Riemann mapping theorem for simply connected sets with smooth boundary

The Riemann mapping theorem states that for any simply connected open set $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, there is a biholomorphic map $H : \Omega \to B_1$. In this section we prove this theorem for simply connected sets Ω with smooth boundaries. The proof follows the original idea of Riemann, which is based on a variational principle. We notice that this proof also provides the up-to-the-boundary regularity of H, which turns out to be as smooth as $\partial\Omega$.

Idea of the proof. The Riemann's proof starts from the following observation. Suppose that we have a map

$$H : \overline{\Omega} \to \overline{B}_1,$$

which is a homeomorphism between $\overline{\Omega}$ and $\overline{B}_1$, and a conformal map between Ω and B_1. Suppose that H can be written in the form

$$H(z) = zh(z),$$

for some holomorphic function h. Since

$$|H(z)| = 1 \quad \text{for all} \quad z \in \partial\Omega,$$

we have that

$$|h(z)| = \frac{1}{|z|} \quad \text{for all} \quad z \in \partial\Omega.$$

Assume that h can be written in the form

$$h = \exp(u + iv),$$

where u and v are real-valued functions and that $L := u + iv$ is holomorphic on Ω. Then u and v are harmonic on Ω. Moreover, since

$$|h(z)| = e^{u(z)},$$

we have that u should satisfy

$$u(z) = -\ln|z| \quad \text{for all} \quad z \in \partial\Omega.$$

Thus, u is the solution to

$$\Delta u = 0 \quad \text{in} \quad \Omega\ , \qquad u = -\ln|z| \quad \text{on} \quad \partial\Omega.$$

In order to prove the Riemann mapping theorem, we invert this process. We start from the harmonic extension defined from the above equation, we reconstruct H and we prove that it is a bijection between $\overline{\Omega}$ and $\overline{B}_1$.

Teorema 1 (Riemann mapping theorem for smooth sets). *Let Ω be a connected bounded open set in $\mathbb{R}^2$ whose boundary $\partial\Omega$ is:*

- *$C^{k,\alpha}$ regular for some $k \geq 1$ and $\alpha \in (0,1)$;*
- *parametrized by a single closed $C^{k,\alpha}$ curve $\sigma : [0,1] \to \mathbb{R}^2$ with $|\sigma'(t)| > 0$ for all $t \in [0,1]$.*

Then, there is a map

$$H : \overline{\Omega} \to \overline{B}_1$$

such that:

(i) *$H : \overline{\Omega} \to \mathbb{R}^2$ is $C^{k,\alpha}$ regular for some $\alpha \in (0,1)$;*
(ii) *H is holomorphic on Ω;*
(iii) *$H : \overline{\Omega} \to \overline{B}_1$ is a homeomorphism;*
(iv) *the inverse $H^{-1} : \overline{B}_1 \to \overline{\Omega}$ is $C^{k,\alpha}$ regular on $\overline{B}_1$ and holomorphic on B_1.*

Proof. Without loss of generality we can suppose that $0 \in \Omega$. We proceed in several steps.

Step 1. Construction of H. Let

$$u : \overline{\Omega} \to \mathbb{R}$$

be a continuous function solution to

$$\Delta u = 0 \quad \text{in} \quad \Omega\ , \qquad u(z) = -\ln|z| \quad \text{on} \quad \partial\Omega\,.$$

Since $\partial\Omega$ is $C^{k,\alpha}$ and since the boundary datum is smooth, we have that

$$u \in C^{k,\alpha}(\overline{\Omega}).$$

Consider the differential form

$$-\partial_y u\,dx + \partial_x u\,dy.$$

This is a closed form since

$$d\Big(- \partial_y u \, dx + \partial_x u \, dy\Big) = \Delta u \, dx \wedge dy = 0.$$

Then, there is a function $v : \Omega \to \mathbb{R}$ (called *harmonic conjugate of* u) such that

$$dv = -\partial_y u(x, y) \, dx + \partial_x u(x, y) \, dy,$$

or equivalently

$$\begin{cases} \partial_x v = -\partial_y u \\ \partial_y v = \partial_x u \end{cases} \qquad \text{in} \qquad \Omega.$$

