Topology from a Remote Point of View

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Pisa, June 2008

The Setting

- nonstandard set theory with $*$-mapping
  (Here: **HST** with $*\colon \mathbb{WF} \to \mathbb{S}$)

- some Saturation-principle
  (Here: $\{\mathfrak{M}_i : i \in I\}$, $I \in \mathbb{WF}$, $\mathfrak{M}_i \in \mathbb{I}$ with fip,
  then $\emptyset \neq \bigcap\{\mathfrak{M}_i : i \in I\}$)

- topological space $(X, \mathcal{T})$ with enlargement $({}^*X, {}^*\mathcal{T})$

- a set y is *standard* iff $y = {*x}$ for some $x \in \mathbb{WF}$

Some Notation

Families

Formally $(M_i)_{i\in I}$ with $M_i \subset X$ is a mapping $M : I \to \mathfrak{P}(X)$, $i \mapsto M_i$.
So ${}^*(M_i)_{i\in I}$ is ${}^*M : {}^*I \to {}^*\mathfrak{P}(X)$ with ${}^*M({}^*i) = {}^*(M(i))$.

Standard Elements

Given a set $I \in \mathbb{WF}$ we write $\boxed{{}^*_\sigma I = {}^*I \cap \mathbb{S}}$ for the subset of standard elements of ${}^*I$. It holds ${}^*_\sigma I = \{{}^*i :\ i \in I\}$.
We use $\boxed{{}^*_{\mathrm{n}}I = {}^*I \setminus {}^*_\sigma I}$ for the subset of nonstandard elements.

Is there some order relation $<$ on I we also use ${}^*I_\infty = \{\mathfrak{i} \in {}^*I :\ \forall i \in I\,({}^*i < \mathfrak{i})\}$ for the elements which are larger than any standard element.

If I is infinite we have by `Saturation` ${}^*I_\infty \neq \emptyset$.

Filters and Monads

- Given a filter $\mathcal{F}$, we call $\mu_{\mathcal{F}} = \bigcap_{F \in \mathcal{F}} {}^*F$ the *filtermonad* of $\mathcal{F}$, which is not empty by `Saturation`.

- For internal $\mathfrak{A} \subset {}^*X$ we have $\mu_{\mathcal{F}} \subset \mathfrak{A} \iff \exists F \in \mathcal{F}\,({}^*F \subset \mathfrak{A})$.

- $\mathcal{F} = \{F \subset X : \mu_{\mathcal{F}} \subset {}^*F\}$

- For internal $\mathfrak{A} \subset {}^*X$ we call $\mathrm{Fil}(\mathfrak{A}) = \{F \subset X : \mathfrak{A} \subset {}^*F\}$ the *discrete filter* generated by $\mathfrak{A}$ and its filtermonad $\delta(\mathfrak{A})$ the *discrete monad* of $\mathfrak{A}$.

Special Filters

- a filter $\mathcal{F}$ is *principal* iff $\mu_{\mathcal{F}} \subset {}^*X$ is a standard set (in that case we have $\mathcal{F} = \mathsf{Fil}({}^*M) = \{F \subset X : M \subset F\}$ for ${}^*M = \mu_{\mathcal{F}}$)

- a filter $\mathcal{F}$ is an ultrafilter iff for every filtermonad $\mu_{\mathcal{G}}$ we have $\mu_{\mathcal{F}} \cap \mu_{\mathcal{G}} \neq \emptyset \Rightarrow \mu_{\mathcal{F}} \subset \mu_{\mathcal{G}}$

Neighbourhood-Filters

From now on $(X, \mathcal{T})$ be a topological space.

