Local in-time existence and regularity of solutions of the Navier-Stokes equations via discretization

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The Navier-Stokes problem

The initial value problem for the Navier-Stokes equations is:

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla) u = -\nabla p + f & \text{in } \mathcal{D} \\ \operatorname{div} u = 0 & \text{in } \mathcal{D} \\ u = u_0 & \text{on } \mathbb{T}^n \times \{0\} \end{cases} \tag{1}$$

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The unique (if it exists) strong solution of (1) is a pair of "sufficiently regular" functions $u : \mathcal{D} \to \mathbb{R}^n$, $p : \mathcal{D} \to \mathbb{R}$ satisfying (1)

Modified Navier-Stokes equations

Consider a cut-off function $\chi^M \in C^\infty([0,\infty))$ such that:

$$\chi^M(r) = \begin{cases} 1 & \text{if } r < M \\ 0 & \text{if } r > 2M \end{cases}$$

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Let $\|u\|_{1,2} = \max_i \|u_{x_i}\|_{L^\infty(\mathbb{T}^n \times [0,T])} + \max_{i,j} \|u_{x_i x_j}\|_{L^\infty(\mathbb{T}^n \times [0,T])}$.

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$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u = -\nabla p + f & \text{in } \mathcal{D} \\ \Delta p = -\chi^M(\|u\|_{1,2})\,\mathrm{tr}(D_x u)^2 & \text{in } \mathcal{D} \\ u = u_0 & \text{on } \mathbb{T}^n \times \{0\} \end{cases} \tag{2}$$

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Note that $\mathrm{div}(u\cdot\nabla)u = \mathrm{tr}(D_x u)^2 = \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$

Discretized space and time

- Discretization of $\mathbb{T}^n$ (for any $n \in \mathbb{N}$):

$$\begin{aligned} \mathbb{T}^n_M &= \Big\{0, h, 2h, \ldots, (M-1)h, 1\Big\}^n \\ &= h\,(\mathbb{Z} \bmod M)^n; \end{aligned}$$

with $M \in \mathbb{N}_1$, and $h = \frac{1}{M}$.

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Given any $x = (m_1, m_2, \ldots, m_n)h$ and $y = (l_1, l_2, \ldots, l_n)h$ in $\mathbb{T}^n_M$, let:

$$x + y = \Big((m_1 + l_1) \bmod M, \ldots, (m_n + l_n) \bmod M\Big)h$$

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- Discretization of time: with $T \in \mathbb{R}^+$ and $K \in \mathbb{N}_1$, let $k = \frac{T}{K}$, and define:

$$I^T_K = \left\{0, k, 2k, \ldots, (K-1)k\right\} = k\,(\mathbb{N} \cap [0, K));$$

Gridfunctions

- Discretization of $\mathbb{T}^n \times [0, T]$: to each triple $d = (M, K, T)$, with $T \in \mathbb{R}^+$ and $M, N \in \mathbb{N}_1$ we associate discretizations as defined above. Let:

$$\mathcal{D}_d = \mathbb{T}^n_M \times \left\{0, k, 2k, \ldots, (K-1)k\right\}$$

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- Gridfunctions:

$$U : \overline{\mathcal{D}}_d \to \mathbb{R}^n$$

$$P : \overline{\mathcal{D}}_d \to \mathbb{R}^n$$

Finite diference operators

- Discretization of the gradient:

$$\nabla_d U(x,t) = \frac{1}{2h}\Big(U(x+he_i,t) - U(x-he_i,t)\Big)_{i=1,\dots,n}$$

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- Discretization of the laplacian:

$$\Delta_d U(x,t) = \frac{1}{h^2}\sum_{i=1}^{n}\Big(U(x+he_i,t) - 2U(x,t) + U(x-he_i,t)\Big)$$

Discretization of $Pu = u_t - \nu\Delta u + (u \cdot \nabla)u$:

$$P_d U(x,t) = \frac{U(x,t+k) - U(x,t)}{k} - \nu\Delta_d U(x,t) + \sum_{i=1}^{n} U_i(x,t)\frac{U(x+he_i,t) - U(x-he_i,t)}{2h}$$

