Ultraproducts and characterizations of classical Banach spaces or lattices.

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1. Ultraproducts of Banach spaces

Let $(X_i)_{i\in I}$ be a family of Banach spaces and $\mathcal{U}$ be an ultrafilter on the set I.
Consider the product $\prod_{i\in I} X_i$, equipped with its natural vector space structure, and the linear subspace of bounded families :

$$V_b = \{(x_i)_{i\in I} : \sup \|x_i\|_{X_i} < \infty\}$$

A semi-norm $\rho_{\mathcal{U}}$ can be defined on V_b by

$$\rho_{\mathcal{U}}((x_i)) = \lim_{i,\mathcal{U}} \|x_i\|_{X_i}$$

Define an equivalence relation of V_b by

$$(x_i) \sim (y_i) \quad \iff \quad \rho_{\mathcal{U}}((x_i - y_i)) = 0$$

The quotient of V_b by this equivalence relation is a vector space on which ρ induces a norm. The resulting normed space is called the $\mathcal{U}$-ultraproduct of the given family (X_i), and denoted by $\prod_{\mathcal{U}} X_i$.

Observe that

$$\prod_{\mathcal{U}} X_i = V_b/N_{\mathcal{U}}$$

where $N_{\mathcal{U}}$ is the linear subspace $N_{\mathcal{U}} = \rho_{\mathcal{U}}^{-1}(0)$.
For $(x_i) \in V_b$ denote by $[x_i]_{\mathcal{U}}$ its equivalence class, then clearly $\|[x_i]_{\mathcal{U}}\| = \lim_{i,\mathcal{U}} \|x_i\|_{X_i}$.

It can be shown that $\prod_{\mathcal{U}} X_i$ is complete (thus a Banach space).

A Banach space X embeds (linearly, isometrically) in any or its ultrapowers by the "diagonal map"

$$D : X \to X_{\mathcal{U}}, x \mapsto [(x)]_{\mathcal{U}}$$

(where (x) is the constant family : $x_i = x$ for all x)

Main examples

Finite dimensional spaces
Any ultrapower $X_{\mathcal{U}}$ of a finite dimensional space X is trivially identifiable to X itself, under the diagonal map. The inverse map is

$$P : X_{\mathcal{U}} \to X, [x_i]_{\mathcal{U}} \mapsto Px = \lim_{i,\mathcal{U}} x_i$$

The class of finite dimensional spaces is thus trivially closed under ultrapowers ; of course it is not closed under ultraproducts. Let us illustrate this point :

Fact. *Every Banach space X is identifiable to a closed subspace of some of an ultraproduct of its finite-dimensional subspaces.*

Indeed let $\mathcal{F}(X)$ be the set of finite dimensional subspaces of X, ordered by inclusion, Φ the filter of cofinal subsets of $\mathcal{F}(X)$, $\mathcal{U}$ an ultrafilter containing Φ.

For $F \in \mathcal{F}(X)$ define

Then
$$D_F : X \to F, D_F(x) = \begin{cases} x & \text{if } x \in F \\ 0 & \text{if not} \end{cases}$$

$$D : X \to \prod_{\mathcal{U}} F, x \mapsto Dx = [D_F(x)]_{\mathcal{U}}$$

is the desired linear isometry.

L_p spaces

By L_p-space we mean any Banach space isometric to some $L_p(\Omega, \mathcal{A}, \mu)$-space. It can be of finite dimension n (space ℓ_p^n), discrete (ℓ_p, more generally $\ell_p(\Gamma)$), nonatomic ($L_p[0,1]$,...)...

Fact. [Krivine] *The class of L_p-spaces is closed under ultraproducts.*

The following corollary is an old illustration (perhaps the first one) of the usefullness of ultraproducts in Banach spaces theory :

Corollary. *A Banach space is linearly isometric to a subspace of some L_p-space iff all of its finite-dimensional subspaces are.*

Remark. Say that two Banach spaces X, Y are *C-isomorphic* if there is a linear isomorphism $T : X \to Y$ with $\|T\| \, \|T^{-1}\| \leq C$. Then the preceding corollary is true with "C-isomorphic" in place of "isometric".

