Ultrapower of $\mathbb{N}$ and Density Problems

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Outline

- Construct nonstandard model of number system by ultrapower construction
- Characterize asymptotic densities in nonstandard model
- Survey the results about asymptotic densities obtained with the help of nonstandard model

Standard Model: $(V, \in)$

$$V_0 = \mathbb{R}$$

$$V_{n+1} = V_n \cup \mathcal{P}(V_n)$$

$$V = \bigcup_{n=0}^{N} V_n$$

where $\mathcal{P}(V_n)$ is the collection of all subsets of V_n and N is a fixed sufficiently large positive (standard) integer.

Standard model contains all number theoretic objects currently under consideration and all number theoretic arguments can be interpreted in the standard model with only membership relation $\in$.

For example, $\leqslant$ on $\mathbb{R}$ can be viewed as a set of some ordered pairs of real numbers (a, b). A pair of real numbers (a, b) can be viewed as the set $\{\{a\}, \{a, b\}\} \in V_2$. Hence $\leqslant \subseteq V_2$, which means $\leqslant \in V_3$. Now the expression "$a \leqslant b$" can be interpreted as "$\{\{a\}, \{a, b\}\} \in \leqslant$".

Nonstandard Model: $({}^*V, {}^*\in)$

Let $V^{\mathbb{N}}$ be the set of all sequences $\langle a_n \rangle$ in V. $V^{\mathbb{N}}$ can be viewed as a (not very useful) extension of V if one identifies each $A \in V$ with a constant sequence $\langle A \rangle$ in $V^{\mathbb{N}}$.

Fix a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Given $\langle a_n \rangle, \langle b_n \rangle \in V^{\mathbb{N}}$, let $\langle a_n \rangle \sim \langle b_n \rangle$ iff $\{n : a_n = b_n\} \in \mathcal{F}$. ($\sim$ is an equivalence relation.)

$$[\langle a_n \rangle] = \{\langle b_n \rangle \in V^{\mathbb{N}} : \langle a_n \rangle \sim \langle b_n \rangle\}.$$

$${}^*V = V^{\mathbb{N}}/\mathcal{F} = \{[\langle a_n \rangle] : \langle a_n \rangle \in V^{\mathbb{N}}\}.$$

$$[\langle a_n \rangle] \,{}^*\!\in [\langle b_n \rangle] \text{ iff } \{n : a_n \in b_n\} \in \mathcal{F}.$$

The map $* : V \mapsto {}^*V$ defined by ${}^*a = [\langle a \rangle]$ is an embedding satisfying $a = b$ iff ${}^*a = {}^*b$ and $a \in b$ iff ${}^*a \,{}^*\!\in {}^*b$.

Note that ${}^*\mathbb{N}$ is the ultrapower of $\mathbb{N}$ modulo $\mathcal{F}$. For each $k \in \mathbb{N}$ we have ${}^*k = [\langle k \rangle] \in {}^*\mathbb{N}$. If $\langle a_n \rangle$ is an increasing sequence in $\mathbb{N}$, we also have $[\langle a_n \rangle] \in {}^*\mathbb{N}$.

We call $({}^*V, {}^*\in)$ a nonstandard model. ${}^*V$ can be considered as an extension of V. For convenience we often drop the symbol $*$ in some occasions when no confusion will be resulted. For example we often write $\in$ for ${}^*\in$, $\leqslant$ for ${}^*\leqslant$, a for ${}^*a$ when $a \in V_0$, etc.

Note that $* : V \mapsto {}^*V$ is not a surjection. Let $a_n = n$. Then $H = [\langle a_n \rangle] \in {}^*\mathbb{N}$ and for every $k \in \mathbb{N}$, $H > k$.

Transfer Principle

For every first–order formula $\varphi(\overline{x})$ and $\overline{a} \in V$, $\varphi(\overline{a})$ is true in V iff $\varphi({}^*\overline{a})$ is true in ${}^*V$.

