Formule di Gauss-Green

Integrale di una funzione su un intervallo variabile

Lemma 1. *Siano $u : [a,b] \to \mathbb{R}$ e $v : [a,b] \to \mathbb{R}$ due funzioni di classe $C^1([a,b])$ e tali che*

$$u(x) < v(x) \quad \text{per ogni} \quad x \in (a,b).$$

Sia $F : \mathbb{R}^2 \to \mathbb{R}$ una funzione di classe C^1. Allora, la funzione

$$(a,b) \ni x \to \int_{u(x)}^{v(x)} F(x,y)\,dy$$

è derivabile in (a,b) e la sua derivata è :

$$\frac{\partial}{\partial x}\int_{u(x)}^{v(x)} F(x,y)\,dy = v'(x)\,F\big(x,v(x)\big) - u'(x)\,F\big(x,u(x)\big) + \int_{u(x)}^{v(x)} \partial_x F(x,y)\,dy.$$

Dimostrazione 1

Sia $x \in (a,b)$. Per ogni $h \in \mathbb{R}$ tale che $x+h \in (a,b)$, calcoliamo

$$\begin{aligned}\int_{u(x+h)}^{v(x+h)} F(x+h,y)\,dy - \int_{u(x)}^{v(x)} F(x,y)\,dy &= \int_{v(x)}^{v(x+h)} F(x+h,y)\,dy \\ &\quad - \int_{u(x)}^{u(x+h)} F(x+h,y)\,dy \\ &\quad + \int_{u(x)}^{v(x)} \Big(F(x+h,y) - F(x,y)\Big)\,dy.\end{aligned}$$

Per il teorema della media integrale abbiamo

$$\int_{v(x)}^{v(x+h)} F(x+h,y)\,dy = \Big(v(x+h) - v(x)\Big)\,F\big(x+h, v(x)+\kappa_v\big)$$

$$\int_{u(x)}^{u(x+h)} F(x+h,y)\,dy = \Big(u(x+h) - u(x)\Big)\,F\big(x+h, u(x)+\kappa_u\big)$$

dove le funzioni

$$\kappa_v = \kappa_v(x,h) \qquad \text{e} \qquad \kappa_u = \kappa_u(x,h)$$

sono tali che

$$|\kappa_v(x,h)| \le \big|v(x+h) - v(x)\big| \qquad \text{e} \qquad |\kappa_u(x,h)| \le \big|u(x+h) - u(x)\big|.$$

Allora abbiamo

$$\begin{aligned}\frac{1}{h}\left(\int_{u(x+h)}^{v(x+h)} F(x+h,y)\,dy - \int_{u(x)}^{v(x)} F(x,y)\,dy\right) &= \frac{1}{h}\Big(v(x+h) - v(x)\Big)\,F\big(x+h, v(x)+k_v\big) \\ &\quad - \frac{1}{h}\Big(u(x+h) - u(x)\Big)\,F\big(x+h, u(x)+k_u\big) \\ &\quad + \int_{u(x)}^{v(x)} \frac{1}{h}\Big(F(x+h,y) - F(x,y)\Big)\,dy\end{aligned}$$

e passando al limite per $h \to 0$, otteniamo

$$\lim_{h\to 0}\frac{1}{h}\Big(v(x+h) - v(x)\Big)\,F\big(x+h, v(x)+k_v\big) = v'(x)\,F\big(x, v(x)\big),$$

$$\lim_{h\to 0}\frac{1}{h}\Big(u(x+h) - u(x)\Big)\,F\big(x+h, u(x)+k_u\big) = u'(x)\,F\big(x, u(x)\big).$$

Usando invece il teorema di Lagrange, abbiamo che per ogni (x,y,h) esiste
$$w = w(x,y,h) \in \mathbb{R}\,, \qquad |w(x,y,h)| \le |h|$$
tale che
$$\frac{1}{h}\Big(F(x+h,y) - F(x,y)\Big) = \partial_x F\big(x+w,y\big).$$
Di conseguenza,
$$\int_{u(x)}^{v(x)} \frac{1}{h}\Big(F(x+h,y) - F(x,y)\Big)\,dy = \int_{u(x)}^{v(x)} \partial_x F\big(x+w(x,y,h),y\big)\,dy$$
Usando l'uniforme continuità di $\partial_x F$ su
$$D = \Big\{(x,y) \ : \ x \in [a,b],\ u(x) \le y \le v(x)\Big\},$$
otteniamo che
$$\lim_{h\to 0} \int_{u(x)}^{v(x)} \frac{1}{h}\Big(F(x+h,y) - F(x,y)\Big)\,dy = \int_{u(x)}^{v(x)} \partial_x F\big(x,y\big)\,dy\,.$$
Di conseguenza,
$$\lim_{h\to 0} \frac{1}{h}\left(\int_{u(x+h)}^{v(x+h)} F(x+h,y)\,dy - \int_{u(x)}^{v(x)} F(x,y)\,dy\right) = v'(x)\,F\big(x,v(x)\big) - u'(x)\,F\big(x,u(x)\big) + \int_{u(x)}^{v(x)} \partial_x F(x,y)\,dy\,,$$
ovvero la funzione
$$x \mapsto \int_{u(x)}^{v(x)} F(x,y)\,dy\,,$$
definita per $x \in [a,b]$, è integrabile sull'intervallo aperto (a,b) e la sua derivata è data da
$$\frac{\partial}{\partial x}\int_{u(x)}^{v(x)} F(x,y)\,dy = v'(x)\,F\big(x,v(x)\big) - u'(x)\,F\big(x,u(x)\big) + \int_{u(x)}^{v(x)} \partial_x F(x,y)\,dy\,. \qquad \square$$

