Ultraproducts in Functional Analysis

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- More important than the ultrafilters are the spaces constructed with the help of ultrafilters.
- The new spaces look locally like the old one.

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$$\|(x_i) + N\| \;=\; \lim_{i,\mathcal{U}} X_i$$

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2) Use that Hilbert spaces are stable under ultraproducts.

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Let X be a infinite dimensional Banach space and $p \ \geq \ 2$ be the infimum over all q such that

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Major open problem in operator algebras: Is the predual of a von Neumann algebra finitely represented in the predual in $B(\ell_2)$?

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2) However, $(N^{\mathcal{U}})_*$ is a two-sided ideal in $\prod_{\mathcal{U}} N_*$.
3) The Chang-Keisler theorem for ultraproducts in the vNa-sense is missing.

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- Popa has very successfully studied defomration/rigidity result in von Neumann algebras.

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A good way to understand this is to ask wheather for a finite set $x_1, ..., x_m \subset N$ there are matrices $y_1, ..., y_m \in M_n$ of $n \times n$ matrices such that

$$|\tau(x_{i_1} \cdots x_{i_k}) - \frac{tr}{n}(y_{i_1} \cdots y_{i_k})| < \varepsilon \,?$$

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Problem 3: Let N be an arbitrary von Neumann algebra. Is there an embedding in $(\prod_{\mathcal{U}} B(H)_*)^*$ (or $B(H)^{**}$) with a normal conditional expectation $E : \prod_{\mathcal{U}} B(H) \to N$?

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Theorem

(94) The four problems are all equivalent.

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Warning: (Nhany-Raynaud)

$$\lim_{i,\mathcal{U}_1} \lim_{j,\mathcal{U}_2} \|x_i + y_j\|_p \neq \lim_{j,\mathcal{U}_1} \lim_{i,\mathcal{U}_1} \|x_i + y_j\|_p$$

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Thanks for listening!