2. More structure : Banach lattices.

An *ordered Banach space* is a Banach space X equipped with an order $\leq$ compatible with both the linear structure and the topology. Equivalently :

$$X_+ := \{x \in X : x \geq 0\} \text{ is a closed convex cone}$$

$$x \leq y \quad \Longleftrightarrow \quad (y - x) \in X_+$$

X is a *Banach lattice* if moreover
– the ordered space $(X, \leq)$ is a lattice, i. e.
$x \vee y := \max(x, y)$ and $x \wedge y := \min(x, y)$ exist for every pair $\{x, y\}$ in X.
In particular we may define $|x| := x \vee (-x)$.
– the norm is compatible with the order i.e.

$$|x| \leq |y| \quad \Longrightarrow \quad \|x\| \leq \|y\|$$

Ultraproducts of Banach Lattices.
An important feature of the operations $\vee$ and $\wedge$ is that they are both separately 1-Lipschitzian with respect to each of their arguments :

$$\|x \vee y - x \vee z\| \leq \|y - z\|, \text{ etc}$$

Given a family $(X_i, \leq_i)_{i \in I}$ and an ultrafilter $\mathcal{U}$ we may thus define operations $\vee$ and $\wedge$ on $\prod_{\mathcal{U}} X_i$ by

$$[x_i]_{\mathcal{U}} \vee [y_i]_{\mathcal{U}} := [x_i \vee y_i]_{\mathcal{U}}; \quad [x_i]_{\mathcal{U}} \wedge [y_i]_{\mathcal{U}} := [x_i \wedge y_i]_{\mathcal{U}}$$

Define a relation $\leq$ on $\prod_{\mathcal{U}} X_i$ by

$$x \leq y \quad \Longleftrightarrow \quad x = x \wedge y$$

It turns out that $(\prod_{\mathcal{U}} X_i, \leq)$ is a Banach lattice, the associated max and min functions of which are $\vee$, resp. $\wedge$. This is the Banach lattice ultraproduct of the family $(X_i, \leq_i)_{i \in I}$.

Examples

L_p Banach lattices
By an L_p Banach lattice we mean a Banach lattice which is linearly and order isometric to some $L_p(\Omega, \mathcal{A}, \mu)$ (equipped with the natural partial order of functions).
The class of L_p Banach lattices coincides (if $1 \leq p < \infty$) with that of *abstract L_p spaces*, i. e. of Banach lattices satisfying the unique axiom

$$(KB_p) \qquad \forall x, \ \|x\|^p = \|x \vee 0\|^p + \|x \wedge 0\|^p$$

(Kakutani-Bohnenblust). We have then clearly :

Fact. *The class of L_p Banach lattices is closed under ultraproducts.*

This fact implies in turn (by forgetting the order structure) the above stated fact that the class of L_p Banach *spaces* is closed under ultraproducts.

Nakano Banach lattices

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $p : \Omega \to [1, \infty)$ be a **bounded** measurable function. The associated *Nakano space* $L_{p(\cdot)}(\Omega, \mathcal{A}, \mu)$ is the linear space of (classes of) measurable functions f such that :

$$\Theta(f) := \int_\Omega |f(\omega)|^{p(\omega)} < \infty$$

Several norms can be considered on $L_{p(\cdot)}$ but probably the most popular is the *Luxemburg norm*

$$\|f\|_{p(.)} = \inf\{c > 0 : \Theta(f/c) \leq 1\}$$

With the Luxemburg norm and the natural order of functions, $L_{p(.)}(\Omega, \mathcal{A}, \mu)$ appears as a Banach lattice.

