ON THE TERMS OF UNLIMITED RANK OF LUCAS SEQUENCES

ABDELMADJID BOUDAOUD

Abstract. Let P, Q be nonzero integers such that $D = P^2 - 4Q$ is different to zero. The sequences of integers defined by

$$\begin{cases} U_n &= PU_{n-1} - QU_{n-2} \quad , \quad U_0 = 0 \quad U_1 = 1 \\ V_n &= PV_{n-1} - QV_{n-2} \quad , \quad V_0 = 2 \quad V_1 = P. \end{cases}$$

are called the Lucas sequences associated to the pair (P, Q) [1,5]. In this paper we prove the following result:

Theorem. If P, Q are such that D is strictly positive. Then for each unlimited n, each of integers U_n and V_n is, to a limited integer near, product of two unlimited integers.

1. Introduction & Rappel

This work is in the frame of the non standard analysis ([2, 3]). In [1] we had asked: Is every unlimited integer equal to the sum of a limited integer and a product of two unlimited integers ? i.e

$$Unlimited = s\tan dard + Unlimited \times Unlimited$$

We had provided in this reference some examples affirming this question.

These examples are as follows [1]

Example 1. *Definition.* A *pseudoprime* (in base 2), also called a *Poulet number,* is a composite odd number n such that

$$2^{n-1} = 1 \, (\text{mod } n) \, .$$

Then. Any unlimited pseudoprime n (in base 2) is the product of two unlimited natural numbers, i.e.

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$$n = \omega_1.\omega_2$$

where $\omega_1 \cong +\infty$, $\omega_2 \cong +\infty$.

Let $a \geq 2$ be a natural number.

Example 2. *Definition.* A composite integer $n > a$ that verifies

$$a^{n-1} = 1 \,(\text{mod } n) \ .$$

is called an a-pseudoprime.
Then. Let $a \geq 2$ be a standard integer. Any unlimited a-pseudoprime n is the product of two unlimited natural numbers, i.e.

$$n = \omega_1.\omega_2$$

where $\omega_1 \cong +\infty$, $\omega_2 \cong +\infty$.

Let $a \geq 2$ be a standard natural number.

Example 3. In the base a any unlimited Euler pseudoprime n (resp. strong pseudoprime) is the product of two unlimited natural numbers, i.e.

$$n = \omega_1.\omega_2$$

where $\omega_1 \cong +\infty$, $\omega_2 \cong +\infty$.

Example 4. *Definition.* A composite integer n that verifies $a^{n-1} = 1 \,(\text{mod } n)$ for every integer a, $1 < a < n$, such that a is relatively prime to n, is called a Carmichael number.
Then. Any unlimited Carmichael number n is the product of two unlimited natural numbers, i.e.

$$n = \omega_1.\omega_2$$

where $\omega_1 \cong +\infty$, $\omega_2 \cong +\infty$.

Example 5. If exists an infinity of even perfect number (n is called a perfect number if $\sigma(n) = 2n$), then we have: Any unlimited even perfect number n is the product of two unlimited natural numbers, i.e.

$$n = \omega_1.\omega_2$$

where $\omega_1 \cong +\infty$, $\omega_2 \cong +\infty$.

Example 6. *Definition.* Let n be a natural number. If $\sigma(n) = 2\,n-1$ then n is called almost perfect.
Then. Any unlimited almost perfect number n is the product of two unlimited natural numbers, i.e.

$$n = \omega_1.\omega_2$$

where $\omega_1 \cong +\infty$, $\omega_2 \cong +\infty$.

In this work we present another example. Let's start with a small preview on Lucas sequences associated to a pair of integers [4, 5]:
Let P, Q be nonzero integers. Consider the polynomial $p(x) = x^2 - Px + Q$; its discriminant is $D = P^2 - 4Q$ and the roots are

$$\alpha = \frac{P+\sqrt{D}}{2}, \beta = \frac{P-\sqrt{D}}{2}. \tag{1.1}$$

Suppose that P and Q are such that D is different of zero. The sequences of integers

$$\begin{cases} U_n(P,Q) = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \quad U_0(P,Q) = 0 \quad U_1(P,Q) = 1 \\ V_n(P,Q) = \alpha^n + \beta^n \quad , \quad V_0(P,Q) = 2 \quad V_1(P,Q) = P \end{cases} \tag{1.2}$$

are called the Lucas sequences associated to the pair (P, Q). We will note by U_n (resp. V_n) the element $U_n(P,Q)$ (resp. $V_n(P,Q)$).
It is demonstrated that for $n \geq 2$:

$$\begin{array}{lcl} U_n & = & PU_{n-1} - QU_{n-2} \quad , \quad U_0 = 0 \quad U_1 = 1 \\ V_n & = & PV_{n-1} - QV_{n-2} \quad , \quad V_0 = 2 \quad V_1 = P \ . \end{array} \tag{1.3}$$

In the particular case $(P,Q) = (1,-1)$, the sequence $(U_n)_{n\geq 0}$ beginning as follows 0 1 1 2 3 5 8 13 ... was first considered by Fibonacci; for the same values the sequence of Lucas numbers $(V_n)_{n\geq 0}$ which is the companion sequence of Fibonacci numbers begins as follows: 2 1 3 4 7 11 18 ... .

