Un esercizio sul calcolo di lim sup e lim inf all'inifinito

Esercizio 1. *Data la funzione* $F : \mathbb{R}^2 \to \mathbb{R}$

$$F(x,y) = \frac{xy}{x^2 + y^4 + 1},$$

calcolare $\limsup\limits_{|(x,y)|\to+\infty} F(x,y)$ *e* $\liminf\limits_{|(x,y)|\to+\infty} F(x,y)$.

Soluzione. In coordinate polari abbiamo

$$F(R\cos\theta, R\sin\theta) = \frac{R^2\cos\theta\sin\theta}{R^2\cos^2\theta + R^4\sin^4\theta + 1}.$$

Consideriamo ora l'equazione

$$\partial_\theta\left[\frac{R^2\cos\theta\sin\theta}{R^2\cos^2\theta + R^4\sin^4\theta + 1}\right] = 0,$$

ovvero

$$\begin{aligned}
0 &= \partial_\theta\Big[\cos\theta\sin\theta\Big]\Big(R^2\cos^2\theta + R^4\sin^4\theta + 1\Big) - \Big(\cos\theta\sin\theta\Big)\partial_\theta\Big[R^2\cos^2\theta + R^4\sin^4\theta + 1\Big] \\
&= \Big[-\sin^2\theta + \cos^2\theta\Big]\Big(R^2\cos^2\theta + R^4\sin^4\theta + 1\Big) - \Big(\cos\theta\sin\theta\Big)\Big[-2R^2\cos\theta\sin\theta + 4R^4\sin^3\theta\cos\theta\Big] \\
&= \Big(-\sin^2\theta + \cos^2\theta\Big) + R^2\cos^2\theta - R^4\sin^4\theta\Big(\sin^2\theta + 3\cos^2\theta\Big),
\end{aligned}$$

che possiamo scrivere anche come

$$\sin^4\theta\Big(3 - 2\sin^2\theta\Big) = \frac{1}{R^2}\Big(1 - \sin^2\theta\Big) + \frac{1}{R^4}\Big(1 - 2\sin^2\theta\Big). \tag{1}$$

Se θ_R è una qualsiasi delle soluzioni di (1), allora

$$\sin^4(\theta_R) \le \frac{1}{R^2} + \frac{1}{R^4} \le \frac{2}{R^2} \qquad \text{e quindi} \qquad \sin^2(\theta_R) \le \frac{2}{R}.$$

Di conseguenza,

$$\begin{aligned}
\sin^4(\theta_R) &= \frac{\frac{1}{R^2}\Big(1 - \sin^2\theta_R\Big) + \frac{1}{R^4}\Big(1 - 2\sin^2\theta_R\Big)}{3\Big(1 - \frac{2}{3}\sin^2\theta_R\Big)} \\
&= \frac{\frac{1}{R^2}\Big(1 + O(1/R)\Big) + \frac{1}{R^4}\Big(1 + O(1/R)\Big)}{3\Big(1 + O(1/R)\Big)} = \frac{1}{R^2}\frac{\Big(1 + O(1/R)\Big)}{3\Big(1 + O(1/R)\Big)},
\end{aligned}$$

e quindi

$$\sin^2(\theta_R) = \frac{1}{\sqrt{3}}\frac{1}{R}\Big(1 + O(1/R)\Big) \qquad \text{e} \qquad \cos^2(\theta_R) = 1 + O(1/R).$$

Di conseguenza, per una qualsiasi soluzione θ_R di (1) si ha

$$\big|F(R\cos\theta_R, R\sin\theta_R)\big| = \frac{R^2\frac{1}{3^{1/4}}\frac{1}{\sqrt{R}}\Big(1 + O(1/R)\Big)^{1/2}\Big(1 + O(1/R)\Big)^{1/2}}{R^2\Big(1 + O(1/R)\Big) + R^4\frac{1}{2R^2}\Big(1 + O(1/R)\Big) + 1} = O\Big(\frac{1}{\sqrt{R}}\Big).$$

Quindi

$$\sup_{\theta\in[0,2\pi]}|F(R\cos\theta, R\sin\theta)| = O\Big(\frac{1}{\sqrt{R}}\Big),$$

e in conclusione

$$\lim_{|(x,y)|\to+\infty} F(x,y) = \limsup_{|(x,y)|\to+\infty} F(x,y) = \liminf_{|(x,y)|\to+\infty} F(x,y) = 0.$$

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