Stochastic Navier-Stokes equations: ideas and results using nonstandard analysis

Nigel J. Cutland
University of York, UK
e-mail: nc507@york.ac.uk

(Joint with Marek Capiński, Jerry Keisler, Kasia Grzesiak, Brendan Enright)

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Aim of the talk: to sketch informally the Loeb space approach and what can be achieved in these areas.

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Set $\mathcal{H} = \{u \in C_0^\infty(D, \mathbb{R}^d)\colon \text{ div } u = 0\}$ with norms $|u|$ and $\|u\|$ derived from

$$(u, v) = \sum_{j=1}^{d} \int_D u^j(x) v^j(x) dx, \qquad ((u, v)) = \sum_{j=1}^{d} \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right)$$

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The sNSe are now formulated as a stochastic differential equation in $\mathbf{H}$:

$$du = [-\nu A u - B(u) + f(t, u)]dt + g(t, u)dw_t$$

Initially regard this as an equation in $\mathbf{V}'$ (the dual of $\mathbf{V}$) although it turns out that solutions live in $\mathbf{H}$ (and in fact in $\mathbf{V}$ for almost all times).

$$du = [-\nu Au - B(u) + f(t,u)]dt + g(t,u)dw_t \tag{1}$$

The equation is understood as a weak *integral* equation :

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$$g : [0,\infty) \times \mathbf{V} \to L(\mathbf{H},\mathbf{H}) \qquad \text{and} \qquad f : [0,\infty) \times \mathbf{V} \to \mathbf{V}'.$$

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Note The pressure has disappeared, because $\nabla p = 0$ in $\mathbf{V}'$.

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Loeb space methods provide a **single** space $\mathbf{\Omega}$ (a Loeb space) and a Wiener process w carrying solutions for **all** (random) initial conditions and **all** f, g.
This makes them powerful for discussing attractors and optimal control theory for sNSe. Loeb spaces are *saturated* and *homogeneous*.

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An example of a non-zero infinitesimal is given by $(1, \frac{1}{2}, \frac{1}{3}, \ldots)\mathcal{U}$.

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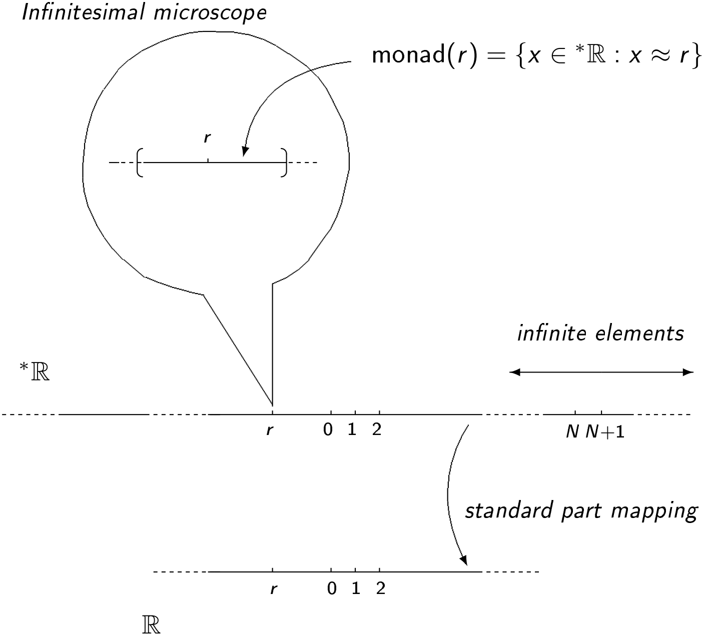
A good way to picture ${}^*\mathbb{R}$ is as follows (note that some features in the diagram are yet to be explained).



The Hyperreals

Now extend *all* sets A, functions f and relations R on $\mathbb{R}$ to ${}^*\mathbb{R}$ pointwise – with the extensions denoted by ${}^*A, {}^*f$ and ${}^*R$.

Examples: ${}^*\mathbb{N}, {}^*\mathbb{Z}$ and ${}^*\mathbb{Q}$, the sets of *hypernatural numbers, hyperintegers* and *hyperrationals* respectively We can talk about an infinite (hyper)natural number N.