- for $\mathfrak{A} \subset {}^*X$ we set $\mathcal{F}(\mathfrak{A}) = \{V \in \mathcal{T} : \mathfrak{A} \subset {}^*V\}$ and call its filtermonad $\mu_{\mathcal{T}}(\mathfrak{A})$ the *neighbourhood-monad* of $\mathfrak{A}$

- for $\mathfrak{A} = \{\mathfrak{a}\}$ we write $\mu_{\mathcal{T}}(\mathfrak{a})$ for the neighbourhood-monad

- we call $\mathfrak{x} \in {}^*X$ *near-standard* if $\mathfrak{x} \in \mu_{\mathcal{T}}({}^*x)$ for some $x \in X$ and *remote* otherwise

- $\mathtt{ns}({}^*X)$ be the set of all near-standard elements of ${}^*X$

- $\mathtt{rmt}({}^*X) = {}^*X \setminus \mathtt{ns}({}^*X)$ be the set of all remote points

Some Topological Results

Is $\overline{M}$ the closure of M, some `Transfer`-principle shows for internal $\mathfrak{A} \subset {}^*X$

$$\overline{\mathfrak{A}} = \{\mathfrak{x} \in {}^*X : \forall^{int}\mathfrak{V} \in {}^*\mathcal{T}\,(\mathfrak{x} \in \mathfrak{V} \Rightarrow \mathfrak{V} \cap \mathfrak{A} \neq \emptyset)\}$$

[Take this as definition for the closure of external sets (such as monads).] Then

- for $M \subset X$ we have $\overline{M} = \{x \in X : \mu_{\mathcal{T}}({*x}) \cap {}^*M \neq \emptyset\}$
- for closed $M \subset X$ we have $\mathtt{rmt}({}^*M) = \mathtt{rmt}({}^*X) \cap {}^*M$
- for internal $\mathfrak{A} \subset \mathtt{rmt}({}^*X)$ we have $\overline{\mathfrak{A}} \subset \mathtt{rmt}({}^*X)$

First Results on Remote Points

Under different additional conditions $\mathtt{rmt}({}^*X)$ is closed under some set-building processes:

- $\mathfrak{x} \in \mathtt{rmt}({}^*X) \iff \delta(\mathfrak{x}) \subset \mathtt{rmt}({}^*X)$

- $(X, \mathcal{T})$ regular: $\mathfrak{x} \in \mathtt{rmt}({}^*X) \iff \mu_{\mathcal{T}}(\mathfrak{x}) \subset \mathtt{rmt}({}^*X)$

- $(X, \mathcal{T})$ regular: $\mathfrak{x} \in \mathtt{rmt}({}^*X) \iff \overline{\mu}_{\mathcal{T}}(\mathfrak{x}) \subset \mathtt{rmt}({}^*X)$

- (X, d) metric space:

$$\mathfrak{x} \in \mathtt{rmt}({}^*X) \iff \{\mathfrak{y} \in {}^*X : {}^*d(\mathfrak{x}, \mathfrak{y}) \approx 0\} \subset \mathtt{rmt}({}^*X)$$

Regularity

The property $\boxed{\mathfrak{x} \in \mathtt{rmt}({}^*X) \iff \mu_{\mathcal{T}}(\mathfrak{x}) \subset \mathtt{rmt}({}^*X)}$ is even equivalent to regularity.

Two results for $(X, \mathcal{T})$ regular:

- For $\mathfrak{A} \subset \mathtt{rmt}({}^*X)$ internal we have $\overline{\mu}_{\mathcal{T}}\,(\mathfrak{A}) \subset \mathtt{rmt}({}^*X)$.

- If for all $\mathfrak{x} \in \mathtt{rmt}({}^*X)$ we have $\overline{\mu}_{\mathcal{T}}\,(\mathfrak{x}) = \mu_{\mathcal{T}}(\mathfrak{x})$, then $(X, \mathcal{T})$ is even normal.

Compactness

$(X, \mathcal{T})$ compact $\iff$ ${}^*X = \mathtt{ns}({}^*X)$ (Robinson)

So:

$(X, \mathcal{T})$ compact $\iff$ $\mathtt{rmt}({}^*X) = \emptyset$.

It follows that closed subsets of compact spaces are compact (see page 6).

