Ultraproducts and Lie algebras: some possible interactions

Sonia L'Innocente

Department of Mathematics
University of Camerino
Italy

Institute of Mathematics
University of Mons-Hainaut
Belgium

UltraMath 2008

Applications of Ultrafilters and Ultraproducts in Mathematics

Department of Mathematics, University of Pisa

01-07 June 2008, Pisa, Italy

Seminar's aim

We want to illustrate the role of the ultraproducts played in two research projects which concern respectively:

1. The decidability of some representations of the universal enveloping algebra, U_k, of $sl_2(k)$ (S.L., A. Macintyre).
2. Some possible exponentiations over $U = U_{\mathbb{C}}$ (S.L., A. Macintyre, F. Point).

Outline

Outline

Our setting

Let k be an algebraically closed field of characteristic 0.

Consider the simple **Lie algebra** $sl_2(k)$ of

all 2×2 trace 0 matrices over k

with the bracket operation $[x, y] = xy - yx$.

Recall that a basis of $sl_2(k)$ is

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So, $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$.

Let U_k denote the universal enveloping algebra of $sl_2(k)$.

Our setting

Let k be an algebraically closed field of characteristic 0.

Consider the simple **Lie algebra** $sl_2(k)$ of

all 2×2 trace 0 matrices over k

with the bracket operation $[x, y] = xy - yx$.

Recall that a basis of $sl_2(k)$ is

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So, $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$.

Let U_k denote the universal enveloping algebra of $sl_2(k)$.

Definition

A **universal enveloping algebra** of $sl_2(k)$ over k is

an associative algebra (with a unit) U_k with

a (Lie algebra) homomorphism $i : sl_2(k) \to U_k$ such that

if A is any associative k-algebra with the homomorphism

$$f : sl_2(k) \to A,$$

then exists a unique homomorphism:

$$\Theta : U_k \to A$$

such that the diagram

$$\begin{array}{ccc} sl_2(k) & \to & U_k \\ \downarrow & \swarrow & \\ A & & \end{array}$$

commutes.

Definition

A **universal enveloping algebra** of $sl_2(k)$ over k is

an associative algebra (with a unit) U_k with

a (Lie algebra) homomorphism $i : sl_2(k) \to U_k$ such that

if A is any associative k-algebra with the homomorphism

$$f : sl_2(k) \to A,$$

then exists a unique homomorphism:

$$\Theta : U_k \to A$$

such that the diagram

$$\begin{array}{ccc} sl_2(k) & \to & U_k \\ \downarrow & \swarrow & \\ A & & \end{array}$$

commutes.

Definition

A **universal enveloping algebra** of $sl_2(k)$ over k is

an associative algebra (with a unit) U_k with

a (Lie algebra) homomorphism $i : sl_2(k) \to U_k$ such that

if A is any associative k-algebra with the homomorphism

$$f : sl_2(k) \to A,$$

then exists a unique homomorphism:

$$\Theta : U_k \to A$$

such that the diagram

$$\begin{array}{ccc} sl_2(k) & \to & U_k \\ \downarrow & \swarrow & \\ A & & \end{array}$$

commutes.

Definition

A **universal enveloping algebra** of $sl_2(k)$ over k is

an associative algebra (with a unit) U_k with

a (Lie algebra) homomorphism $i : sl_2(k) \to U_k$ such that

if A is any associative k-algebra with the homomorphism

$$f : sl_2(k) \to A,$$

then exists a unique homomorphism:

$$\Theta : U_k \to A$$

such that the diagram

$$\begin{array}{ccc} sl_2(k) & \to & U_k \\ \downarrow & \swarrow & \\ A & & \end{array}$$

commutes.