Here are some results that are known [1, 5] :

$$V_{2n} = (V_n)^2 - 2Q^n. \tag{1.4}$$

Let p be a prime integer, then

$$(1.5) \qquad \begin{cases} U_p = \left(\dfrac{D}{p}\right) Mod\,(p) & : \text{ pour } p \geq 3 \\ V_p = P \;\; Mod\,(p) & : \text{ pour } p \geq 2 \end{cases}$$

Where $\left(\dfrac{D}{p}\right)$ is the Legendre symbol that is, according to the relation between p and D, one of the values -1, 0, $+1$. In addition if n, $k \geq 1$, then

$$(1.6) \qquad U_n \mid U_{nk}, \qquad V_n \mid V_{nk} \text{ if } k \text{ is odd.}$$

Moreover

$$(1.7) \qquad \begin{cases} U_n\,(-P, Q) &=& (-1)^{n-1}\, U_n\,(P, Q) \\ V_n\,(-P, Q) &=& (-1)^{n}\, V_n\,(P, Q)\ . \end{cases}$$

Fermat's Little Theorem. If p is a prime number and if a is an integer, then

$$(1.8) \qquad a^p \equiv a\,[p]\,.$$

In particular, if p does not divide a then $a^{p-1} \equiv 1\,[p]$.
External recurrence principle. For all internal or external formula $F\,(n)$ we have ([2]):

$$(1.9) \qquad \left[F\,(0) \quad \text{and} \quad \forall^{st} n \ \left(F\,(n) \Longrightarrow F\,(n+1)\right)\right] \Longrightarrow \forall^{st} n \ F\,(n)$$

Notations. Let x, y and a be real numbers (integers or non)
1°) $x \cong 0$ (resp. $x \cong +\infty$) signifies that x is an infinitesimal (resp. x is a positive unlimited). We have an analogous significance for $x \cong -\infty$.
2°) We say that x is equivalent to y if $x - y \cong 0$.
3°) $x \ncong a$ signifies that x is not equivalent to a.
4°) We say that x is appreciable if it is not an infinitesimal nor an unlimited.
5°) The inequalities $x \underset{\cong}{>} y$ (resp. $x \underset{\ncong}{>} y$) mean that x is strictly superior and equivalent to y (resp. that x is superior and is not equivalent to y). We have an analogous significance for $\underset{\cong}{<}$ and $\underset{\ncong}{<}$.

2. Main result

Now the example of which I spoke before is formulated by the following result.

Theorem. *If P, Q are such that $D > 0$. Then for each unlimited n, each of integers U_n and V_n is, to a limited integer near, product of two unlimited integers.*

Let P and Q be such that $D > 0$ and let $n \cong +\infty$. Put $\lambda = \dfrac{P}{\sqrt{D}}$. To prove this result, we will have need to the following lemmas.

Lemma 1.
1^0) $\alpha \neq \beta$, *Max* $(|\alpha|, |\beta|) \underset{\ncong}{>} 1$ *and*

$$\frac{\beta}{\alpha} = \frac{\lambda - 1}{\lambda + 1}, \quad \frac{\alpha}{\beta} = \frac{\lambda + 1}{\lambda - 1}. \tag{2.1}$$

2^0) *If* $P > 0$ *then:*
i) $|\alpha| > |\beta|$, *ii)* $\dfrac{\beta}{\alpha} \underset{\cong}{<} 1 \iff \lambda \cong +\infty$, *iii)* $\dfrac{\beta}{\alpha} \underset{\cong}{>} -1 \iff \lambda \underset{\cong}{>} 0$ *and*
iv) $\dfrac{\beta}{\alpha} \ncong \pm 1$ if and only if λ is appreciable positive.

3^0) *If* $P < 0$ *then:*
i) $|\alpha| < |\beta|$, *ii)* $\dfrac{\alpha}{\beta} \underset{\cong}{<} 1 \iff \lambda \cong -\infty$, *iii)* $\dfrac{\alpha}{\beta} \underset{\cong}{>} -1 \iff \lambda \underset{\cong}{<} 0$ *and*
iv) $\dfrac{\alpha}{\beta} \ncong \pm 1$ if and only if λ is appreciable negative.