Locally Finite Families

- *Def.(standard):*
$(M_i)_{i\in I}$ is locally finite $\iff$ for every $x \in X$ there is a neighbourhood U with $\{i \in I : M_i \cap U \neq \emptyset\}$ is finite.

- *Nonstandard:*
$(M_i)_{i\in I}$ is locally finite $\iff \bigcup_{\mathfrak{i} \in {}^*_{\mathtt{n}}I} {}^*M(\mathfrak{i}) \subset \mathtt{rmt}({}^*X)$ (see page 2)

- *Conclusion:*
$(M_i)_{i\in I}$ locally finite $\Rightarrow (\overline{M}_i)_{i\in I}$ locally finite

Paracompactness

Def. (nonstandard)
$(X,\mathcal{T})$ paracompact $\iff$ For every internal subset $\mathfrak{A} \subset \mathtt{rmt}({}^*X)$ there is a l.f. open covering $(U_i)_{i\in I}$ of X with ${}^*(U_i) \cap \mathfrak{A} = \emptyset$ for every $i \in I$. That means

$$\mathtt{ns}({}^*X) \subset \bigcup_{i\in I} {}^*(U_i) \quad \text{but} \quad \mathfrak{A} \cap \Big(\bigcup_{i\in I} {}^*(U_i)\Big) = \emptyset$$

It follows: X paracompact and $A \subset X$ closed then A paracompact (see again page 6).

Also easy: X paracompact then X regular (see page 7).

More Paracompactness

If we replace "internal subset" by "filtermonad" in the definition on page 11 and take this as premise, we get paracompactness as conclusion, i.e.

For every filtermonad $\mu \subset \texttt{rmt}({}^*X)$ there is a l.f. open covering $(U_i)_{i\in I}$ of X with ${}^*(U_i) \cap \mu = \emptyset$ for every $i \in I$.

$$\Downarrow$$

For every internal subset $\mathfrak{A} \subset \texttt{rmt}({}^*X)$ there is a l.f. open covering $(U_i)_{i\in I}$ of X with ${}^*(U_i) \cap \mathfrak{A} = \emptyset$ for every $i \in I$.

In fact these statements are equivalent.

More on l.f. Families

Let $(X, \mathcal{T})$ be regular.

- If for every filtermonad $\mu \subset \texttt{rmt}({}^*X)$ there is a l.f. covering $(U_i)_{i \in I}$ of X with ${}^*(U_i) \cap \mu = \emptyset$ for every $i \in I$
  then
  for every filtermonad $\mu' \subset \texttt{rmt}({}^*X)$ there is a l.f. closed covering $(A_i)_{i \in I}$ of X with ${}^*(A_i) \cap \mu' = \emptyset$ for every $i \in I$.

- If for every filtermonad $\mu \subset \texttt{rmt}({}^*X)$ there is a l.f. closed covering $(A_i)_{i \in I}$ of X with ${}^*(A_i) \cap \mu = \emptyset$ for every $i \in I$
  then
  for every filtermonad $\mu' \subset \texttt{rmt}({}^*X)$ there is a l.f. open covering $(O_i)_{i \in I}$ of X with ${}^*(O_i) \cap \mu' = \emptyset$ for every $i \in I$.

Continuous, closed, surjective Mappings

Let $(Y, \mathcal{S})$ another topological space, $p\colon X \to Y$ continuous, closed, surjective and $\mathfrak{y} \in {}^*Y$.

- $\mu_{\mathcal{S}}(\mathfrak{y}) = {}^*p(\mu_{\mathcal{T}}({}^*p^{-1}(\mathfrak{y}))$

- ${}^*p^{-1}(\mu_{\mathcal{S}}(\mathfrak{y})) = \mu_{\mathcal{T}}({}^*p^{-1}(\mathfrak{y}))$

- Let X additionally be paracompact, then:

  $${}^*p^{-1}(\mathfrak{y}) \subset \mathtt{rmt}({}^*X) \Rightarrow \mathfrak{y} \in \mathtt{rmt}({}^*Y)$$