Ultrafilters, Closure operators and the Axiom of Choice

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It is well known that, in a topological space, the open sets can be characterized using filter convergence. In ZF, we cannot replace filters by ultrafilters. It can be proven that the ultrafilter convergence determines the open sets for every topological space if and only if the *Ultrafilter Theorem* holds. More, we can also prove that the Ultrafilter Theorem is equivalent to the fact that $u_X = k_X$ for every topological space X, where k is the usual Kuratowski closure operator and u is the ultrafilter closure, with $u_X(A) := \{x \in X : (\exists \mathcal{U} \text{ ultrafilter in } X)[\mathcal{U} \text{ converges to } x \text{ and } A \in \mathcal{U}]\}$.

These facts arise two different questions that we will try to answer in this talk.

1. Under which set theoretic conditions the equality $u = k$ is true in some subclasses of topological spaces, such as first countable spaces, metric spaces or $\{\mathbb{R}\}$.

2. Is there any topological space X for which $u_X \neq k_X$, but the open sets are characterized by the ultrafilter convergence?

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every filter over a set can extended to an ultrafilter.

CUF – Countable Ultrafilter Theorem:
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CC – the Axiom of Countable Choice.
Every countable family of non-empty sets has a choice function.

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$(X, \mathcal{T})$ – topological space $\qquad A \subseteq X$

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Theorem 1 For all X, $u_X = k_X$.

Theorem 2 For all X, $\hat{u}_X = k_X$.

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The Ultrafilter Theorem is not equivalent to $u = \hat{u}$.

Diagonal Ultrafilter

UX – the set of all ultrafilters in X.

Let $\mathfrak{X} \in U^2X$ and $\mathcal{U} \in UX$, $\mathfrak{X} \to \mathcal{U}$ if
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$$m_X(\mathfrak{X}) := \{A \subseteq X : \mathfrak{X} \in U^2A\}$$

Proposition [ZF] $\mathfrak{X} \to \mathcal{U} \to x \Longrightarrow m_X(\mathfrak{X}) \to x$

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Is there any model of $\mathbf{ZF}$ where these three conditions are satisfied?

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CC+ $\mathbb{N}$ has a free ultrafilter $\Longrightarrow$ **CUF**

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AC($\mathbb{R}$)$\Rightarrow$ **CC**($\mathbb{R}$)+ $\mathbb{N}$ has a free ultrafilter $\Rightarrow$ **CUF($\mathbb{R}$)**