For example, ${}^*\leqslant$ is a dense linear order on ${}^*\mathbb{R}$. In fact, $({}^*\mathbb{R}; +, \cdot, \leqslant, 0, 1)$ is a real closed ordered field with infinitely large numbers such as $[\langle n \rangle]$ and infinitesimally small positive numbers such as $[\langle \frac{1}{n} \rangle]$.

A is **standard** if $A = [\langle a \rangle]$ for some $a \in V$.

A is **internal** if $A = [\langle a_n \rangle]$ for some $a_n \in V$ with $n = 0, 1, \ldots$.

A is **external** if it is not internal.

An integer H in ${}^*\mathbb{N} \setminus \mathbb{N}$ is called a **hyperfinite integer**. If H is a hyperfinite integer, then $[\langle a_n \rangle] = H$ implies that the sequence a_n must be unbounded in $\mathbb{N}$.

For any $a, b \in {}^*\mathbb{N}$, the term $[a, b]$ will exclusively represent an interval of **integers**.

Example Let $A = [a, b] \subseteq \mathbb{N}$. Then $[\langle A \rangle]$ can be viewed as the same interval as $[a, b]$. If $A_n = [1, n]$, then $[\langle A_n \rangle] = [1, H]$ is a hyperfinite interval, where $H = [\langle n \rangle]$. Note that every bounded internal subset $[\langle A_n \rangle]$ of ${}^*\mathbb{N}$ has a maximal element $[\langle \max A_n \rangle]$. Hence $\mathbb{N}$ is an external subset of ${}^*\mathbb{N}$.

Standard Part Map

Note that we view $\mathbb{R}$ as an (external) subset of ${}^*\mathbb{R}$. Let $r, s \in {}^*\mathbb{R}$.

$r \approx 0$ iff $|r| < \frac{1}{k}$ for all $k \in \mathbb{N}$ and

$r \approx s$ iff $r - s \approx 0$.

r is called an infinitesimal if $r \approx 0$.

$r \lessapprox s$ ($r \gtrapprox s$) if $r < s$ ($r > s$) or $r \approx s$.

$r \ll s$ ($r \gg s$) if $r < s$ ($r > s$) and $r \not\approx s$.

$Fin({}^*\mathbb{R}) = \{r \in {}^*\mathbb{R} : |r| < n \text{ for some } n \in \mathbb{N}\}$.

Proposition 1 For each $r \in Fin({}^*\mathbb{R})$ there is a unique $\alpha \in \mathbb{R}$ such that $r \approx \alpha$.

The standard part map is the function
$st : Fin({}^*\mathbb{R}) \mapsto \mathbb{R}$ such that $st(r) = \alpha$ iff $r \approx \alpha$.

Densities of an Infinite Subset of $\mathbb{N}$

Let $A \subseteq \mathbb{N}$ and $x, y \in \mathbb{N}$. Let $A(x, y) = |A \cap [x, y]|$ and $A(x) = A(1, x)$.

Shnirel'man density of A

$$\sigma(A) = \inf_{x \geqslant 1} \frac{A(x)}{x}.$$

Lower asymptotic density of A

$$\underline{d}(A) = \liminf_{x \to \infty} \frac{A(x)}{x}.$$

Upper asymptotic density of A

$$\overline{d}(A) = \limsup_{x \to \infty} \frac{A(x)}{x}.$$

Upper Banach density of A

$$BD(A) = \lim_{x \to \infty} \sup_{k \in \mathbb{N}} \frac{A(k, k + x)}{x + 1}.$$

Clearly

$$0 \leqslant \sigma(A) \leqslant \underline{d}(A) \leqslant \overline{d}(A) \leqslant BD(A) \leqslant 1.$$

Nonstandard Characterizations

Let $A \subseteq \mathbb{N}$ in V.

Proposition 2 $\underline{d}(A) \geqslant \alpha$ iff for every hyperfinite integer H, ${}^*A(H)/H \gtrapprox \alpha$.

Proposition 3 $\overline{d}(A) \geqslant \alpha$ iff there exists a hyperfinite integer H such that ${}^*A(H)/H \gtrapprox \alpha$.

