2-forme chiuse in $\mathbb{R}^3$

Teorema 1. *Sia Ω un rettangolo aperto in $\mathbb{R}^3$. Se*

$$\alpha = a(x,y,z)\,dy\wedge dz + b(x,y,z)\,dz\wedge dx + c(x,y,z)\,dx\wedge dy\,,$$

una 2-forma chiusa di classe C^1 in Ω, allora α è esatta.

Dimostrazione: Cerchiamo una 1-forma

$$A(x,y,z)dx + B(x,y,z)\,dy + C(x,y,z)\,dz,$$

tale che

$$\begin{aligned}
a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy &= d\Big(A\,dx + B\,dy + C\,dz\Big)\\
&= \Big(\partial_y A\,dy + \partial_z A\,dz\Big)\wedge dx\\
&\quad + \Big(\partial_x B\,dx + \partial_z B\,dz\Big)\wedge dy\\
&\quad + \Big(\partial_x C\,dx + \partial_y C\,dy\Big)\wedge dz\\
&= \Big(\partial_y C - \partial_z B\Big)dy\wedge dz + \Big(\partial_z A - \partial_x C\Big)dz\wedge dx + \Big(\partial_x B - \partial_y A\Big)dx\wedge dy.
\end{aligned}$$

In altri termini cerchiamo tre funzioni A, B e C tali che

$$\begin{cases}\partial_y C - \partial_z B = a\\ \partial_z A - \partial_x C = b\\ \partial_x B - \partial_y A = c\end{cases}$$

Definiamo

$$A(x,y,z) := \frac{1}{3}\int_0^z b(x,y,z')\,dz' - \frac{1}{3}\int_0^y c(x,y',z)\,dy'$$

$$B(x,y,z) := \frac{1}{3}\int_0^x c(x',y,z)\,dx' - \frac{1}{3}\int_0^z a(x,y,z')\,dz'$$

$$C(x,y,z) := \frac{1}{3}\int_0^y a(x,y',z)\,dy' - \frac{1}{3}\int_0^x b(x',y,z)\,dx'$$

Calcoliamo ora

$$\begin{aligned}
\partial_y C(x,y,z) - \partial_z B(x,y,z) &= \frac{1}{3}a(x,y,z) - \frac{1}{3}\int_0^x \partial_y b(x',y,z)\,dx' - \frac{1}{3}\int_0^x \partial_z c(x',y,z)\,dx' + \frac{1}{3}a(x,y,z)\\
&= \frac{2}{3}a(x,y,z) - \frac{1}{3}\int_0^x \Big(\partial_y b(x',y,z) + \partial_z c(x',y,z)\Big)\,dx'\\
&= \frac{2}{3}a(x,y,z) + \frac{1}{3}\int_0^x \partial_x a(x',y,z)\,dx'\\
&= \frac{2}{3}a(x,y,z) + \frac{1}{3}a(x,y,z) - \frac{1}{3}a(0,y,z)\\
&= a(x,y,z) - \frac{1}{3}a(0,y,z)
\end{aligned}$$

Analogamente,

$$\partial_z A(x,y,z) - \partial_x C(x,y,z) = b(x,y,z) - \frac{1}{3}b(x,0,z),$$

$$\partial_x B(x,y,z) - \partial_y A(x,y,z) = c(x,y,z) - \frac{1}{3}c(x,y,0).$$

Quindi, basta dimostrare che le 2-forme

$$a(0,y,z)\,dy\wedge dz\ , \qquad b(x,0,z)\,dz\wedge dx\ , \qquad c(x,y,0)\,dx\wedge dy\ ,$$

sono esatte. Infatti, prendendo

$$\varphi_a(x,y,z) = \int_0^y a(0,y',z)\,dy'$$

si ha

$$d\Big(\varphi_a(x,y,z)\,dz\Big) = \Big(\partial_x\varphi_a\,dx + \partial_y\varphi_a\,dy\Big)\wedge dz = \partial_y\varphi_a\,dy\wedge dz = a(0,y,z)\,dy\wedge dz.$$

Analogamente, anche le 2-forme

$$b(x,0,z)\,dz\wedge dx \qquad \text{e} \qquad c(x,y,0)\,dx\wedge dy\ ,$$

sono esatte. □