Proof. 1^0) Because $\alpha = \dfrac{P + \sqrt{D}}{2}$ and $\beta = \dfrac{P - \sqrt{D}}{2}$ we have $\alpha \neq \beta$. Let's study, according to the following cases, the different values of α and β

i) $P > 1$: in this case $\alpha \underset{\ncong}{>} 1$.

ii) $P = 1$: In this case Q must be strictly negative and therefore $\alpha \underset{\ncong}{>} 1$.

iii) $P = -1$: In this case Q must be strictly negative and therefore $|\beta| \underset{\ncong}{>} 1$.

iv) $P < -1$: In this case $|\beta| \underset{\ncong}{>} 1$.

Therefore:

$$Max\left(|\alpha|,|\beta|\right) \underset{\not\cong}{>} 1.$$

By (1.1), $\dfrac{\beta}{\alpha}=\dfrac{\lambda-1}{\lambda+1}$ and $\dfrac{\alpha}{\beta}=\dfrac{\lambda+1}{\lambda-1}$.

2^0) If $P>0$ then it is immediate that $|\alpha|>|\beta|$. Furthermore, $\lambda>0$ and the rest of the proof is legible on the graph of $\dfrac{\beta}{\alpha}(\lambda)=\dfrac{\lambda-1}{\lambda+1}$ in the intervalle $[0,+\infty[$ where one sees the growth of this function of -1 to $+1$.

3^0) This is similar to $2°$). □

Remark. By (1.7), we make proofs of the four following lemmas only when $P>0$. In this case α is positive and, according to lemma 1, $\alpha>|\beta|$; consequently by (1.2) $U_i=\alpha^{i-1}\left(\dfrac{1-(\beta/\alpha)^i}{1-(\beta/\alpha)}\right)>0$, $V_i=\alpha^i\left(1+\left(\dfrac{\beta}{\alpha}\right)^i\right)>0 \quad \text{for } i\geq 1$.

The following lemma (lemma 2) shows how the values of $|U_n|$ *and* $|V_n|$ increase depending on n

Lemma 2. *Each of $|U_n|$ and $|V_n|$ is in the form of $\omega.n$ where ω is an unlimited.*

Proof. By (1.2),

$$\left\{\begin{array}{lcl} U_n & = & \alpha^{n-1}\left(\dfrac{1-(\beta/\alpha)^n}{1-(\beta/\alpha)}\right) \\ V_n & = & \alpha^n\left(1+\left(\dfrac{\beta}{\alpha}\right)^n\right). \end{array}\right. \tag{2.2}$$

Via the discution of possible values of the report $\dfrac{\beta}{\alpha}$ one completes the proof of the proposal in question, where $\dfrac{\beta}{\alpha}=1-\phi$ with $\phi\underset{\cong}{>}0$ or $\dfrac{\beta}{\alpha}=-1+\phi$ with $\phi\underset{\cong}{>}0$ or $\dfrac{\beta}{\alpha}\not\cong\pm 1$.

The following lemma (lemma 3) concerning the report of two terms of U_n and the report of two terms of V_n

Lemma 3. *If n of the form n_1n_2 with $n_1 > 1$ and $n_2 > 1$. Then* $\frac{|U_{n_1n_2}|}{|U_{n_1}|}$ and $\frac{|V_{n_1n_2}|}{|V_{n_1}|}$ are *unlimited.*

Proof. Seen that $n \cong +\infty$ at least n_1 or n_2 is an unlimited. By (1.2),

$$(2.3) \qquad \begin{cases} \dfrac{U_{n_1n_2}}{U_{n_1}} = \dfrac{\alpha^{n_1n_2}}{\alpha^{n_1}} \left(\dfrac{1-(\beta/\alpha)^{n_1n_2}}{1-(\beta/\alpha)^{n_1}} \right) \\ \dfrac{V_{n_1n_2}}{V_{n_1}} = \dfrac{\alpha^{n_1n_2}}{\alpha^{n_1}} \left(\dfrac{1+(\beta/\alpha)^{n_1n_2}}{1+(\beta/\alpha)^{n_1}} \right). \end{cases}$$

Also here the proof of this lemma is done through the discution of the possible values of the report $\frac{\beta}{\alpha}$ which are $\frac{\beta}{\alpha} = 1 - \phi$ with $\phi \underset{\cong}{>} 0$ or $\frac{\beta}{\alpha} = -1 + \phi$ with $\phi \underset{\cong}{>} 0$ or $\frac{\beta}{\alpha} \ncong \pm 1$.

Now we demonstrate that the $|U_i|$ and $|V_i|$ increase with i.