We will use these algebraic properties of U_k:

- U_k has a $\mathbb{Z}$-graded k-algebra. Let $U_{\kappa,m}$ be the subalgebra of elements of grade m. We have

$$\begin{aligned} U_k &= \bigoplus_{m\in\mathbb{Z}} U_{k,\,m}\,; \\ \text{for } m>0,\ U_{k,\,m} &= x^m U_{k,0} = U_{k,0}x^m\,; \\ \text{for } m<0,\ U_{k,\,m} &= y^{|m|} U_{k,0} = U_{k,0}y^{|m|}\,. \end{aligned}$$

- A key role is played by the **Casimir operator** of U_k:

$$c = 2xy + 2yx + h^2$$

which generates the center of U_k

- By PBW basis of U_k, we can see that the 0-component of U_k

$$U_{k\,0} = k[c,h]$$

We will use these algebraic properties of U_k:

- U_k has a $\mathbb{Z}$-graded k-algebra. Let $U_{\kappa,m}$ be the subalgebra of elements of grade m. We have

$$U_k = \bigoplus_{m\in\mathbb{Z}} U_{k,\,m}\,;$$

$$\text{for } m>0,\ U_{k,\,m} = x^m U_{k,0} = U_{k,0}x^m\,;$$

$$\text{for } m<0,\ U_{k,\,m} = y^{|m|} U_{k,0} = U_{k,0}y^{|m|}\,.$$

- A key role is played by the **Casimir operator** of U_k:

$$c = 2xy + 2yx + h^2$$

which generates the center of U_k

- By PBW basis of U_k, we can see that the 0-component of U_k

$$U_{k\,0} = k[c, h]$$

Simple finite dim. representations

Let λ be a positive integer number.
Any simple $(\lambda + 1)$-dim. $sl_2(k)$-module V_λ decomposes as a direct sum

$$V_\lambda = \bigoplus_{j=0}^{\lambda} V_{\lambda,j}$$

of eigenspaces $V_{\lambda,j} = \{v \in V_\lambda\ hv = (\lambda - 2j)v\}$ of h, called the *weight spaces* of V_λ and denoted $Ker(h - (\lambda - 2j))$.

$V_{\lambda,0} = \{v \in V_\lambda : hv = \lambda v \text{ and } xv = 0\}$, often denoted $Ker(x)$ and called *highest* weight space of V_λ

$V_{\lambda,\lambda} = \{v \in V_\lambda : hv = -\lambda v \text{ and } yv = 0\}$, often denoted $Ker(y)$ and called *lowest* weight space of V_λ.

Outline

On the language of left U_k-modules, we focus on suitable linear transformation of V_λ.

We consider

1. the ring of definable scalars, U'_k, of all simple finite dimensional U_k-modules whose elements are pp-definable endomorphisms of each V_λ.
2. As proved by Herzog, U'_k is von Neuman regular ring.

Pseudo-finite dim. representations

Let $Th(L\text{-}fd)$ denote the theory of the class of finite dim. representations of U_k.

A representation M of U_k is called **pseudo-finite dimensional** (from now on **PFD**) iff

$$M \models Th(L\text{-}fd)$$

i.e. M satisfies all sentences (of the language of U_k-modules) true in every finite dimensional representation.

Pseudo-finite dim. representations

Let $Th(L\text{-}fd)$ denote the theory of the class of finite dim. representations of U_k.

A representation M of U_k is called **pseudo-finite dimensional** (from now on **PFD**) iff

$$M \models Th(L\text{-}fd)$$

i.e. M satisfies all sentences (of the language of U_k-modules) true in every finite dimensional representation.

General fact

M is PFD if and only if M is elementary equivalent to an ultraproduct of finite dimensional modules.

We discuss some aspects of ultraproducts of finite dimensional modules.

Property

For M PFD U_k-module, $Cas(M)$ may be $\{0\}$, where

$$Cas(M) = \{\lambda : Ker(c - (\lambda^2 + 2\lambda)) \neq 0\}.$$

To see this, take M equal to the ultraproduct $\prod_{\lambda \in \mathbb{N}} V_\lambda / D$, where D is nonprincipal.

General fact

M is PFD if and only if M is elementary equivalent to an ultraproduct of finite dimensional modules.

We discuss some aspects of ultraproducts of finite dimensional modules.