Proposition 4 $BD(A) \geqslant \alpha$ iff there is a hyperfinite interval $[k, k+H-1] \subseteq {}^*\mathbb{N}$ such that ${}^*A(k, k+H-1)/H \gtrapprox \alpha$.

Proposition 5 If $BD(A) \geqslant \alpha$, then there is $x \in {}^*\mathbb{N}$ such that $\sigma(({}^*A - x) \cap \mathbb{N}) \geqslant \alpha$.

Proposition 6 If there is $x \in {}^*\mathbb{N}$ such that $\underline{d}(({}^*A - x) \cap \mathbb{N}) \geqslant \alpha$, then $BD(A) \geqslant \alpha$.

Level One Applications:

Buy-One-Get-One-Free Scheme

There is a theorem about upper Banach density parallel to each theorem about Shnirel'man density or lower asymptotic density.

Mann's Theorem

Let $A, B \subseteq N$. If $0 \in A \cap B$, then

$$\sigma(A+B) \geqslant \min\{\sigma(A)+\sigma(B),\, 1\}.$$

Parallel Theorem

For any $A, B \subseteq N$,

$$BD(A+B+\{0,1\}) \geqslant \min\{BD(A)+BD(B),\, 1\}.$$

Can we improve this result?

Kneser's Theorem Let $A, B \subseteq \mathbb{N}$. If

$$\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B),$$

then there are $g > 0$ and $G \subseteq [0, g-1]$ such that

(1) $\underline{d}(A+B) \geqslant \underline{d}(A) + \underline{d}(B) - \frac{1}{g}$,
(2) $A + B \subseteq G + g\mathbb{N}$, and
(3) $(G + g\mathbb{N}) \smallsetminus (A+B)$ is finite.

Parallel Theorem Let $A, B \subseteq \mathbb{N}$. If

$$BD(A+B) < BD(A) + BD(B),$$

then there are $g > 0$ and $G \subseteq [0, g-1]$ such that

(1) $BD(A+B) \geqslant BD(A) + BD(B) - \frac{1}{g}$,
(2) $A + B \subseteq G + g\mathbb{N}$,
(3) and there is a sequence of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$ and $(A+B) \cap [a_n, b_n] = (G + g\mathbb{N}) \cap [a_n, b_n]$.

Can we improve this result?

A set $B \subseteq \mathbb{N}$ is called a basis of order h if

$$h * B = \underbrace{B + B + \cdots + B}_{h} = \mathbb{N}.$$

Plünnecke's Theorem Let B be a basis of order h and $A \subseteq \mathbb{N}$. Then

$$\sigma(A + B) \geqslant \sigma(A)^{1-\frac{1}{h}}.$$

Parallel Theorem 1 Let B be a basis of order h and $A \subseteq \mathbb{N}$. Then

$$BD(A + B) \geqslant BD(A)^{1-\frac{1}{h}}.$$

A set $B \subseteq \mathbb{N}$ is called an piecewise basis of order h if there is a sequence a_n of non-negative integers such that

$$[0, n] \subseteq h * ((B - a_n) \cap \mathbb{N}).$$

Parallel Theorem 2 Let B be a piecewise basis of order h and $A \subseteq \mathbb{N}$. Then

$$BD(A + B) \geqslant BD(A)^{1-\frac{1}{h}}.$$

Can we improve this result?

Level Two Applications

Kneser's Theorem for *BD*. If $BD(A+B) < BD(A)+BD(B) = \alpha+\beta$, then there are $g>0$ and $G \subseteq [0,g-1]$ such that