Lemma 4. *For every $i \geq 2$ $|U_i| < |U_{i+1}|$ & $|V_i| < |V_{i+1}|$*

Finally

Lemma 5. *If (P, Q) is not standard then $\frac{|V_2|}{|V_1|} \cong +\infty$.*

Demonstration of the theorem

Case of U_n

We distinguish two cases

I) n premier. By (1.5) $U_n = \left(\frac{D}{n}\right) Mod\,(n)$. Hence $U_n = \pm 1 + kn$. Since, according to lemma 2, $|U_n|$ is in the form of ωn with ω is an unlimited real, the integer k must be unlimited. This finishes the proof for this case.

II) $n = n_1n_2$ where $n_1 \geq n_2 > 1$. By (1.6)
$U_n = CU_{n_1}$ where C is an integer which is, according to lemma 3, unlimited. On the other hand, seen that $n_1 \cong +\infty$ $U_{n_1}$is also, according to lemma 4, unlimited. So the proof is finished for the case of U_n.

Case of V_n

We distinguish four cases

I) $n = p \cong +\infty$ **prime.** we have two cases to consider

a) P limited. By (1.5), $V_p = P\ Mod(p)$ i.e.$V_p = P + kp$. Since P is limited, k must be, according to lemma 2, unlimited.

b) P unlimited. In this case by (1.6), $V_1 \mid V_p$ i.e $V_p = V_1 N$. By lemma 4, we have

$$|V_2| < |V_3| < ... < |V_n| < ... .$$

By lemma 5, $\frac{|V_2|}{|V_1|} \cong +\infty$. Then $\frac{|V_2|}{|V_1|} < \frac{|V_p|}{|V_1|}$ and therefore $\frac{|V_p|}{|V_1|} \cong +\infty$. This signifies that N is unlimited and finish the demonstration for this case because $V_1 = P$ and $|P| \cong +\infty$.

II) $n = 2^s p$ **where** $s \geq 1$ **limited,** $p \cong +\infty$ **prime**

a) P and Q are all both limited. Put for every $s \geq 1$:
$A(s) \equiv \ll$ For $n = 2^s p\ :\ V_n = g_1 + g_2 p$ where g_1(resp. g_2) is a limited (resp. is an unlimited) integer $\gg$.
We have $A(1)$; indeed:
Let $n = 2p$. By (1.4)

$$\begin{aligned} V_n &= V_{2p} \\ &= (V_p)^2 - 2Q^p \ . \end{aligned}$$

The application of (1.5) and (1.8) leads to:

$$\begin{aligned} V_{2p} &= (P + kp)^2 - 2(Q + lp) \\ &= P^2 - 2Q + tp \ . \end{aligned}$$

Put $g_1 = P^2 - 2Q$ and $g_2 = t$. Then g_1 is limited and, according to lemma 2, g_2 is unlimited. Hence $A(1)$. Let $s \geq 1$ be a limited integer and suppose $A(s)$. Let's demonstrate $A(s+1)$:

$$\begin{aligned} V_{2^{s+1}p} &= V_{2(2^s p)} \\ &= (V_{2^s p})^2 - 2Q^{2^s p}. \end{aligned}$$

Because we have $A(s)$ and by (1.8) we deduct

$$V_{2^{s+1}.p} = (g_1 + g_2 p)^2 - 2\left(Q^{2^s} + fp\right).$$

Hence

$$V_{2^{s+1}.p} = g_1^2 - 2Q^{2^s} + \overline{f}p.$$

Seen that g_1, Q and s are limited, the integer $\overline{f}$, according to lemma 2, must be unlimited and this means that we have $A(s+1)$. Then by (1.9), $\forall^{st} s \geq 1\ A(s)$.

b) P or Q is unlimited. In this case by (1.6) $V_{2^s} \mid V_{2^s p}$, i.e. $V_{2^s p} = V_{2^s}.c$. By lemma 3, c is an unlimited integer. By lemma 5 $|V_2| \cong +\infty$ and by lemma 4 $|V_2| < |V_3| < |V_4| < ....$ Hence V_{2^s} is an unlimited. this finishes the demonstration for this case.

III) $n = n_1 n_2$ **where one of** n_1,n_2 **is odd greater or equal to** 3, **the other is unlimited.**
Suppose $n_1 \geq 3$ odd and $n_2 \cong +\infty$. then

$$V_{n_1 n_2} = V_{n_2} C$$

where by (1.6) C is an integer which, according to lemma 3, is unlimited. since $n_2 \cong +\infty$, then, by lemma 4,$V_{n_2}$is unlimited. This finishes the proof for this case.