Properties of ${}^*\mathbb{R}$ are given systematically by the following:

Theorem (Transfer Principle)

Let φ be any first order statement. Then

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A *first order* statement φ (respectively ${}^*\varphi$): refers to elements of $\mathbb{R}$ (respectively ${}^*\mathbb{R}$), both fixed and variable, and to fixed relations and functions f, R (respectively ${}^*f, {}^*R$), with quantification ($\forall x$, $\exists y$) only for *elements*.

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Definition (Standard Part)

If x is a finite hyperreal the unique real $r \approx x$ is called the **standard part** of x, written $r = {}^\circ x = \mathrm{st}(x)$.

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Repeat the above construction to give ${}^*A$ for any mathematical object or structure A; e.g. ${}^*M$ for a metric space with ${}^*d : {}^*M \times {}^*M \to {}^*\mathbb{R}$.

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Remark The standard part mapping extends to the "nearstandard" elements of any extension metric (or topological) space $^*M$ - in particular the space $^*\mathbf{H}$. It is easy to show that elements U in $^*\mathbf{H}$ with $|U|$ finite are *nearstandard* in the *weak* topology.

LOEB MEASURES

A *Loeb measure space* is a measure constructed from a nonstandard (i.e. *internal*) measure (essentially it is an ultraproduct of measures).

Suppose that an internal set Ω and an internal algebra $\mathcal{A}$ of subsets of Ω, are given, μ is a finite internal finitely additive measure on $\mathcal{A}$;

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Theorem (Loeb 1975)

There is a unique σ-additive extension of $^\circ\mu$ to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. The completion of this measure is the ***Loeb measure corresponding to*** *μ, denoted μ_L and the completion of $\sigma(\mathcal{A})$ is the Loeb σ -algebra, denoted by $L(\mathcal{A})$.*

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Similar relationships connect internal (i.e. nonstandard) stochastic integrals to standard stochastic integrals on the Loeb space.

LOEB SPACE SOLUTIONS TO STOCHASTIC NSe

(1) Use standard SDE methods + Transfer to solve the Galerkin approximation to the sNSe in dimension N ($N \in {}^*\mathbb{N}$ **infinite**)

$$dU(\tau) = [-\nu^* AU(\tau) + {}^*B_N(U) + {}^*f_N(\tau, U(\tau))]d\tau + {}^*g_N(\tau, U(\tau))dW_\tau$$

U is an internal stochastic processes $U : {}^*[0, T] \times \Omega \to \mathbf{H}_N \subset {}^*\mathbf{H}$ on an internal space $\mathbf{\Omega}_0 = (\Omega, \mathcal{A}, \mathcal{P})$ with internal Wiener process W in $\mathbf{H}_N$

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(2) Establish an **energy estimate**. There is a **finite** constant E (independent of N) such that

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(5) Show that this u solves the sNSe on the Loeb space corresponding to $\mathbf{\Omega}_0$ i.e. $\mathbf{\Omega} = (\Omega, L(\mathcal{A}), \mathcal{P}_L)$ with filtration derived from that on $\mathbf{\Omega}_0$

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Theorem (Capiński & NJC (1991))

There is an adapted probability space Ω *carrying an* H*-valued Wiener process* w *such that for any (*L^2*-random)* $u_0 \in H$ *and* f, g *(continuous with linear growth) there is a (weak) solution of the stochastic Navier–Stokes equations.*

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That is, an adapted stochastic process $u : [0, \infty) \times \Omega \to H$ *such that for a.a.* ω

(i) $u \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap C(0, T; H_{weak})$ *for all* $T < \infty$,

(ii) for all $t \geq 0$

$$u(t) = u_0 + \int_0^t [\nu A u(s) - B(u(s)) + f(s, u(s))] ds + \int_0^t g(s, u(s)) dw_s$$

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Loeb space methods give new results for each of (2) - (4) for sNSe for drift and noise of the form $f(u)$ and $g(u)$

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Making this precise gives:

Theorem

(Capiński & NJC 1999) For special forms of the noise term $g(u)$ in the 2D sNSe there is a stochastic attractor $A(\omega)$ (compact in the strong topology of $\mathbf{H}$).

Precise definition and proof - too long and complicated!

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This has the crucial semi-flow property $S_{t_1} \circ S_{t_2} = S_{t_1 + t_2}$ along with $S_0 u = u$.