For every $z = (x, y) \in \Omega$, $v(x, y)$ can be computed by integrating the differential form $-\partial_y u \, dx + \partial_x u \, dy$ over any curve $\sigma = (\sigma_1, \sigma_2) : [0, 1] \to \Omega$ with

$$\sigma(0) = (0, 0) \quad \text{and} \quad \sigma(1) = (x, y),$$

precisely:

$$v(x, y) = \int_0^1 \Big(\sigma_1'(t)\partial_y u(\sigma_1(t), \sigma_2(t)) + \sigma_2'(t)\partial_x u(\sigma_1(t), \sigma_2(t))\Big) \, dt,$$

which in particular implies that also the function $v : \overline{\Omega} \to \mathbb{R}$ is in $C^{k,\alpha}(\overline{\Omega})$. Finally, we have constructed maps

$$L : \overline{\Omega} \to \mathbb{C} = \mathbb{R}^2 \ , \qquad L(z) = u(z) + iv(z) \, ,$$

and

$$h : \overline{\Omega} \to \mathbb{C} = \mathbb{R}^2 \ , \qquad h(z) = \exp\Big(u(z) + iv(z)\Big) \, ,$$

which are both of class $C^{k,\alpha}(\overline{\Omega}, \mathbb{C})$ and holomorphic in Ω. This implies that the map

$$H : \overline{\Omega} \to \mathbb{C} = \mathbb{R}^2 \ , \qquad H(z) = zh(z)$$

is of class $C^{k,\alpha}(\overline{\Omega}, \mathbb{C})$ holomorphic in Ω.

Step 2. ***H*** **has values in** $\overline{B}_1$**.** Precisely, we will show that

$$H(\partial\Omega) \subset \partial B_1 \qquad \text{and} \qquad H(\Omega) \subset B_1.$$

First we notice that by construction we have

$$|h(z)| = \exp(u(z)) = \exp(-\ln|z|) = \frac{1}{|z|} \quad \text{on} \quad \partial\Omega.$$

This implies that

$$|H(z)| = 1 \quad \text{for all} \quad z \in \partial\Omega.$$

Now, since the function $\ln|z|$ is subharmonic in Ω and u is harmonic in Ω, we have that

$$u(z) + \ln|z| \text{ is subharmonic in } \Omega \text{ and continuous on } \overline{\Omega}.$$

Since $u(z) + \ln|z|$ vanishes on $\partial\Omega$, the strong maximum principle now yields

$$u(z) + \ln|z| < 0 \quad \text{for} \quad z \in \Omega.$$

This implies that

$$1 > \exp\Big(u(z) + \ln|z|\Big) = |z| \exp(u(z)) = |z||h(z)| = |H(z)| \quad \text{for all} \quad z \in \Omega.$$

Step 3. ***H*** **is onto.** We notice that:

- since $H : \overline{\Omega} \to \mathbb{C}$ is continuous, $H(\overline{\Omega})$ is a closed set;
- since $H : \Omega \to \mathbb{C}$ is holomorphic, the set $H(\Omega)$ is open.

In particular, in order to prove that

$$H(\overline{\Omega}) = \overline{B}_1,$$

it is sufficient to prove that

$$B_1 \subset H(\overline{\Omega}).$$

Suppose by contradiction that there is a point

$$w \in B_1 \setminus H(\overline{\Omega}).$$

For every $t \in [0, 1]$, we set

$$w_t := tw$$

and we define t_* to be the largest $t \in [0,1]$ such that $w_{t_*} \in H(\overline{\Omega})$.

We notice that:

- by assumption $w \notin H(\overline{\Omega})$, so we have that $t_* < 1$;
- $H(0) = 0$ so $t_* > 0$.

Let $z_{t_*} \in \overline{\Omega}$ be such that $H(z_{t_*}) = w_{t_*}$. Since $H(\Omega)$ is open, we necessarily have that $z_{t_*} \in \partial\Omega$. But this is impossible since $H(\partial\Omega) \subset \partial B_1$. This proves that $H(\overline{\Omega}) = \overline{B}_1$.

Step 4. $H : \Omega \to B_1$ is injective. Notice that by construction

$$|h| = e^u \neq 0 \quad \text{in} \quad \Omega.$$

Thus, 0 is the unique zero of the function

$$H : \Omega \to \mathbb{C} \ , \qquad H(z) = zh(z),$$

and 0 has multiplicity 1. Thus, for every connected open set $0 \ni D \Subset \Omega$ whose boundary ∂D is parametrized by a single closed regular C^1 curve $\gamma : [0,1] \to \mathbb{C}$ with $\gamma' \neq 0$, we have that:

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{\partial_z H(z)}{H(z)}\, dz.$$

Consider the map

$$z_0 \mapsto \frac{1}{2\pi i} \int_\gamma \frac{\partial_z H(z)}{H(z) - H(z_0)}\, dz$$

defined for every $z_0 \in D$. Since this map is continuous and has values in $\mathbb{N}$ we have that

$$\frac{1}{2\pi i} \int_\gamma \frac{\partial_z H(z)}{H(z) - H(z_0)}\, dz = 1 \quad \text{for every} \quad z_0 \in D.$$

This shows that H is injective as map from D to $\mathbb{C}$. Since D is arbitrary, we get that H is injective as a map from Ω to B_1.