IV) $n = 2^{\omega+1}$ **with** $\omega \cong +\infty$

a) Q is even ($Q = 2t$, $t \in \mathbb{Z}^*$). We have $V_n = V_{2^{\omega+1}} = (V_{2^\omega})^2 - 2\,(Q)^{2^\omega}$. By considering $2^\omega = 2.2^{\omega-1}$ and by applying (1.4), we obtain $V_{2^\omega} = V_{2.2^{\omega-1}} = V_{2^{\omega-1}}^2 - 2Q^{2^{\omega-1}}$. Hence, by replacing V_{2^ω} by its value gotten in this last equality,

$$\begin{aligned} V_n = V_{2^{\omega+1}} &= \left(V_{2^{\omega-1}}^2 - 2Q^{2^{\omega-1}}\right)^2 - 2Q^{2^\omega} \\ &= (V_{2^{\omega-1}})^4\, Mod\left(Q^{2^{\omega-1}}\right). \end{aligned} \tag{2.4}$$

Similarly, by considering $V_{2^{\omega-1}} = V_{2.2^{\omega-2}}$, we get

$$V_n = V_{2^{\omega+1}} = (V_{2^{\omega-2}})^8\, Mod\left(Q^{2^{\omega-2}}\right). \tag{2.5}$$

Thus if $f \cong +\infty$ is an integer such that $\omega - f \cong +\infty$ then the process that has permitted to write V_n according to formulas (2.4) and (2.5) will, after successive iterations, permit to write

$$V_n = V_{2\omega+1} = \left(V_{2^{\omega-f}}\right)^{2^{f+1}} Mod\left(Q^{2^{\omega-f}}\right). \tag{2.6}$$

where $V_{2^{\omega-f}}$ is an unlimited. Now if $V_{2^{\omega-f}}$ is even then

$$V_{2\omega+1} = 2^{\gamma}.t$$

where $\gamma = \min\left(2^{f+1}, 2^{\omega-f}\right)$ and t is integer. This signifies that we can put $V_{2\omega+1}$ in the form of $2^{\gamma_1}.2^{\gamma_2}.t$ where γ_1 and γ_2 are two unlimited integers of which the sum is γ.
If $V_{2^{\omega-f}}$ is odd, then
$V_{2\omega+1} - 1 = \left[\left(V_{2^{\omega-f}}\right)^{2^{f+1}} - 1\right] + kQ^{2^{\omega-f}}$. Since $\left(V_{2^{\omega-f}}\right)^{2^{f+1}} - 1$ is a difference of two squares, then

$$V_{2\omega+1} - 1 = \left[\left(V_{2^{\omega-f}}\right)^{2^f} - 1\right]\left[\left(V_{2^{\omega-f}}\right)^{2^f} + 1\right] + kQ^{2^{\omega-f}}.$$

Also $\left(V_{2^{\omega-f}}\right)^{2^f} - 1$ is a difference of two squares, consequently

$$V_{2\omega+1} - 1 = \left[\left(V_{2^{\omega-f}}\right)^{2^{f-1}} - 1\right]\left[\left(V_{2^{\omega-f}}\right)^{2^{f-1}} + 1\right]\left[\left(V_{2^{\omega-f}}\right)^{2^f} + 1\right] + kQ^{2^{\omega-f}}.$$

By this way we can write $V_{2\omega+1} - 1$ as follows

$$\begin{aligned} V_{2\omega+1} - 1 = &\left[\left(V_{2^{\omega-f}}\right)^{2^{f-t}} - 1\right]\left[\left(V_{2^{\omega-f}}\right)^{2^{f-t}} + 1\right]\left[\left(V_{2^{\omega-f}}\right)^{2^{f-(t-1)}} + 1\right] + ... \\ &....... + \left[\left(V_{2^{\omega-f}}\right)^{2^{f-1}} + 1\right]\left[\left(V_{2^{\omega-f}}\right)^{2^f} + 1\right] + kQ^{2^{\omega-f}} \end{aligned}$$

where t is an integer verifying $1 \leq t < f$.

Let's take $t_0 \cong +\infty$ such that $t_0 < f$ and $t_0 + 2 < 2^{\omega-f}$. This is possible, indeed: since that $Min\left(f, 2^{\omega-f}\right) \cong +\infty$ therefore we can choose an integer $s \cong +\infty$ and $s \leq Min\left(f, 2^{\omega-f}\right)$. Let's take $t_0 = s - 3$. seen that $Q^{2^{\omega-f}}$ contains the factor $2^{2^{\omega-f}}$ and the product $\left[\left(V_{2^{\omega-f}}\right)^{2^{f-t_0}} - 1\right]\prod_{i=0}^{t_0}\left[\left(V_{2^{\omega-f}}\right)^{2^{f-i}} + 1\right]$ contains 2^k where $k \geq t_0 + 2$, then

$$V_{2\omega+1} - 1 = 2^{t_0+2}N$$

where N is an integer. Therefore

$$V_n - 1 = V_{2\omega+1} - 1 = 2^{t_1}2^{t_2}N\ .$$

where t_1 and t_2 are two unlimited positive integers of which the sum is $t_0 + 2$.

b) Q is odd ($Q = 2t + 1$, $t \in \mathbb{Z}$).