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$$= \frac{1}{\lambda h^2}\Bigg(U(x,t+k) - (1-2n\nu\lambda)U(x,t) - \lambda \sum_{i=1}^{n}\left(\left(\nu - \frac{h}{2}U_i(x,t)\right)U(x+he_i,t) + \left(\nu + \frac{h}{2}U_i(x,t)\right)U(x-he_i,t)\right)\Bigg)$$

where $\lambda = \frac{k}{h^2} = \frac{TM^2}{K} \in \mathbb{R}^+$,

The finite-diference problem

$$\begin{cases} P_d U(x,t) = -\nabla_d P + f(x,t) & \text{in } \mathcal{D}_d \\ \Delta_d P = -\chi^M(\|U\|^d_{1,2}) \displaystyle\sum_{i,j=1}^{n} \delta^0_{h,j} U_i \, \delta^0_{h,i} U_j & \text{in } \mathcal{D}_d \\ U(x,0) = u_0(x) & \text{on } \mathbb{T}^n_M. \end{cases}$$

where

$$\delta^0_{h,i} U_j(x,t) = \frac{1}{2h}\Big(U_j(x+he_i,t) - U_j(x-he_i,t)\Big)$$

Explicit solution of the finite-diference problem

Solving $P_d U(x,t) = -\nabla_d P(x,t) + f(x,t)$ for $U(x, t+k)$ yields:

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$$\begin{aligned} U(x,t+k) &= \left(1 - 2n\nu\lambda\right) U(x,t) \\ &+\lambda \sum_{i=1}^{n} \left(\left(\nu - \frac{h}{2} U_i(x,t) \right) U(x+he_i,t) \right. \\ &\qquad \left. + \left(\nu + \frac{h}{2} U_i(x,t) \right) U(x-he_i,t) \right) \\ &+\lambda h^2 \bar{f}(x,t) \end{aligned}$$

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where $\bar{f}(x,t) = -\nabla_d P(x,t) + f(x,t)$.

Iteration function

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with

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Solution of the discrete problem (3)

$$\begin{cases} U(x,t+k) = \Phi(U,U)(x,t) + \lambda h^2 \overline{f}(x,t) & (x,t) \in \mathcal{D}_d \\ \Delta_h P(x,t) = -\chi^M(\|U\|_{1,2}^d) \sum_{i,j=1}^{n} \delta_{j,h} U_i(x,t)\, \delta_{i,h} U_j(x,t) & (x,t) \in \mathcal{D}_d \\ U(x,0) = u_0(x,0) & x \in \mathbb{T}_M^n. \end{cases}$$

where $\overline{f}(x,t) = -\nabla_d P(x,t) + f(x,t)$.

Stability conditions

Stability conditions are needed to ensure that the iterates behave nicely. These conditions ensure that $\phi(U, U)(x, t)$ is a weighted average of the values of U at (x, t) and its neighbouring points.

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- Stability condition: $\lambda < \dfrac{1}{2n\nu}$
- Need also $\dfrac{h}{2} U_i(x, t)$ "small": h infinitesimal.

Estimate for discretized Poisson equation

The solutions of

$$\Delta_d P = \chi^M(\|U\|_{1,2}^d) \sum_{i,j=1}^{n} \delta_{h,j}^0 U_i \; \delta_{h,i}^0 U_j$$

can be estimated using the Maximum Principle for Δ_d on an appropriate domain (A. Brandt's method).

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Lemma

For some $C = C(n) > 0$ finite):

$$\|P\|^d_{0,2} \leq CM^2$$

Properties of the iteration function

We now work in $\langle V(\mathbb{R}), {}^*V(\mathbb{R}), *\rangle$

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Lemma

Let h be a positive infinitesimal. Let $U, V, W, Z \in (\mathbb{R}^n)^{\overline{\mathcal{D}}_d}$. If there exists an $M \in {}^*\mathbb{R}$ such that, for all $(x,t) \in \overline{\mathcal{D}}_d$, $|U(x,t)| \leq M$ and $V(x,t)$ is finite, then, for all $(x,t) \in \overline{\mathcal{D}}_d$:

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(a) $|\Phi(U,V)(x,t)| \leq M$

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(a) $|\Phi(U,V)(x,t)| \leq M$

(b) $\Phi(U,W)(x,t) - \Phi(V,Z)(x,t) = \Phi(U-V,W)(x,t) - \frac{\lambda h}{2} \sum_{i=1}^{n} \Big(Z_i(x,t) - W_i(x,t)\Big)\Big(V(x+he_i,t) - V(x-he_i,t)\Big).$

