Algebras of Generalized Functions and Nonstandard Analysis

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1. Generalized functions: introduction and motivation
   - Linear generalized functions (distributions)
   - Nonlinear generalized functions

2. Improving generalized functions by means of ultrafilters
   - Idea of construction
   - Properties

Linear generalized functions: Dirac's δ-impulse

- Physical interpretation: singular object with an infinite concentration at the origin $x = 0$, e.g. mass distribution of a unit point mass.
- Formal property: $\int_{\mathbb{R}^n} \delta(x)\varphi(x)\,dx = \varphi(0)$, for each $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$. $(*)$

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Observation 1

- The map $\mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathbb{R}$: $\varphi \mapsto \varphi(0)$ is a continuous linear map.
- This map captures the essence of the formal property $(*)$.

Observation 2

- For any (locally integrable) function f, the map $\mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathbb{R}$: $\varphi \mapsto \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$ is a continuous linear map.
- This map determines f completely (up to measure zero).

$\mathcal{C}_c^\infty(\mathbb{R}^n) = \{$ smooth functions with compact support $\}$

Linear generalized functions: distributions

Definition

A continuous linear map $\mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a (Schwartz) **distribution**.

There exists a natural definition of partial differentiation on distributions, extending the classical definition for $\mathcal{C}^1$-functions.
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Applications

- Justification of formulas containing derivatives of nondifferentiable functions used by physicists
- Theory of partial differential equations (PDEs):
  every linear PDE with constant coefficients has a distributional solution (L. Ehrenpreis, B. Malgrange, 1955).
- Formulation of Quantum Field Theory.

Multiplication of distributions

- Linear operations $(+, \partial_j, \int)$ can be defined naturally on distributions.
- Products and other nonlinear operations have no natural counterpart on the space of distributions.

Example: δ^2, $\sqrt{\delta}$ do not make sense as distributions.

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Yet:

- In theoretical physics, formal products of distributions are used (e.g., in quantum field theory, general relativity).
- Nonlinear PDEs with singular (discontinuous or distributional) data occur as models of real-world phenomena (e.g. in geophysics).

Need for a mathematical theory.

The algebra $\mathcal{G}$ of nonlinear generalized functions

Idea

- A (Colombeau) **nonlinear generalized function** $\in \mathcal{G}$ is constructed by means of a net (=family) of $\mathcal{C}^\infty$-functions.
- $\mathcal{G}$ should contain the space of distributions.
- A product in $\mathcal{G}$ should be defined that coincides with the product of (sufficiently regular) usual functions.

$\mathcal{G}$ will be a differential algebra provided with an embedding (=injective morphism) of the space of distributions.

The algebra $\mathcal{G}$ of nonlinear generalized functions

Construction of $\mathcal{G}$ (J.F. Colombeau):
$(\mathcal{C}^\infty)^{(0,1)} :=$ {nets of smooth functions indexed by a parameter $\varepsilon \in (0,1)$}.

To ensure an embedding of distributions with good properties, the nets are restricted by a growth condition:

$$\mathcal{A} = \{(u_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty)^{(0,1)} :$$
$$(\forall K \subset\subset \mathbb{R}^n)(\forall \alpha \in \mathbb{N}^n)(\exists N \in \mathbb{N})(\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \le \varepsilon^{-N}, \text{ for small } \varepsilon)\}.$$

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Two nets are identified if their difference belongs to the differential ideal

$$\mathcal{I} = \{(u_\varepsilon)_\varepsilon \in \mathcal{A} : \\ (\forall K \subset\subset \mathbb{R}^n)(\forall \alpha \in \mathbb{N}^n)(\forall m \in \mathbb{N})(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{m}, \text{ for small } \varepsilon)\}.$$

By definition, $\mathcal{G} = \mathcal{A}/\mathcal{I}$.

The algebra $\mathcal{G}$ of nonlinear generalized functions

Distributions are embedded into $\mathcal{G}$ by smoothing. The embedding preserves the vector space operations and ∂_j.

Theorem (Nonlinear operations in $\mathcal{G}$)

If $F \in \mathcal{C}^\infty(\mathbb{R}^m)$ with all derivatives of polynomial growth and $u_1, \ldots, u_m \in \mathcal{G}$, the composition $F(u_1, \ldots, u_m) \in \mathcal{G}$ is well-defined and coincides with the usual composition if $u_1, \ldots, u_m \in \mathcal{C}^\infty$.

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The theorem is optimal, in the following sense:

Theorem (Schwartz impossibility result)

One cannot construct a differential algebra $\mathcal{A}$ containing the distributions such that the product $u_1 \cdot u_2$ in $\mathcal{A}$ coincides with the usual product, if u_1, $u_2 \in \mathcal{C}^k$ (for fixed $k \in \mathbb{N}$).

The ring $\widetilde{\mathbb{R}}$ of generalized numbers

Let $u \in \mathcal{G}$.

- $\int_{\mathbb{R}^n} u(x)\, dx$ can be defined as a generalized number.
- The point value $u(a)$ at $a \in \mathbb{R}^n$ can be defined as a generalized number.
- The set of generalized numbers $\widetilde{\mathbb{R}}$ coincides with the set of generalized functions in $\mathcal{G}$ with zero gradient.
- $\widetilde{\mathbb{R}}$ is a non-archimedean partially ordered ring that contains $\mathbb{R}$.

Example: $\delta(0) \in \widetilde{\mathbb{R}}$, $\int_{\mathbb{R}^n} \delta^2(x)\, dx \in \widetilde{\mathbb{R}}$ are infinitely large numbers.

