c \neq 0$ .

Finally, we describe a couple of applications to endomorphisms of complex surfaces:

COROLLARY 8.2. *Let  $S$  be a smooth compact one-dimensional submanifold of a complex surface  $M$ , and take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . Assume that  $f$  is tangential, or that  $S \setminus \text{Sing}(f)$  is comfortably embedded in  $M$ , and let  $X$  denote  $X_f$ ,  $H_{\sigma,f}$  or  $H_{\sigma,f}^1$  as usual; assume moreover that  $X \neq O$ . Then*

- (i) *if  $c_1(N_S) \neq 0$  then  $\chi(S) - \nu_f c_1(N_S) > 0$ ;*
- (ii) *if  $c_1(N_S) > 0$  then  $S$  is rational,  $\nu_f = 1$  and  $c_1(N_S) = 1$ .*

*Proof.* The well-known theorem about the localization of the top Chern class at the zeroes of a global section (see, e.g., [Su, Th. III.3.5]) yields

$$(8.1) \quad \sum_{x \in \text{Sing}(X)} N(X; x) = \chi(S) - \nu_f c_1(N_S),$$

where  $N(X; x)$  is the multiplicity of  $x$  as a zero of  $X$ . Now, If  $c_1(N_S) \neq 0$  then by Theorem 6.2 the set  $\text{Sing}(X)$  is not empty. Therefore the sum in (8.1) must be strictly positive, and the assertions follow. □

*Definition 8.2.* Let  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . We say that a point  $p \in S$  is *weakly attractive* if there are infinite orbits arbitrarily close to  $p$ , that is, if for every neighborhood  $U$  of  $p$  there is  $q \in U$  such that  $f^n(q) \in U \setminus S$  for all  $n \in \mathbb{N}$ . In particular, this happens if there is an infinite orbit converging to  $p$ .

Then we can prove the following

PROPOSITION 8.3. *Let  $S$  be a smooth compact one-dimensional submanifold of a complex surface  $M$ , and take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . If  $f$  is tangential then there are at most  $\chi(S) - \nu_f c_1(N_S)$  weakly attractive points for  $f$  on  $S$ .*

*Proof.* By (8.1) the sum of zeros of the section  $X_f$  (counting multiplicity) is equal to  $\chi(S) - \nu_f c_1(N_S)$ . Thus the number of zeros (not counting multiplicity) is at most  $\chi(S) - \nu_f c_1(N_S)$ . The assertion then follows from Proposition 8.1. □

Finally, the previous index theorems allow a classification of the smooth curves which are fixed by a holomorphic map and are dynamically trivial.

THEOREM 8.4. *Let  $S$  be a smooth compact one-dimensional submanifold of a complex surface  $M$ , and take  $f \in \text{End}(M, S)$ ,  $f \neq \text{id}_M$ . Moreover assume that  $\text{sp}(df_p) = \{1\}$  for some  $p \in S$ . If there are no weakly attractive points for  $f$  on  $S$  then only one of the following cases occurs:*

- (i)  $\chi(S) = 2$ ,  $c_1(N_S) = 0$ , or

(ii)  $\chi(S) = 2$ ,  $c_1(N_S) = 1$ ,  $\nu_f = 1$ , or

(iii)  $\chi(S) = 0$ ,  $c_1(N_S) = 0$ .

*Proof.* Since  $N_S$  is a line bundle over a compact curve  $S$ , the action of  $df$  on  $N_S$  is given by multiplication by a constant, and since  $df_p$  has only the eigenvalue 1 then this constant must be 1. If  $f$  were nontangential then by Proposition 8.1.(ii) all but a finite number of points of  $S$  would be weakly attractive. Therefore  $f$  is tangential. By [A2, Cor. 3.1] (or [Br, Prop. 7.7]) if there is a point  $q \in S$  so that  $\text{Res}(X_f, N_S, p) \notin \mathbb{Q}^+$  then  $q$  is weakly attractive. Thus the sum of the residues is nonnegative and by Theorem 6.2 it follows that  $c_1(N_S) \geq 0$ . Thus (8.1) yields

$$(8.2) \quad \chi(S) \geq \nu_f c_1(N_S) \geq 0.$$

Therefore the only possible cases are  $\chi(S) = 0, 2$ . If  $\chi(S) = 0$  then (8.2) implies  $c_1(N_S) = 0$ . Assume that  $\chi(S) = 2$ . Thus  $c_1(N_S) = 0, 1, 2$ . However if  $c_1(N_S) = 1$  and  $\nu_f = 2$  or if  $c_1(N_S) = 2$  (and necessarily  $\nu_f = 1$ ) then (8.1) would imply that  $X_f$  has no zeroes, and thus  $c_1(N_S) = 0$  by Theorem 6.2.  $\square$

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