Un altro esercizio sul calcolo di lim sup e lim inf all'infinito

Esercizio 1. *Data la funzione* $F : \mathbb{R}^2 \to \mathbb{R}$

$$F(x,y) = \frac{xy + x}{x^2 + y^2 + 1},$$

calcolare $\limsup_{|(x,y)|\to+\infty} F(x,y)$ *e* $\liminf_{|(x,y)|\to+\infty} F(x,y)$.

Soluzione. In coordinate polari abbiamo

$$F(R\cos\theta, R\sin\theta) = \frac{R^2\cos\theta\sin\theta + R\cos\theta}{R^2+1}.$$

Ora scriviamo

$$F(R\cos\theta, R\sin\theta) = f(R,\theta) + g(R,\theta),$$

dove

$$f(R,\theta) = \frac{R^2\cos\theta\sin\theta}{R^2+1} \qquad \text{e} \qquad g(R,\theta) = \frac{R\cos\theta}{R^2+1}\ .$$

Abbiamo che

$$|g(R,\theta)| \le \frac{R}{1+R^2} \qquad \text{per ogni} \qquad \theta \in [0,2\pi].$$

Di conseguenza,

$$\sup_{\theta\in[0,2\pi]} f(R,\theta) - \frac{R}{1+R^2} \le \sup_{\theta\in[0,2\pi]} F(R\cos\theta, R\sin\theta) \le \sup_{\theta\in[0,2\pi]} f(R,\theta) + \frac{R}{1+R^2}.$$

$$\inf_{\theta\in[0,2\pi]} f(R,\theta) - \frac{R}{1+R^2} \le \inf_{\theta\in[0,2\pi]} F(R\cos\theta, R\sin\theta) \le \inf_{\theta\in[0,2\pi]} f(R,\theta) + \frac{R}{1+R^2}.$$

Passando al limite per $R \to +\infty$, otteniamo che

$$\limsup_{R\to+\infty}\Big\{\sup_{\theta\in[0,2\pi]} F(R\cos\theta, R\sin\theta)\Big\} = \limsup_{R\to+\infty}\Big\{\sup_{\theta\in[0,2\pi]} f(R,\theta)\Big\},$$

$$\liminf_{R\to+\infty}\Big\{\inf_{\theta\in[0,2\pi]} F(R\cos\theta, R\sin\theta)\Big\} = \liminf_{R\to+\infty}\Big\{\inf_{\theta\in[0,2\pi]} f(R,\theta)\Big\}.$$

Quindi, basta calcolare

$$\limsup_{R\to+\infty}\Big\{\sup_{\theta\in[0,2\pi]} f(R,\theta)\Big\} \qquad \text{e} \qquad \liminf_{R\to+\infty}\Big\{\inf_{\theta\in[0,2\pi]} f(R,\theta)\Big\}.$$

Osserviamo che

$$f(r,\theta) = \frac{R^2\cos\theta\sin\theta}{R^2+1} = \frac{R^2}{R^2+1}\frac{\sin(2\theta)}{2},$$

e, siccome $\sup_{\theta\in[0,2\pi]} \sin(2\theta) = 1$ e $\inf_{\theta\in[0,2\pi]} \sin(2\theta) = -1$,

$$\sup_{\theta\in[0,2\pi]} f(R,\theta) = \frac{1}{2}\frac{R^2}{R^2+1} \qquad \text{e} \qquad \inf_{\theta\in[0,2\pi]} f(R,\theta) = -\frac{1}{2}\frac{R^2}{R^2+1}.$$

Passando al limite per $R \to +\infty$,

$$\limsup_{R\to+\infty}\Big\{\sup_{\theta\in[0,2\pi]} f(R,\theta)\Big\} = \frac{1}{2} \qquad \text{e} \qquad \liminf_{R\to+\infty}\Big\{\inf_{\theta\in[0,2\pi]} f(R,\theta)\Big\} = -\frac{1}{2}.$$

In conclusione

$$\limsup_{|(x,y)|\to+\infty} F(x,y) = \frac{1}{2} \qquad \text{e} \qquad \liminf_{|(x,y)|\to+\infty} F(x,y) = -\frac{1}{2}.$$

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