Ultrafilters in generalized function theory

$\widetilde{\mathbb{R}}$ is a **partially ordered** ring with **zero divisors**.

- Hard to interpret: the value of a generalized function can be a number not comparable with a real number?
- Hard to obtain results: e.g., the Hahn-Banach theorem, a basic tool in functional analysis, does not hold for Banach spaces over $\widetilde{\mathbb{R}}$.

By means of ultrafilters, the algebraic properties of nonlinear generalized functions can be improved (M. Oberguggenberger, T. Todorov, 1998).

An improved version of $\mathcal{G}$: idea of construction

Let $\mathcal{U}$ be a nontrivial ultrafilter on $(0,1)$.
In the spirit of ultrafilter-models of nonstandard analysis, an algebra of generalized functions $\mathcal{G}_\mathcal{U} := \mathcal{A}_\mathcal{U}/\mathcal{I}_\mathcal{U}$ can be defined, where

$$\mathcal{A}_\mathcal{U} = \{(u_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty)^{(0,1)} : \\ (\forall K \subset\subset \mathbb{R}^n)(\forall \alpha \in \mathbb{N}^n)(\exists N \in \mathbb{N})(\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \le \varepsilon^{-N}, \quad \mathcal{U}\text{-a.e.})\},$$

$$\mathcal{I}_\mathcal{U} = \{(u_\varepsilon)_\varepsilon \in \mathcal{A}_\mathcal{U} : \\ (\forall K \subset\subset \mathbb{R}^n)(\forall \alpha \in \mathbb{N}^n)(\forall m \in \mathbb{N})(\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \le \varepsilon^{m}, \quad \mathcal{U}\text{-a.e.})\}.$$

It can be checked that this modification does not destroy the desirable properties of $\mathcal{G}$ (in particular, the good embedding of the distributions).

An improved version of $\mathcal{G}$: properties

Within $\mathcal{G}_{\mathcal{U}}$:

- The generalized numbers are isomorphic with the nonstandard **field of asymptotic numbers** ${}^{\rho}\mathbb{R}$ (A. Robinson, 1972).
- ${}^{\rho}\mathbb{R}$ is a totally ordered, real closed field.
- $\mathcal{G}_{\mathcal{U}}$ is isomorphic with an algebra of **pointwise**, infinitely differentiable **functions** ${}^{\rho}\mathbb{R}^{n} \to {}^{\rho}\mathbb{R}$.
- The **Hahn-Banach theorem** holds for Banach spaces over ${}^{\rho}\mathbb{R}$.

Using principles from nonstandard analysis, problems can be solved more easily.

The full algebra $\mathcal{G}_{full}$ of nonlinear generalized functions

Embedding of distributions in $\mathcal{G}$

- Fix a particular net $(\varphi_\varepsilon)_\varepsilon$ that approximates δ.
- The embedded image of a distribution T is the net $(T \star \varphi_\varepsilon)_\varepsilon$, approximating T.

The choice of the net $(\varphi_\varepsilon)_\varepsilon$ is not unique and represents one particular way to approximate δ. If one is free to choose an approximation to solve a particular problem, $\mathcal{G}$ can be used.

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If the solution of a problem needs to be independent of the approximation, the so-called **full algebra** $\mathcal{G}_{full}$ (J.-F. Colombeau, 1983) is used.
$u \in \mathcal{G}_{full}$ is a net of smooth functions **indexed by** $\mathcal{C}_c^\infty(\mathbb{R}^n)$ (up to a certain identification).

Embedding of distributions in $\mathcal{G}_{full}$ (canonical)

- The embedded image of a distribution T is the net $(T \star \varphi)_\varphi$.

An improved version of $\mathcal{G}_{full}$

$$\boxed{\begin{array}{ccc} \mathcal{G} & \to & \mathcal{G}_{\mathcal{U}} \\ \mathcal{G}_{full} & \to & ? \end{array}}$$

- $\mathcal{G}_{full} = \mathcal{A}_{full}/\mathcal{I}_{full}$, but $\mathcal{A}_{full}$, $\mathcal{I}_{full}$ do not lend themselves to an interpretation as sets of nets in which a certain growth property holds modulo a filter on $\mathcal{C}_c^\infty(\mathbb{R}^n)$.
- Adapting the definition of $\mathcal{G}_{full}$ to this requirement causes technical difficulties: it is no longer clear that the nets representing distributions $(T \star \varphi)_\varphi \in \mathcal{A}_{full}$!
- By a careful choice of an ultrafilter $\mathcal{U}$ on $\mathcal{C}_c^\infty(\mathbb{R}^n)$, one can ensure that $(T \star \varphi)_\varphi \in \mathcal{A}_{\mathcal{U},full}$. The resulting algebra $\mathcal{G}_{\mathcal{U},full}$ satisfies both the good algebraic properties of $\mathcal{G}_{\mathcal{U}}$ and the good (canonical) embedding properties of $\mathcal{G}_{full}$ (T. Todorov, H. Vernaeve, 2007[1]).

[1] *Full algebra of generalized functions and nonstandard asymptotic analysis*, to appear in *Logic And Analysis*, arXiv:0712:2603.

Summary

- To describe singular physical phenomena, generalized functions (distributions) were introduced.
- When nonlinear operations are used, a more general theory of nonlinear generalized functions is needed.
- Ultrafilters can be used to improve the algebraic properties of nonlinear generalized function algebras.

Reference for the theory of Colombeau nonlinear generalized functions:

M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer, *Geometric Theory of Generalized Functions*, Kluwer, 2001.