Property

For M PFD U_k-module, $Cas(M)$ may be $\{0\}$, where

$$Cas(M) = \{\lambda : Ker(c - (\lambda^2 + 2\lambda)) \neq 0\}.$$

To see this, take M equal to the ultraproduct $\prod_{\lambda \in \mathbb{N}} V_\lambda / D$, where D is nonprincipal.

General fact

M is PFD if and only if M is elementary equivalent to an ultraproduct of finite dimensional modules.

We discuss some aspects of ultraproducts of finite dimensional modules.

Property

For M PFD U_k-module, $Cas(M)$ may be $\{0\}$, where

$$Cas(M) = \{\lambda : Ker(c - (\lambda^2 + 2\lambda)) \neq 0\}.$$

To see this, take M equal to the ultraproduct $\prod_{\lambda \in \mathbb{N}} V_\lambda / D$, where D is nonprincipal.

Our aim.

We want to prove the decidability of the theory of PFD-modules.

The main strategy

Since U_k' is von Neumann regular ring, we should prove that U_k' is recursive.

Main result

We construct explicitly a commutative extension of $U_{k\,0}$ *assuming some plausible conjectures about the decision problem for integer points on curves.*

Our aim.

We want to prove the decidability of the theory of PFD-modules.

The main strategy

Since U'_k is von Neumann regular ring, we should prove that U'_k is recursive.

Main result

We construct explicitly a commutative extension of $U_{k\,0}$ *assuming some plausible conjectures about the decision problem for integer points on curves.*

Main idea for constructing U'_k

We focus on $U_{k,0}$.

The heart of the matter is the generation of idempotents, and especially those corresponding to the kernels of elements of $U_{k,0}$.

$\forall p \in U_{k,0}$ and $\forall M \in FinDim$, define $Ker(p) = \{m \in M : p \cdot m = 0\}$.

Our first idempotents

$$e_p \text{ and } 1 - e_p$$

are the projections respectively onto $Ker(p)$ and $Image(p)$ relative to M.

Main idea for constructing U'_k

We focus on $U_{k,0}$.

The heart of the matter is the generation of idempotents, and especially those corresponding to the kernels of elements of $U_{k,0}$.

$\forall p \in U_{k,0}$ and $\forall M \in FinDim$, define
$Ker(p) = \{m \in M : p \cdot m = 0\}$.

Our first idempotents

$$e_p \text{ and } 1 - e_p$$

are the projections respectively onto $Ker(p)$ and $Image(p)$ relative to M.

To generate other idempotents, it is enough to study the solutions of the equation:

$$p(\lambda^2 + 2\lambda, \lambda - 2j) = 0.$$

We will call $p \in U_{k,0}$ **standard** iff there are finitely many solutions of

$$p(\lambda^2 + 2\lambda, \lambda - 2j) = 0 \quad \forall M \in FinDim,$$

p **nonstandard** if the are infinitely many solutions.

p as affine curve

Let $p \in U_{k,0}$, so $p = p(c, h)$ where $p(x_1, x_2) \in k[x_1, x_2]$. Consider the affine plane curve $\mathcal{C}_p$ defined by $p(x_1, x_2) = 0$.

We use some methods from **diophantine geometry**.

To generate other idempotents, it is enough to study the solutions of the equation:

$$p\,(\lambda^2 + 2\lambda, \lambda - 2j) = 0.$$

We will call $p \in U_{k,\,0}$ **standard** iff there are finitely many solutions of

$$p\,(\lambda^2 + 2\lambda, \lambda - 2j) = 0 \quad \forall M \in FinDim,$$

p **nonstandard** if the are infinitely many solutions.

p as affine curve

Let $p \in U_{k,\,0}$, so $p = p(c, h)$ where $p\,(x_1, x_2) \in k[x_1, x_2]$. Consider the affine plane curve $\mathcal{C}_p$ defined by $p\,(x_1, x_2) = 0$.

We use some methods from **diophantine geometry**.

Outline

1. Our Setting
2. Decidability and U_k-modules
3. **Exponentiation over $U = U_{\mathbb{C}}$**
4. References

Exponentiation

Restrict our attention on $\mathbb{C}$. Let $U = U_{\mathbb{C}}$.