Process attractors for sNSe

For $d = 3$ uniqueness is unknown. To overcome this for the deterministic NSe, Sell replaced the phase space $\mathbf{H}$ by $\mathbf{W}$ = all solution paths for the *deterministic* Navier–Stokes equations. The semigroup action S_t on $\mathbf{W}$ is time translation: if $u = u(\cdot) \in \mathbf{W}$ then $S_t u = v \in \mathbf{W}$ is given by

$$(S_t u)(s) = u(t+s).$$

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Theorem (Sell (1996))

There is global attractor $A \subseteq \mathbf{W}$ for the 3-dimensional (deterministic) Navier–Stokes equations.

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This turns out to be asking too much. We need a weaker definition. In the following, if u is a stochastic process then $\mathrm{Law}(u)$ is defined to be the probability law (on path space) of the coupled process (u, w).

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Even for this weaker definition, existence requires a rather large probability space

Theorem

(NJC & H.J.Keisler,2004)There is a Loeb space $\mathbf{\Omega}$ *and a natural class of solutions* X *that has a process attractor* A*. The class* X *contains solutions to the sNSe for all* L^2 *random initial conditions.*
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Remark It can be shown that if $\mathbf{\Omega}$ is any sufficiently rich space (for example if $\mathbf{\Omega}$ is a Loeb space) then any process attractor A is **not** compact.

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This idea has been applied to the sNSe in a variety of settings, always involving a Loeb space so that solutions for all controls live on the same probability space.

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This idea has been applied to the sNSe in a variety of settings, always involving a Loeb space so that solutions for all controls live on the same probability space. Results have been obtained for **2D** systems of the form

$$u(t) = u_0 + \int_0^t \{-\nu Au(s) - B(u(s)) + f(s,u,\theta(s,u))\}\,ds + \int_0^t g(s,u)\,dw(s)$$

with θ Hölder continuous, or with θ having no feedback in u, or with the feedback consisting of cumulative digital observations of the solution at a fixed finite number of times.

For the **3D equations** results are only for systems with no feedback: i.e. $\theta = \theta(t)$. The possible non-uniqueness of solutions requires a large space to work in - one containing all possible solutions for a given control to allow initially the existence of an optimal solution for a given control.

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A typical situation in either 2D or 3D is that we have a nonstandard control Θ (possibly θ_N) and a nonstandard solution for it:

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where $U : {}^*[0, T] \to {}^*\mathbf{H}$ or $\mathbf{H}_N$. Then we "standardise" the control to give $\theta = {}^\circ\Theta$ and as in the basic existence proof show that it is possible to take $u(t,\omega) = {}^\circ U(t,\omega)$ It remains to prove that u is a solution for control θ and $J(\theta) = {}^\circ J(\Theta)$ to give optimality.

Details: NJC & K.Grzesiak: Stochastics (2005) and AMO (2007).

Application 3: NON-HOMOGENEOUS (i.e. non-constant density) STOCHASTIC NSe with multiplicative noise

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(3) Loeb space methods (NJC & Brendan Enright): solve the *stochastic* equations with general *multiplicative* noise for $d = 2, 3$ assuming $M \geq \rho_0 \geq m > 0$.

Definition of a weak solution The velocity and the density will be stochastic processes living on an adapted probability space $\mathbf{\Omega}$.

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Note. $g = 0$ gives Kazhikhov's definition for the deterministic equations.

Theorem (NJC & Brendan Enright, JDE 2006) *Suppose that* $u_0 \in H$ *and* $\rho_0 \in L^\infty(D)$ *with* $0 < m \leq \rho_0(x) \leq M$, *and* f, g *satisfy natural continuity and growth conditions. Then there is a weak solution* (ρ, u) *to the stochastic nonhomogeneous Navier-Stokes equations with*

$$\mathbb{E}\left(\sup_{t \leq T} |u(t)|^2 + \nu \int_0^T ||u(t)||^2 \, dt\right) < \infty$$

and for almost all ω, *for all* t

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$$\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$$

where $P = \Pi_L$, $\mathcal{F} = L(\mathcal{A})$ and $(\mathcal{F}_t)_{t\geq 0}$ is the filtration obtained from $(\mathcal{A}_\tau)_{\tau\geq 0}$.

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(c) the equation for $u(t, \omega)$ *holds for* ***all*** $T_0 \leq T$.

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1. No need for limiting arguments and specialized compactness theorems to get a convergent subsequence from a sequence of finite dimensional Galerkin approximations. In fact the specialized compactness theorems (and the appropriate topology) are discovered as by-products.

2. The richness of Loeb spaces means that all activity can take place in a single underlying probability space - not only convenient but essential for formulating some ideas - eg process attractors and optimal controls in 3D.