(1) $BD(A+B) \geqslant \alpha+\beta-\frac{1}{g}$,

(2) $A+B \subseteq G+g\mathbb{N}$, and

(3) for any two sequences of intervals $[a_n^{(i)}, b_n^{(i)}] \subseteq \mathbb{N}$ for $i=0,1$ with $\lim_{n\to\infty}(b_n^{(i)}-a_n^{(i)}) = \infty$,

$$\lim_{n\to\infty} \frac{A(a_n^{(0)}, b_n^{(0)})}{b_n^{(0)}-a_n^{(0)}+1} = \alpha,\ \lim_{n\to\infty} \frac{B(a_n^{(1)}, b_n^{(1)})}{b_n^{(1)}-a_n^{(1)}+1} = \beta,$$

$$\text{and } 0 < \inf_{n\in\mathbb{N}} \frac{b_n^{(0)}-a_n^{(0)}}{b_n^{(1)}-a_n^{(1)}} \leqslant \sup_{n\in\mathbb{N}} \frac{b_n^{(0)}-a_n^{(0)}}{b_n^{(1)}-a_n^{(1)}} < \infty,$$

there exists $[c_n^{(i)}, d_n^{(i)}] \subseteq [a_n^{(i)}, b_n^{(i)}]$ such that

$$\lim_{n\to\infty} \frac{d_n^{(i)}-c_n^{(i)}}{b_n^{(i)}-a_n^{(i)}} = 1 \text{ and for every } n\in\mathbb{N}$$

$$\begin{aligned}(A+B) &\cap [c_n^{(0)}+c_n^{(1)}, d_n^{(0)}+d_n^{(1)}] \\ &= (G+g\mathbb{N}) \cap [c_n^{(0)}+c_n^{(1)}, d_n^{(0)}+d_n^{(1)}].\end{aligned}$$

Definition Let $B \subseteq \mathbb{N}$ and $h \in \mathbb{N}$.

- B is a lower asymptotic basis of order h if $\underline{d}(h * B) = 1$.
- B is a upper asymptotic basis of order h if $\overline{d}(h * B) = 1$.
- B is a upper Banach basis of order h if $BD(h * B) = 1$.

Remarks (1) B is a basis of order h iff $0 \in B$ and $\sigma(h * B) = 1$.

(2) A piecewise basis of order h is an upper Banach basis of order h but not vice versa.

Theorem 1 (Plünnecke's inequality for $\underline{d}$)

Let B be a lower asymptotic basis of order h and $A \subseteq \mathbb{N}$. Then

$$\underline{d}(A+B) \geqslant \underline{d}(A)^{1-\frac{1}{h}}.$$

Theorem 2 (Plünnecke's inequality not true for $\overline{d}$)

There exists an upper asymptotic basis B of order 2 and a set A with $\overline{d}(A) = \frac{1}{2}$ such that $\overline{d}(A+B) = \overline{d}(A)$.

Theorem 3 (Plünnecke's inequality for BD)

Let B be an upper Banach basis of order h and $A \subseteq \mathbb{N}$. Then

$$BD(A+B) \geqslant BD(A)^{1-\frac{1}{h}}.$$

Inverse Theorem for $\overline{d}$

Let $A \subseteq \mathbb{N}$, $0 \in A$, $\gcd(A) = 1$, and $0 < \overline{d}(A) = \alpha < \frac{1}{2}$. Then $\overline{d}(A+A) \geqslant \frac{3}{2}\alpha$. If $\overline{d}(A+A) = \frac{3}{2}\alpha$, then either (a) there exist $k > 4$ and $c \in [1, k-1]$ such that $\alpha = \frac{2}{k}$ and

$$A \subseteq k\mathbb{N} \cup (c + k\mathbb{N})$$

or (b) for every increasing sequence $\langle h_n : n \in \mathbb{N}\rangle$ with

$$\lim_{n\to\infty} \frac{A(0,\ h_n)}{h_n + 1} = \alpha,$$

there exist two sequences $0 \leqslant c_n \leqslant b_n \leqslant h_n$ such that

$$\lim_{n\to\infty} \frac{A(b_n, h_n)}{h_n - b_n + 1} = 1,$$

$$\lim_{n\to\infty} \frac{c_n}{h_n} = 0,$$

and

$$[c_n + 1, b_n - 1] \cap A = \emptyset$$

for every $n \in \mathbb{N}$.