Our aim We define some possible exponentiations over U.

1. First, we describe the exponential map

$$\mathrm{EXP}_{\lambda} : U \longrightarrow GL_{\lambda+1}(\mathbb{C})$$

   for each $\lambda \in \omega - \{0\}$.

2. Then, we discuss the exponential map

$$\mathrm{EXP} : U \rightarrow \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$$

   where $\mathcal{V}$ be a non-principal ultrafilter on ω

Exponentiation

Restrict our attention on $\mathbb{C}$. Let $U = U_{\mathbb{C}}$.

Our aim We define some possible exponentiations over U.

1. First, we describe the exponential map
$$\mathrm{EXP}_\lambda : U \longrightarrow GL_{\lambda+1}(\mathbb{C})$$
for each $\lambda \in \omega - \{0\}$.
2. Then, we discuss the exponential map
$$\mathrm{EXP} : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$$
where $\mathcal{V}$ be a non-principal ultrafilter on ω

Our strategy

We will use:

- The *matrix characterization* of every simple U-modules V_λ by the map $\Theta_\lambda : U \to M_{\lambda+1}$ (where $M_{\lambda+1} = \mathrm{End}(V_\lambda)$).
- the natural matrix exponential map defined over $M_{\lambda+1}(\mathbb{C})$

$$\exp : \ M_{\lambda+1}(\mathbb{C}) \longrightarrow GL_{\lambda+1}(\mathbb{C})$$

such that $\forall A \in M_{\lambda+1}(\mathbb{C})$,

$$exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I_{\lambda+1} + A + \frac{A^2}{2} + \frac{A^3}{3!} + \ldots)$$

where $I_{\lambda+1}$ denote the $(\lambda+1) \times (\lambda+1)$ identity matrix.

Definition: the map EXP_λ

Let $\lambda \in \omega - \{0\}$ (later λ will range in ω).
We can define a **new exponential map** over U:

$$\mathrm{EXP}_\lambda : U \xrightarrow{\Theta_\lambda} M_{\lambda+1}(\mathbb{C}) \xrightarrow{\exp} GL_{\lambda+1}(\mathbb{C})$$

$$\mathrm{EXP}_\lambda(u) = \exp(\Theta_\lambda(u)), \qquad \forall u \in U.$$

Proposition

We can prove that the map EXP_λ is surjective.

Question.

Which is the value of $\mathrm{EXP}_\lambda(u)$ for every $u \in U$? What is its kernel?

Definition: the map EXP_λ

Let $\lambda \in \omega - \{0\}$ (later λ will range in ω).
We can define a **new exponential map** over U:

$$\mathrm{EXP}_\lambda : U \xrightarrow{\Theta_\lambda} M_{\lambda+1}(\mathbb{C}) \xrightarrow{\exp} GL_{\lambda+1}(\mathbb{C})$$

$$\mathrm{EXP}_\lambda(u) = \exp(\Theta_\lambda(u)), \qquad \forall u \in U.$$

Proposition

We can prove that the map EXP_λ is surjective.

Question.

Which is the value of $\mathrm{EXP}_\lambda(u)$ for every $u \in U$? What is its kernel?

Definition: the map EXP_λ

Let $\lambda \in \omega - \{0\}$ (later λ will range in ω).
We can define a **new exponential map** over U:

$$\mathrm{EXP}_\lambda : U \xrightarrow{\Theta_\lambda} M_{\lambda+1}(\mathbb{C}) \xrightarrow{\exp} GL_{\lambda+1}(\mathbb{C})$$

$$\mathrm{EXP}_\lambda(u) = \exp(\Theta_\lambda(u)), \qquad \forall u \in U.$$

Proposition

We can prove that the map EXP_λ is surjective.

Question.

Which is the value of $\mathrm{EXP}_\lambda(u)$ for every $u \in U$? What is its kernel?