Estimate for $\|U\|_{L^\infty_d(A)}$

Let $\|U\|_{L^\infty_d(A)} = \max_{(x,t)\in A} |U(x,t)|$

Lemma

If U is the solution of the discrete problem then

$$\begin{aligned} \|U\|_{L^\infty_d(\overline{\mathcal{D}}_d)} &\leq \|u_0\|_{L^\infty} + T\|\bar{f}\|_{L^\infty_d} \\ &\leq \|u_0\|_{L^\infty} + T\left(CM^2 + \|f\|_{L^\infty}\right) \end{aligned}$$

Estimate for $[U]_{L_d^\infty(A)}$

Let $[U]_{L_d^\infty(A)} = \max_{(x,t),(y,t)\in A,\ x\neq y} \dfrac{|U(x,t) - U(y,t)|}{|x-y|}$

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Lemma

Let

$$F_0 = \left(\frac{1}{\sqrt{n}}[\bar{f}]_{L_d^\infty(\overline{\mathcal{D}}_d)}\right)^{1/2} \leq \left(\frac{CM^2+[f]_{L^\infty}}{\sqrt{n}}\right)^{1/2}$$

$$L_0 = [u_0]_{L_d^\infty(\mathbb{T}_M^n)}$$

If U is the solution of the discrete problem then for any $T < \frac{1}{\sqrt{n}(F_0+L_0)}$, $[U]_{L_d^\infty(\overline{\mathcal{D}}_d)}$ is uniformly bounded by a constant depending only on T, u_0, f, n and M.

Estimate for $[[U]]_{L^\infty_d(A)}$

- Let $[[U]]_{L^\infty_d(A)} = \max_{(x,t),(x,t+k)\in A} \dfrac{|U(x,t+k) - U(x,t)|}{k}$

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- Let $[[U]]_{L_d^\infty(A)} = \max_{(x,t),(x,t+k)\in A} \dfrac{|U(x,t+k) - U(x,t)|}{k}$
- Let $L_2 = \max\left([[u_0]]_{L_d^\infty(\mathbb{T}_M^n)}, \frac{1}{n}([[\bar{f}]]_{L_d^\infty(\overline{\mathcal{D}}_d)})^{1/2}\right)$

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- Using the regularity of u_0:

$$\frac{U(x,k)-U(x,0)}{k} = \nu \sum_{i=1}^{n} \delta_{h,i,i}^0 U$$

$$- \textstyle\sum_{i=1}^n U_i \delta_{h,i}^0 U + \bar{f}(x,0)$$

$$\approx \nu \Delta u_0(x) - (u_0 \cdot \nabla) u_0(x) - \nabla p(x,0) + f(x,0)$$

This gives us an estimate for $[[u_0]]_{L_d^\infty(\mathbb{T}_M^n)}$ in terms of the initial data.

Estimate for $[[U]]_{L_d^\infty(A)}$

Lemma

If U is the solution of the discrete problem then: For any $T < \frac{1}{\sqrt{n}(F_0+L_0)}$, $[[U]]_{L_d^\infty(A)}$ is uniformly bounded by a constant (dependent only on n, u_0, f, M and T).

First existence result

A good candidate for solution is:

$$u(\mathrm{st}\, x, \mathrm{st}\, t) = \mathrm{st}\, U(x, t)$$

$$p(\mathrm{st}\, x, \mathrm{st}\, t) = \mathrm{st}\, P(x, t)$$

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Theorem

Let u, p and T be as above. Then u is a strong solution (i.e, at least $C^{2,1}$) of the modified Navier-Stokes problem (2) on $\mathbb{T}^n \times [0,T]$.

Sketch of the proof

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- Estimate the diference between U and the (smooth) solution of the classical parabolic problem:

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- Let q satisfy

$$\Delta q = -\chi^M(\|u\|_{0,2})\,\mathrm{tr}(D_x u)^2$$

Then $\Delta_d q - \Delta_d p \approx 0$. Use the maximum principle to conclude that $P - {}^*q$ is infinitesimal. Then (the standard) p is equal to the $C^{3,\alpha}$ function q.

Main result

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Proof: u, p solve the modified problem and are $C^{2,1,\alpha}$. By uniform continuity of u and its first and second derivatives we conclude that for any $M > \|u_0\|_{0,2}$ there is a $T > 0$ such that $\|u\|_{0,2} \leq M$. Then, for $0 \leq t \leq T$, the modified problem is equivalent to the original problem.