Put $n_o = 2^\omega$. If $Q = \pm 1$, then by (1.4)

$$\begin{aligned} V_n = V_{2n_0} &= (V_{n_0})^2 - 2Q^{n_0} \\ &= (V_{n_0})^2 - 2 \end{aligned}$$

because n_o is even. This end the proof because V_{n_0} is, by lemma 2, an unlimited. Therefore we suppose $Q \neq \pm 1$ and we distinguish the following cases

1°) P **is even.** In this case, by induction, we show easily that the elements V_n ($n \geq 0$) are even. Moreover $V_2 \neq 2$, because otherwise $P^2 - 2Q = 2$ and therefore $D = P^2 - 4Q = 2 - 2Q$. Hence the fact that $D > 0$ means $2 - 2Q > 0$ i.e. $Q < 0$ ($Q \in \mathbb{Z}^*$). This contradicts $P^2 - 2Q = 2$. By the same way we show that $V_2 \neq -2$.

Now we demonstrate that $V_n - 2$ equal to the product of two unlimited integers. Indeed by (1.4)

$$\begin{aligned} V_n = V_{2^{\omega+1}} &= V_{2n_0} \\ &= V_{n_0}^2 - 2Q^{n_0}. \end{aligned}$$

Then

$$\begin{aligned} V_{2n_0} - 2 &= V_{n_0}^2 - 4 - 2Q^{n_0} + 2 \\ &= (V_{n_0} - 2) . (V_{n_0} + 2) - 2(Q^{n_0} - 1) . \end{aligned}$$

Seen that $Q^{n_0} - 1$ is the difference between two squares,

$$V_{2n_0} - 2 = (V_{n_0} - 2)(V_{n_0} + 2) - 2\left(Q^{n_0/2} - 1\right)\left(Q^{n_0/2} + 1\right). \tag{2.7}$$

Because n_0 is divisible by 2, the application of (1.4) to $V_{n_0} - 2$ permit to write $V_{n_0} - 2 = V_{2(n_0/2)} - 2 = V_{(n_0/2)}^2 - 4 - 2\left(Q^{n_0/2} - 1\right)$. Then from this and by (2.7) we have

$$V_{2n_0} - 2 = \left[V_{(n_0/2)}^2 - 4 - 2\left(Q^{n_0/2} - 1\right)\right](V_{n_0} + 2) - 2\left(Q^{n_0/2} - 1\right)\left(Q^{n_0/2} + 1\right).$$

Seen that $V_{(n_0/2)}^2 - 4$ and $\left(Q^{n_0/2} - 1\right)$ are differences between squares, it ensues

$$\begin{aligned} V_{2n_0} - 2 = &\left(V_{(n_0/2)} - 2\right)\left(V_{(n_0/2)} + 2\right)(V_{n_0} + 2) \\ &-2\left(Q^{n_0/4} - 1\right)\left(Q^{n_0/4} + 1\right)(V_{n_0} + 2) \\ &-2\left(Q^{n_0/4} - 1\right)\left(Q^{n_0/4} + 1\right)\left(Q^{n_0/2} + 1\right). \end{aligned} \tag{2.8}$$

Because $n_0/2$ is divisible by 2, the application of (1.4) to $V_{n_0/2} - 2$ permit to write

$$V_{(n_0/2)}-2 = V^2_{(n_0/4)}-2Q^{n_0/4}-4+2$$
$$= \left(V^2_{(n_0/4)}-4\right)-2\left(Q^{n_0/4}-1\right).$$

By replacing $V_{(n_0/2)}-2$ by $\left(V^2_{(n_0/4)}-4\right)-2\left(Q^{n_0/4}-1\right)$ and by observing that $V^2_{(n_0/4)}-4$ and $Q^{n_0/4}-1$ are differences between squares, we get

(2.9)
$$\begin{aligned} V_{2n_0}-2 = & \left(V_{(n_0/4)}-2\right)\left(V_{(n_0/4)}+2\right)\left(V_{(n_0/2)}+2\right)\left(V_{n_0}+2\right) \\ & -2\left(Q^{n_0/8}-1\right)\left(Q^{n_0/8}+1\right)\left(V_{(n_0/2)}+2\right)\left(V_{n_0}+2\right) \\ & -2\left(Q^{n_0/8}-1\right)\left(Q^{n_0/8}+1\right)\left(Q^{n_0/4}+1\right)\left(V_{n_0}+2\right) \\ & -2\left(Q^{n_0/8}-1\right)\left(Q^{n_0/8}+1\right)\left(Q^{n_0/4}+1\right)\left(Q^{n_0/2}+1\right). \end{aligned}$$