Because of the intrinsic characterization of U, we are not able to give immediately a satisfactory answer.
But, we can easily calculate:

$$\begin{aligned} \mathrm{EXP}_\lambda(x) &= \exp(\Theta_\lambda(x)) = \exp(X_{\lambda+1}) = \\ &= 1_{\lambda+1} + X_{\lambda+1} + \frac{X_{\lambda+1}^2}{2} + \ldots + \frac{X_{\lambda+1}^\lambda}{\lambda!}; \end{aligned}$$

$$\begin{aligned} \mathrm{EXP}_\lambda(y) &= \exp(\Theta_\lambda(y)) = \exp(Y_{\lambda+1}) = \\ &= 1_{\lambda+1} + Y_{\lambda+1} + \frac{Y_{\lambda+1}^2}{2} + \ldots + \frac{Y_{\lambda+1}^\lambda}{\lambda!}; \end{aligned}$$

$$\begin{aligned} \mathrm{EXP}_\lambda(h) &= \exp(\Theta_\lambda(h)) = \exp(H_{\lambda+1}) = \\ &= \mathrm{diag}(e^\lambda, e^{\lambda-2}, \ldots, e^{-\lambda+2}, e^{-\lambda}); \end{aligned}$$

$$\begin{aligned} \mathrm{EXP}_\lambda(c) &= \exp(\Theta_\lambda(c)) = \exp(\mathrm{diag}(\lambda^2 + 2\lambda, \ldots, \lambda^2 + 2\lambda)) = \\ &= \mathrm{diag}(e^{\lambda^2+2\lambda}, \ldots, e^{\lambda^2+2\lambda}) \end{aligned}$$

About our question, first let us observe:

Lemma

Let $U = \oplus_{m \in \mathbb{Z}} U_m$. We can prove that Θ_λ maps:

(i) any element u_0 of $U_0 = \mathbb{C}[c, h]$ onto a diagonal matrix,

(ii) any element $u_m \in U_m$ of positive degree m, $u_m = x^m u_0$ (with $u_0 \in U_0$), onto the upper triangular matrix (with $l = (\lambda + 1) - m$ nonzero complex entries $\star_l$) if $m \leq \lambda$:

$$\begin{pmatrix} 0 & 0 & \star_1 & 0 \dots & 0 \\ 0 & 0 & 0 & \star_2 \dots & 0 \\ \vdots & \vdots & 0 & & \star_l \\ \vdots & \vdots & & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

otherwise (when $m \geq \lambda + 1$) $\Theta_\lambda(u_m)$ is null.
A similar thing involves any element of negative degree $-m$.

Remark

Any element $u_0 \in U_0$ belongs to the kernel of EXP_λ if and only if

$$\bigwedge_{0 \le j \le \lambda} p\left(\lambda^2 + 2\lambda, \lambda - 2j\right) \in 2\pi i\mathbb{Z}$$

We can get a partial answer to our question.

Proposition

EXP_λ maps any element u of U onto $SL_{\lambda+1}(\mathbb{C})$ if the following condition is satisfied

$$\mathrm{tr}(\Theta_\lambda(u)) \in 2\pi i\mathbb{Z}.$$

In particular, if $u \in \oplus_{m \neq 0} U_m$, then its image by EXP_λ lies always in $SL_{\lambda+1}(\mathbb{C})$.

Remark

Any element $u_0 \in U_0$ belongs to the kernel of EXP_λ if and only if

$$\bigwedge_{0 \leq j \leq \lambda} p\left(\lambda^2 + 2\lambda, \lambda - 2j\right) \in 2\pi i\mathbb{Z}$$

We can get a partial answer to our question.

Proposition

EXP_λ maps any element u of U onto $SL_{\lambda+1}(\mathbb{C})$ if the following condition is satisfied

$$\mathrm{tr}(\Theta_\lambda(u)) \in 2\pi i\mathbb{Z}.$$

In particular, if $u \in \oplus_{m \neq 0} U_m$, then its image by EXP_λ lies always in $SL_{\lambda+1}(\mathbb{C})$.

A further aim

Let $\mathcal{V}$ be a non-principal ultrafilter on ω and consider the ultraproducts $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ and $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ as structures on the language of Lie algebras.

