Idempotent ultrafilters and finer topologies on $\beta\mathbb{N}$

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Outline

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Strongly summable and union ultrafilter

Classical Definitions

- For a sequence $(x_n)_{n\in\mathbb{N}}$ in a some semigroup $(S,\cdot)$ let $FP(x_n) := \{\prod_{i\in F} x_i \mid \emptyset \neq F \subseteq S \text{ finite}\}$.
- $u \in \beta\mathbb{N}$ is **strongly summable**, if it has a base of FS-sets.

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Strongly summable and union ultrafilter

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Some properties I

Let u be (very) strongly summable.

Algebra (Hindman)

- u is idempotent (in the semigroup $(\beta\mathbb{N}, +)$).
- If $p + q = u$ (in $\beta\mathbb{N}$), then $\left\{ \begin{array}{l} p = u + z \\ q = u - z \end{array} \right\}$ for some $z \in \mathbb{Z}$.
- The maximal subgroup in $\beta\mathbb{N}$ with identity u is minimal ($\cong \mathbb{Z}$).

Some properties II

Let u be (very) strongly summable.

Combinatorics (Blass)

- u-*prod*(ω) has exactly 5 constellations and 3 skies.
- u has a Ramsey property for partitions of finite sums.
- $\min(u)$ and $\max(u)$ are selective ultrafilters.

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Daguenet(-Teissier)'s topological families

Topological families

A family Φ of filters on $\mathbb{N}$ is a topological family, if it

1. includes the countably generated filters
2. is closed under images and preimages (for all $f : \mathbb{N} \to \mathbb{N}$)
3. is closed under countable (compatible) unions.

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Then the sets of ultrafilters extending filters in Φ generate a (finer) topology on $\beta\mathbb{N}$.

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Examples

- Countably generated filters
- F_σ filters
- $\undertilde{\mathbf{\Sigma}}^1_1$ filters

Finer Topologies

Let Φ be a topological family and consider $\beta\mathbb{N}$ with the Φ-topology.

Baire Category Theorem (Daguenet)

The intersection of ω_1-many open dense sets is dense (ω_1-comeager).

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Ultrafilters and finer topologies

Consider the F_σ-topology on $\beta\mathbb{N}$.

Application(Daguenet)

- The set of P-points is the intersection of $2^{\aleph_0}$ open dense sets.
- Under CH: the set of P-points with no image being selective/rapid/"property C" is ω_1-comeager.

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Sums of filters

Let F, G, H be filters; $\overline{F}, \overline{G}, \overline{H}$ the corresponding subsets of $\beta\mathbb{N}$.

(Combinatorical) sum

$F + G := \{A \subseteq \mathbb{N} \mid (\exists V \in F, (W_v)_{v \in V} \text{ in } G) \bigcup_{v \in V} v + W_v \subseteq A\}$

Topological characterization

$F + G \supseteq H$ iff
For all $q \in \overline{F}$, $(p_n)_{n \in \omega}$ in $\overline{G}$: $\quad q\text{-}\lim_{n \in \mathbb{N}}(n + p_n) \in \overline{H}$.

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Idempotent filters

Idempotency

F is idempotent, if $F + F \supseteq F$.

Addition and finer Topologies

There are (many) idempotent filters in every topological family Φ.

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Examples in every topological family (for $\beta\mathbb{N}$)

1. "Adequate partial semigroups"
2. FS_∞-filters
3. $\min^{-1}$, $\max^{-1}$ filters (!)

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Topological families revisited

Let Φ be a topological family and consider the Φ-topology.

Idempotent ultrafilters

Assume CH and let $A \subseteq \beta\mathbb{N}$ be ω_1-comeager.
There exists dense $D \subseteq E(\beta\mathbb{N})$, such that
$\{\min(e) \mid e \in D\}$ and $\{\max(e) \mid e \in D\}$ are dense in A.

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Examples revisited

- (Blass) For countably generated filters: The set of (very) strongly summable ultrafilters (with selective image).
- For F_σ-filters: The set of idempotent ultrafilters with min and max strong P-points.
- For $\underset{\sim}{\Sigma}^1_1$-filters: The set of idempotent ultrafilters with min and max not $\geq_{RK}$ any P-point or any rapid.

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Questions

- Can "dense" be replaced by "ω_1-comeager "?
- What is the position of these ultrafilters in the partial order of idempotent ultrafilters?
- What other algebraic properties do these new idempotent ultrafilters have?
- What other topological families are there and may they help with the above?

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The End

Thank You!