So the process consisting, every time to apply (1.4) and to put the difference between two squares as a product of two factors, leads to the following general formulate

$$\begin{aligned} & V_n-2=V_{2n_0}-2= \\ & \left(V_{n_0/2^{i-1}}-2\right)\left(V_{n_0/2^{i-1}}+2\right)...\left(V_{n_0/2}+2\right)\left(V_{n_0}+2\right) \\ & -2\left(Q^{n_0/2^i}-1\right)\left(Q^{n_0/2^i}+1\right)\left(V_{n_0/2^{i-2}}+2\right)...\left(V_{n_0/2}+2\right)\left(V_{n_0}+2\right) \\ & -2\left(Q^{n_0/2^i}-1\right)\left(Q^{n_0/2^i}+1\right)\left(Q^{n_0/2^{i-1}}+1\right)\left(V_{n_0/2^{i-3}}+2\right)..\left(V_{n_0}+2\right) \\ & -2\left(Q^{n_0/2^i}-1\right)\left(Q^{n_0/2^i}+1\right)\left(Q^{n_0/2^{i-1}}+1\right)\left(Q^{n_0/2^{i-2}}+1\right)\left(V_{n_0/2^{i-4}}+2\right) \\ & \quad ...\left(V_{n_0}+2\right) \\ & \quad .... \\ & -2\left(Q^{n_0/2^i}-1\right)\left(Q^{n_0/2^i}+1\right)\left(Q^{n_0/2^{i-1}}+1\right)...\left(Q^{n_0/2^2}+1\right)\left(V_{n_0}+2\right) \\ & -2\left(Q^{n_0/2^i}-1\right)\left(Q^{n_0/2^i}+1\right)\left(Q^{n_0/2^{i-1}}+1\right)...\left(Q^{n_0/2^2}+1\right)\left(Q^{n_0/2}+1\right). \end{aligned}$$

(2.10)

This formula is general in the following sense: If we replace i by 1 we recover (2.7) and by 2 we recover (2.8) etc... .

Take $i_0 \cong +\infty$ such that $\dfrac{n_0}{2^{i_0}} \geq 1$. The formula (2.10) is formed by i_0+1 terms where each term is a product of i_0+1 nonzero factors of which each is a multiple of 2. This because on the one hand the integers $V_{n_0/2^j}$ $(0 \leq j \leq i_0-1)$ appearing in the formula is even and, according to lemme 4, different of ± 2 following the fact that V_2 is different from these values. On the other hand Q is odd different of ± 1. Then in (2.10), we can put 2^{i_0+1} as a common factor between terms constituting $V_{2n_0}-2$. From this

$$V_n - 2 = V_{2n_0} - 2 = 2^{t_1} 2^{t_2} t$$

where t_1 and t_2 are two unlimited positive integers of which the sum is $i_0 + 1$ and t is an integer.

2°) P is odd. In this case we demonstrate by induction that V_{2^n} $(n \geq 1)$ is odd; indeed: $V_{2^1} = V_2 = P^2 - 2Q$ this signifies that V_2 is odd. Suppose thatV_{2^n}, $n \geq 1$ is odd. $V_{2^{n+1}} = (V_{2^n})^2 - 2Q^{2^n}$ then $V_{2^{n+1}}$ is also odd. On the other hand$V_2 \neq 1$ because otherwise $P^2 - 2Q = 1$, then the fact that $D = P^2 - 4Q = 1 - 2Q > 0$ signifies $Q < 0$ and this contradicts $P^2 - 2Q = 1$. By the same way $V_2 \neq -1$.

By (1.4)

$$\begin{aligned} V_n = V_{2^{\omega+1}} &= V_{2n_0} \\ &= V_{n_0}^2 - 2Q^{n_0}. \end{aligned}$$

Then $V_{2^{\omega+1}} + 1 = V_{n_0}^2 - 1 + 2 - 2Q^{n_0}$

$$\begin{aligned} V_{2^{\omega+1}} + 1 &= V_{n_0}^2 - 1 + 2 - 2Q^{n_0} \\ &= (V_{n_0} - 1)(V_{n_0} + 1) + 2(1 - Q^{n_0}) \end{aligned}$$

So

$$\begin{aligned} V_{2^{\omega+1}} + 1 &= (V_{n_0} + 1)(V_{n_0} - 1) \\ &\quad + 2\left(1 - Q^{n_0/2}\right)\left(1 + Q^{n_0/2}\right). \end{aligned} \tag{2.11}$$

Let's calculate, with (1.4),$V_{n_0} + 1$:

$$\begin{aligned} V_{n_0} + 1 &= V_{2(n_0/2)} + 1 \\ &= \left[V_{(n_0/2)}^2 - 1 + 2 - 2Q^{(n_0/2)}\right] \\ &= \left[\left(V_{(n_0/2)} - 1\right)\left(V_{(n_0/2)} + 1\right) + 2\left(1 - Q^{(n_0/2)}\right)\right]. \end{aligned}$$