We will focus on the map EXP from U to $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ defined as follows:

$$\mathrm{EXP} : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$$

$$\mathrm{u} \to [EXP_{\lambda}(\mathrm{u})]_{\mathcal{V}} \qquad \forall u \in U$$

by composing the injective map $[\Theta_{\lambda}] : U \to \prod_{V_{\lambda}} M_{\lambda+1}(\mathbb{C})$, with the map $[\exp]_{\mathcal{V}} : \prod_{V_{\lambda}} M_{\lambda+1}(\mathbb{C}) \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$.

A further aim

Let $\mathcal{V}$ be a non-principal ultrafilter on ω and consider the ultraproducts $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ and $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ as structures on the language of Lie algebras.

We will focus on the map EXP from U to $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ defined as follows:

$$\mathrm{EXP} : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$$

$$\mathrm{u} \to [\mathrm{E}XP_{\lambda}(\mathrm{u})]_{\mathcal{V}} \qquad \forall u \in U$$

by composing the injective map $[\Theta_{\lambda}] : U \to \prod_{V_{\lambda}} M_{\lambda+1}(\mathbb{C})$, with the map $[\exp]_{\mathcal{V}} : \prod_{V_{\lambda}} M_{\lambda+1}(\mathbb{C}) \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$.

Note that *EXP* satisfies the properties stated for each EXP_λ. Moreover,

- $EXP(\oplus_{m\neq 0} U_m) \subset \prod_{\mathcal{V}} SL_{\lambda+1}(\mathbb{C})$;
- $EXP(U_0) \subset \prod_{\mathcal{V}} Diag_{\lambda+1}(\mathbb{C})$.

Connection with standard and nonstandard idempotents

Let U' be the ring of definable scalars of U (described at the beginning).
$\forall p = p(c, h) \in U_0$, let e_p denote the idempotent corresponding to the projection on $\mathrm{Ker}(\Theta_\lambda(u))$ on V_λ $(\forall \lambda \in \omega)$.
By using all results obtained in this setting, we can observe:

1. if $p \in U_0$ is standard, then $[\Theta_\lambda(u_0)]$ is invertible in $\prod_{V_\lambda} M_{\lambda+1}(\mathbb{C})$.
2. if p is non-standard, so for some non-principal ultrafilter the image of e_p in the ultraproduct will be of the form $[e_p] = [(\mathrm{diag}(0, 1, 1, \ldots 1, \ldots 1, 1, 0)]$.

Question

Which elements of U' can we identify in $\prod_{V_\lambda} M_{\lambda+1}(\mathbb{C})$?

Connection with standard and nonstandard idempotents

Let U' be the ring of definable scalars of U (described at the beginning).
$\forall p = p(c, h) \in U_0$, let e_p denote the idempotent corresponding to the projection on $\mathrm{Ker}(\Theta_\lambda(u))$ on V_λ $(\forall \lambda \in \omega)$.
By using all results obtained in this setting, we can observe:

1. if $p \in U_0$ is standard, then $[\Theta_\lambda(u_0)]$ is invertible in $\prod_{V_\lambda} M_{\lambda+1}(\mathbb{C})$.
2. if p is non-standard, so for some non-principal ultrafilter the image of e_p in the ultraproduct will be of the form $[e_p] = [(\mathrm{diag}(0, 1, 1, \ldots 1, \ldots 1, 1, 0)]$.

Question

Which elements of U' can we identify in $\prod_{V_\lambda} M_{\lambda+1}(\mathbb{C})$?

We focus on the following query.

Question

What is the kernel of EXP?

Proposition

Let $p = p(c, h) \in U_0$, with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. Write $p(x_1, x_2)$ in the form $\frac{1}{2\pi i} q(x_1, x_2)$. Then, if $p \in ker(\text{EXP})$, then $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$.