Dimostrazione 2

Osserviamo che per ogni $x \in (a,b)$
$$\int_{u(x)}^{v(x)} F(x,y)\,dy = \int_0^1 F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v(x)-u(x)\big)\,dt\,.$$
Per il lemma della derivazione sotto il segno di integrale abbiamo che
$$\begin{aligned}
&\partial_x\left[\int_{u(x)}^{v(x)} F(x,y)\,dy\right] \\
&\quad= \int_0^1 \partial_x\Big[F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v(x)-u(x)\big)\Big]\,dt \\
&\quad= \int_0^1 F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v'(x)-u'(x)\big)\,dt \\
&\qquad\quad+ \int_0^1 \partial_x F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v(x)-u(x)\big)\,dt \\
&\qquad\qquad+ \int_0^1 \Big(u'(x) + t\big(v'(x)-u'(x)\Big)\partial_y F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v(x)-u(x)\big)\Big]\,dt. \\
&\quad= \int_{u(x)}^{v(x)} \partial_x F(x,y)\,dy + \int_0^1 F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v'(x)-u'(x)\big)\,dt \\
&\qquad\qquad+ \int_0^1 \Big(u'(x) + t\big(v'(x)-u'(x)\Big)\partial_y F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v(x)-u(x)\big)\Big]\,dt \\
&\quad= \int_{u(x)}^{v(x)} \partial_x F(x,y)\,dy + u'(x)\int_0^1 \partial_y F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\big(v(x)-u(x)\big)\,dt \\
&\qquad\quad+ \big(v'(x)-u'(x)\big)\int_0^1 \Big[F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big) + t\big(v'(x)-u'(x)\Big)\partial_y F\Big(x, u(x) + t\big(v(x)-u(x)\big)\Big)\Big]\,dt\,,
\end{aligned}$$

che possiamo scrivere anche come

$$\begin{aligned}\partial_x\left[\int_{u(x)}^{v(x)}F(x,y)\,dy\right]&=\int_{u(x)}^{v(x)}\partial_x F(x,y)\,dy\\&\qquad+\Big(v'(x)-u'(x)\Big)\int_0^1\partial_t\Big[tF\Big(x,u(x)+t\big(v(x)-u(x)\big)\Big)\Big]\,dt\\&\qquad+u'(x)\int_0^1\partial_t\Big[F\Big(x,u(x)+t\big(v(x)-u(x)\big)\Big)\Big]\,dt\\&=\int_{u(x)}^{v(x)}\partial_x F(x,y)\,dy+\Big(v'(x)-u'(x)\Big)F\big(x,v(x)\big)+u'(x)\Big[F\big(x,v(x)\big)-F\big(x,u(x)\big)\Big]\\&=\int_{u(x)}^{v(x)}\partial_x F(x,y)\,dy+v'(x)F\big(x,v(x)\big)-u'(x)F\big(x,u(x)\big).\quad\square\end{aligned}$$

Formule di Gauss-Green

Teorema 2 (Formule di Gauss-Green)**.** *Siano*

$$u:[a,b]\to\mathbb{R}\qquad e\qquad v:[a,b]\to\mathbb{R}\,,$$

due funzioni di classe C^1 su $[a,b]$ tali che

$$u(x)<v(x)\quad\text{per ogni}\quad x\in(a,b)\,.$$

Sia D il dominio normale semplice determinato dalle funzioni u e v

$$D=\Big\{(x,y)\ :\ x\in[a,b],\ u(x)\le y\le v(x)\Big\}\,.$$

Allora, per ogni funzione $F:\mathbb{R}^2\to\mathbb{R}$ di classe C^1 abbiamo

$$\begin{aligned}\iint_D\partial_y F(x,y)\,dx\,dy&=\int_a^b\int_{u(x)}^{v(x)}\partial_y F(x,y)\,dy\,dx=\int_a^b\Big(F\big(x,v(x)\big)-F\big(x,u(x)\big)\Big)\,dx\ ;\\\iint_D\partial_x F(x,y)\,dx\,dy&=\int_a^b\int_{u(x)}^{v(x)}\partial_x F(x,y)\,dy\,dx=\int_a^b u'(x)\,F\big(x,u(x)\big)\,dx-\int_a^b v'(x)\,F\big(x,v(x)\big)\,dx\\&\qquad+\int_{u(b)}^{v(b)}F(b,y)\,dy-\int_{u(a)}^{v(a)}F(a,y)\,dy\,.\end{aligned}$$

Inoltre, le formule di Gauss-Green si possono scrivere anche come

$$\iint_D\partial_x F(x,y)\,dy\,dx=\int_\gamma F(x,y)\,dy\qquad e\qquad\iint_D\partial_y F(x,y)\,dy\,dx=-\int_\gamma F(x,y)\,dx,$$

dove γ è la curva semplice chiusa e C^1 a tratti che parametrizza il bordo ∂D in senso antiorario.