When $p(\cdot)$ is a constant function $= p$ then
$L_{p(\cdot)}(\Omega, \mathcal{A}, \mu) = L_p(\Omega, \mathcal{A}, \mu)$
Set $\bar{p} = \operatorname{ess\,supp}(\omega)$.

Theorem. [L. P. Poitevin] *Let* $1 \leq D < \infty$*. The class of Nakano Banach lattices (and thus of Nakano Banach spaces) with* $\bar{p} \leq D$ *is closed under ultraproducts.*

Remark : define the *essential range* $R_{p(\cdot)}$ of $p(\cdot)$ as the set of points $t \in \mathbb{R}_+$ such that $\mu(p^{-1}(t-\varepsilon, t+\varepsilon)) > 0$ for every $\varepsilon > 0$. This is a compact subset of $[1, +\infty)$. Poitevin has proved in his thesis (2006) that $R_{p(\cdot)}$ is invariant under lattice-isometries and that given any compact set K, the classes $\mathcal{N}_{\subset K}$ and $\mathcal{N}_{=K}$ of Nakano Banach lattices with $R_{p(\cdot)} \subset K$, resp. $R_{p(\cdot)} = K$ are closed under ultraproducts.

Vector-valued L_p*-spaces*

Given $(\Omega, \mathcal{A}, \mu)$, $p \in [1, \infty)$ and E a Banach space let $L_p(\Omega, \mathcal{A}, \mu; E)$ be the space of Bochner-measurable functions $\Omega \to E$, such that $\int \|f(\omega)\|_E^p d\mu(\omega) < \infty$, equipped with the norm $\|f\| = (\int \|f(\omega)\|_E^p d\mu(\omega))^{1/p}$.

We shall limit ourselves to the cases
$E = L_q$ (abstract L_q-space) : then $L_p(E)$ has a natural structure of Banach lattice.
Consider the class $(\mathbf{L_p L_q})$ of Banach lattices linearly and order isometric to some $L_p(L_q)$-space ;
It turns out that (for $p \neq q$) this classes are *not*

closed under ultraproducts (even under ultrapowers). However some enlarged class that we describe now is closed.
If X is a Banach lattice, an order ideal Y in X is a linear subspace such that

$$y \in Y, |x| \leq |y| \quad \Longrightarrow \quad x \in Y$$

If $X = L_p(\Omega, \mathcal{A}, \mu; L_q(\Omega', \mathcal{A}', \mu'))$, elements of X can be viewed as measurable functions on $\Omega \times \Omega'$ (w. r. to the product σ-algebra) ; if the measures μ, μ' are σ-finite, a closed order ideal in X has the form

$$Y_A = \{\chi_A f : f \in X\}$$

for some measurable $A \subset \Omega \times \Omega'$.

Theorem. [M. Levy, Y. R., 1986] *Let* $\mathbf{BL_pL_q}$ *be the class of Banach lattices order isometric to some closed order ideal in a space* $L_p(L_q)$*. Then* $\mathbf{BL_pL_q}$ *is closed under ultraproducts.*

3. Ultra-roots

Definition. *Given two Banach spaces X, Y we say that X is a ultra-root of Y iff Y is linearly isometric to some ultrapower $X_{\mathcal{U}}$ of X.*

Similarly, if X, Y are two Banach lattices, then X is a ultra-root of Y iff Y is linearly and order isometric to some ultrapower $X_{\mathcal{U}}$ of X.

A class $\mathcal{C}$ of Banach spaces (resp. lattices) is axiomatizable iff it is closed under ultraproducts and ultra-roots.

Remark. The last sentence above is just a definition.
Recall however that Henson and Iovino have elaborated a language of "positive bounded formulas", in which any class $\mathcal{C}$ which is closed under ultrapowers and ultra-roots amits an axiomatisation (= is characterized by a set T of sentences) :

$$X \in \mathcal{C} \quad \Longleftrightarrow \quad X \models T$$

(Conversely given a set T of axioms, the class of Banach spaces (resp. lattices) satisfying it is closed under ultraproducts, but perhaps not under ultra-roots : it is necessary to pass to some set T^+ of all "approximations" of sentences in T.)