Proof

Let $q[x_1, x_2] = \sum_{k=0}^{d} q_k[x_1].x_2^k$ and assume that $q(c, h) \in ker(\text{EXP})$. Then, the set $\{\lambda \in \omega : \bigwedge_{0 \leq 2.j \leq \lambda} \; q(\lambda^2 + \lambda, \lambda - 2j) \in 2.\pi.i.\mathbb{Z}\} \in V_\lambda \quad (\star)$.
Note that it is enough to express hypothesis $(\star)$ for $\lambda > d$.

We focus on the following query.

Question

What is the kernel of EXP?

Proposition

Let $p = p(c, h) \in U_0$, with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. Write $p(x_1, x_2)$ in the form $\frac{1}{2\pi i} q(x_1, x_2)$. Then, if $p \in ker(\text{EXP})$, then $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$.

Proof

Let $q[x_1, x_2] = \sum_{k=0}^{d} q_k[x_1].x_2^k$ and assume that $q(c, h) \in ker(\text{EXP})$. Then, the set $\{\lambda \in \omega : \bigwedge_{0 \leq 2.j \leq \lambda} \; q(\lambda^2 + \lambda, \lambda - 2j) \in 2.\pi.i.\mathbb{Z}\} \in V_\lambda \quad (\star)$.
Note that it is enough to express hypothesis $(\star)$ for $\lambda > d$.

Proposition

Let p be as above where $q(x_1, x_2) = \sum_{k=0}^{d} q_k(x_1)x_2^k$, with $q_k(x) \in \mathbb{Q}[x_1]$. Then, $p \in \mathrm{Ker(EXP)}$ for all non-principal ultrafilter $\mathcal{V}$ if and only if $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$ and for each $0 \leq k \leq d$, $q_k(0) \in \mathbb{Z}$.

Further questions

We would like to put a topology on U in such a way that EXP is continuous.

1. Does U embed as a closed subspace of $\prod_{\mathcal{V}_\lambda} M_{\lambda+1}(C)$?
2. Can we put on $\prod_{\mathcal{V}_\lambda} GL_{\lambda+1}(\mathbb{C})$ (respectively on $EXP(U)$) the structure of a *Lie group*, or simply of a topological group? Is $EXP(U)$ connected?

Proposition

Let p be as above where $q(x_1, x_2) = \sum_{k=0}^{d} q_k(x_1)x_2^k$, with $q_k(x) \in \mathbb{Q}[x_1]$. Then, $p \in \mathrm{Ker}(\mathrm{EXP})$ for all non-principal ultrafilter $\mathcal{V}$ if and only if $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$ and for each $0 \leq k \leq d$, $q_k(0) \in \mathbb{Z}$.

Further questions

We would like to put a topology on U in such a way that *EXP* is continuous.

1. Does U embed as a closed subspace of $\prod_{V_\lambda} M_{\lambda+1}(C)$?
2. Can we put on $\prod_{V_\lambda} GL_{\lambda+1}(\mathbb{C})$ (respectively on $EXP(U)$) the structure of a *Lie group*, or simply of a topological group? Is $EXP(U)$ connected?

Outline

S. L'Innocente and A. Macintyre.
Towards Decidability of the Theory of Pseudo-Finite Dimensional Representations of $sl_2(\mathbb{C})$; I,
In: A. Ehrenfeucht, V.W. Marek, M. Srebrny. Andrzej Mostowski and Foundational Studies. *IOS Press*, 2008, 235-260

S. L'Innocente and A. Macintyre.
Towards Decidability of the Theory of Pseudo-Finite Dimensional Representations of $sl_2(\mathbb{C})$; II,
In progress

S. L'Innocente, A. Macintyre and F. Point.
Possible exponentiations over the universal enveloping algebra of $sl_2(\mathbb{C})$.
In progress.

J. Dixmier.
Enveloping Algebras,
North Holland, 1977

K. Erdmann and M. Wildon.
Introduction to Lie algebras.
Springer, SUMS series, 2006.

M. Hindry and J. Silverman.
Diophantine geometry.
An introduction. Graduate Texts in Mathematics, 201,
Springer, 2000

M. Prest.
Purity, Spectra and Localisation.
In preparation.