Now by replacing in (2.11) by the value of $V_{n_0} + 1$ we get

$$\begin{aligned} V_{2^{\omega+1}} + 1 &= \left[\left(V_{(n_0/2)} - 1\right)\left(V_{(n_0/2)} + 1\right) + 2\left(1 - Q^{(n_0/2)}\right)\right](V_{n_0} - 1) \\ &\quad + 2\left(1 - Q^{n_0/2}\right)\left(1 + Q^{n_0/2}\right) \\ &= \left(V_{(n_0/2)} - 1\right)\left(V_{(n_0/2)} + 1\right)(V_{n_0} - 1) \\ &\quad + 2\left(1 - Q^{(n_0/2)}\right)(V_{n_0} - 1) \\ &\quad + 2\left(1 - Q^{(n_0/2)}\right)\left(1 + Q^{(n_0/2)}\right). \end{aligned}$$

Then

$$\begin{aligned} V_{2^{\omega+1}} + 1 &= \left(V_{(n_0/2)} - 1\right)\left(V_{(n_0/2)} + 1\right)(V_{n_0} - 1) \\ &\quad + 2\left(1 - Q^{(n_0/4)}\right)\left(1 + Q^{(n_0/4)}\right)(V_{n_0} - 1) \\ &\quad + 2\left(1 - Q^{(n_0/4)}\right)\left(1 + Q^{(n_0/4)}\right)\left(1 + Q^{(n_0/2)}\right). \end{aligned} \tag{2.12}$$

So for $i \geq 0$, the general formulates is

$V_n + 1 = V_{2^{\omega+1}} + 1 =$
$\left(V_{n_0/2^i} + 1\right)\left(V_{n_0/2^i} - 1\right)\left(V_{n_0/2^{i-1}} - 1\right) ... \left(V_{n_0/2} - 1\right)\left(V_{n_0} - 1\right)$
$+2\left(1 - Q^{n_0/2^{i+1}}\right)\left(1 + Q^{n_0/2^{i+1}}\right)\left(V_{n_0/2^{i-1}} - 1\right)\left(V_{n_0/2^{i-2}} - 1\right) ... \left(V_{n_0} - 1\right)$
$+2\left(1 - Q^{n_0/2^{i+1}}\right)\left(1 + Q^{n_0/2^{i+1}}\right)\left(1 + Q^{n_0/2^i}\right)\left(V_{n_0/2^{i-2}} - 1\right) ... \left(V_{n_0} - 1\right)$
$+2\left(1 - Q^{\frac{n_0}{2^{i+1}}}\right)\left(1 + Q^{\frac{n_0}{2^{i+1}}}\right)\left(1 + Q^{\frac{n_0}{2^i}}\right)\left(1 + Q^{\frac{n_0}{2^{i-1}}}\right)\left(V_{\frac{n_0}{2^{i-3}}} - 1\right) .. \left(V_{n_0} - 1\right)$
$+$..................................
$+2\left(1 - Q^{n_0/2^{i+1}}\right)\left(1 + Q^{n_0/2^{i+1}}\right)\left(1 + Q^{n_0/2^i}\right)\left(1 + Q^{n_0/2^{i-1}}\right) ... \left(1 + Q^{n_0/2^2}\right)$
$\left(V_{n_0} - 1\right)$
$+2\left(1 - Q^{n_0/2^{i+1}}\right)\left(1 + Q^{n_0/2^{i+1}}\right)\left(1 + Q^{n_0/2^i}\right)\left(1 + Q^{n_0/2^{i-1}}\right) ... \left(1 + Q^{n_0/2}\right).$

(2.13)

This formula is general in the following sense: where if we replace i by 0 we recover (2.11) and by 1 we recover (2.12) etc... .

Take $i_0 \cong +\infty$ such that $\dfrac{n_0}{2^{i_0}} \geq 2$. The formula (2.13) is formed by i_0+2 terms where each term is a product of $i_0 + 2$ nonzero factors of which each is a multiple of 2. This because on the one hand the integers $V_{n_0/2^j}$ $(0 \leq j \leq i_0)$ appearing in the formula is odd and, according to lemme 4, different of ± 1 following the fact that V_2 is different from these values. On the other hand Q is odd different of ± 1. Then in (2.13), we can put 2^{i_0+2} as a common factor between terms constituting $V_{2n_0} + 1$. From this

$$V_n + 1 = V_{2n_0} + 1 = 2^{t_1}.2^{t_2}.t$$

where t_1 and t_2 are two unlimited positive integers of which the sum is $i_0 + 2$ and t is an integer.□

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Département de Mathématiques, Université de M'sila, Algérie
E-mail address: boudaoudab@yahoo.fr