Examples (old)

L_p-Banach lattices

Fact. *For a given $1 \leq p < \infty$ the class of L_p Banach lattices is axiomatisable.*

Indeed it is closed under ultraproducts and substructures (=sublattices), a fortiori under ultraroots.

The Kakutani-Bohnenblust axiom gives a *characterization* of this class, which can be transcripted in an *axiomatization* in Henson-Iovino language.

L_p-Banach spaces

Fact. [Henson] *The class of L_p Banach spaces is axiomatisable.*

For $1 < p < \infty$ it relies on the fact that the unit ball of any closed linear subspace of an L_p space is compact in the "weak topology".
If $Y_{\mathcal{U}} = X = L_p$-space then $Y \subset X$ (by the "diagonal embedding" and one can define a linear bounded surjection :

$$P : X \to Y,\ [x_i]_{\mathcal{U}} \mapsto Px = \operatorname*{weaklim}_{i,\mathcal{U}} x_i$$

P is a linear norm one projection, and a celebrated theorem by Douglas and Ando states that its range has to be linearly isometric to some L_p-space.

A*characterization* of L_p-Banach spaces (which can be transcripted to HI's language) is the following :

X is a L_p-space iff it is a $\mathcal{L}_{p,1+}$-space, that is :

$\forall \varepsilon > 0, \forall F \in \mathcal{F}(X), \exists G \in \mathcal{F}(X)$ with $F \subset G$ and G is $(1+\varepsilon)$-isomorphic to some finite ℓ_p^d space (the dimension$d$of which is controlled by $\dim F$ and ε).

Examples (new)

Nakano Banach lattices

Theorem. [Poitevin 2006] *Let $D \in [1, \infty)$. The class of Nakano Banach lattices $L_{p(\cdot)}$ with $\bar{p} \leq D$ is axiomatizable. More generally given a compact set $K \subset [1, \infty)$ in the classes $\mathcal{N}_{\subset K}$ and $\mathcal{N}_{=K}$ are axiomatizable.*

Characterization of $\mathcal{N}_{\subset K}$:

Definition. *Let $\mathcal{F}$ be a class of Banach lattices.*

We say that a Banach lattice X is a script $(1^+, \mathcal{F})$-lattice *if for every $\varepsilon > 0$ and every finite system $(x_1, ..., x_n)$ of positive disjoint elements there exists a finite-dimensional sublattice F of X which is $1+\varepsilon$-isomorphic to a member of $\mathcal{F}$, and* $\operatorname{dist}(x_j, F) < \varepsilon$, *for $j = 1, \ldots n$.*

Observe that a d-dimensional Nakano space is the space $\mathbb{R}^d$ equipped with a modular

$$\Theta_{\mathbf{p}}(x) = \sum_{j=1}^{d} |x_j|^{p_j} \quad \text{if } x = (x_1, \ldots, x_d)$$

Its essential range is $K_{\mathbf{p}} = \{p_1, \ldots, p_d\}$.

Theorem. [L. Poitevin, Y. R.] *Members of $\mathcal{N}_{\subset K}$ are exactly the script $(1^+, \mathcal{N}_{\subset K})$-Banach lattices.*

Class $\mathbf{BL_pL_q}$ of closed order ideals in $L_p(L_q)$-Banach lattices

Theorem. [Henson, Y.R. 2007] *For $1 \leq p, q < \infty$ the class $\mathbf{BL_pL_q}$ is axiomatizable. Members of $\mathbf{BL_pL_q}$ are exactly the script $(1^+, \mathbf{BL_pL_q})$-Banach lattice.*

Observe that a finite dimensional Banach lattice is simply a finite p-direct sum of finite dimensional ℓ_q spaces.