Step 5. Behavior of H at the boundary. Let

$$g(x,y) = \ln|z| = \frac{1}{2}\ln(x^2 + y^2).$$

We notice that since $u + g : \overline{\Omega}$ is continuous in $\overline{\Omega}$, subharmonic and strictly negative in Ω and vanishes on $\partial\Omega$, the Hopf maximum principle implies that

$$\nabla(u + g) \neq 0 \quad \text{on} \quad \partial\Omega,$$

which can be written as

$$\nabla u(x,y) + \frac{(x,y)}{x^2 + y^2} \neq 0 \quad \text{for} \quad (x,y) \in \partial\Omega,$$

or in terms of $z = x + iy$ as

$$(\partial_x u - i\partial_y u) + \frac{1}{z} \neq 0 \quad \text{for} \quad z \in \partial\Omega.$$

We next notice that

$$\begin{aligned}\partial_z L(z) &= \frac{1}{2}(\partial_x - i\partial_y)(u + iv) \\ &= \frac{1}{2}(\partial_x u + \partial_y v) + i\frac{1}{2}(\partial_x v - \partial_y u) \\ &= \partial_x u - i\partial_y u,\end{aligned}$$

so the above condition yields

$$\partial_z L(z) + \frac{1}{z} \neq 0 \quad \text{for} \quad z \in \partial\Omega.$$

Now, we notice that

$$\begin{aligned}\partial_z H(z) &= h(z) + z\partial_z h(z) \\ &= h(z)\Big(1 + \frac{z\partial_z h(z)}{h(z)}\Big) \\ &= h(z)\Big(1 + z\partial_z L(z)\Big),\end{aligned}$$

for every $z \in \overline{\Omega}$. This implies that

$$\partial_z H(z) \neq 0 \quad \text{for} \quad z \in \partial\Omega,$$

which gives that the map

$$H : \overline{\Omega} \to \overline{B}_1$$

is invertible around any boundary point $z \in \partial\Omega$. This proves two things:

- first, since $H : \Omega \to B_1$ is injective, then also $H : \overline{\Omega} \to \overline{B}_1$ should be injective;
- second, the inverse map $H^{-1} : \overline{B}_1 \to \overline{\Omega}$ is $C^{k,\alpha}$ regular up to the boundary.

This concludes the proof. □

In the proof of the Riemann mapping theorem we have used the following well-known formula for the number of zeros of a holomorphic function.

Teorema 2 (Number of zeros of a holomorphic function)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and let $\Phi : \Omega \to \mathbb{C}$ be a holomorphic function. Let $D \Subset \Omega$ be a bounded connected open set such that:*

- *$\Phi \neq 0$ on ∂D;*
- *the boundary ∂D of D is C^1 regular and is parametrized by a positively oriented closed regular C^1 curve $\gamma : [0,1] \to \mathbb{C}$ with $\gamma' \neq 0$.*

Then, the number $N(\Phi, D)$ of zeros (counted with their multiplicity) of Φ in D given by the formula

$$N(\Phi, D) = \frac{1}{2\pi i} \int_\gamma \frac{\partial_z \Phi(z)}{\Phi(z)}\, dz.$$

Proof. Let $N := N(\Phi, D)$ and let $z_1, \dots, z_N$ be the zeros of Φ in D (counted with their multiplicity). Then, Φ can be written as

$$\Phi(z) = (z - z_1)(z - z_2)\dots(z - z_N)\Psi(z),$$

where $\Psi : \Omega \to \mathbb{C}$ is a holomorphic function, which is non-zero in a neoghborhood of $\overline{D}$. One can easily check that

$$\frac{\partial_z \Phi(z)}{\Phi(z)} = \frac{\partial_z \Psi(z)}{\Psi(z)} + \sum_{j=1}^{N} \frac{1}{z - z_j}.$$

Now, since $\frac{\partial_z \Psi}{\Psi}$ is holomorphic in a neighborhood of $\overline{D}$, we have that

$$\frac{1}{2\pi i} \int_\gamma \frac{\partial_z \Psi(z)}{\Psi(z)}\, dz = 0,$$

so that

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{\partial_z \Phi(z)}{\Phi(z)}\, dz &= \frac{1}{2\pi i} \int_\gamma \frac{\partial_z \Psi(z)}{\Psi(z)}\, dz + \sum_{j=1}^{N} \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_j} \\ &= \sum_{j=1}^{N} \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_j}. \end{aligned}$$

Now the claim follows since, by the Cauchy formula, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_j} = 1,$$

for every $z_j \in D$. □