ULTRAPOWER OF $\mathbb{N}$ AND DENSITY PROBLEMS

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Lower asymptotic density, upper asymptotic density, and upper Banach density of an infinite subset of $\mathbb{N}$ have simple characterizations in the ultrapower of $\mathbb{N}$. This simplicity offers insights when dealing with problems involving densities. In this talk we survey some results on densities and explain how these characterizations play a crucial role in the derivation of the results.

Shnirel'man density has probably been studied the most. Recent research has revealed that the behavior of lower asymptotic density and the behavior of upper Banach density are very similar to that of Shnirel'man density while the behavior of upper asymptotic density is not. One of the research tools used for upper Banach density is the buy-one-get-one-free scheme, which indicates that for every existing theorem about Shnirel'man density or lower asymptotic density there is a parallel theorem about upper Banach density. The consequences of this scheme usually are not in their optimal form. We will then discuss some improvements on Kneser's Theorem and Plünnecke's Inequality for upper Banach density. For upper asymptotic density we will survey a result on the structural property of a set $A \subseteq \mathbb{N}$ when the upper asymptotic density of $A + A$ reaches its minimal possible value.

The following are definitions of densities: Let $A \subseteq \mathbb{N}$, $a, b \in \mathbb{N}$, and $A(a, b) = |A \cap [a, b]|$.

$$\text{Shnirel'man density } \ \sigma(A) = \inf_{n \geqslant 1} \frac{A(1,n)}{n},$$

$$\text{lower asymptotic density } \ \underline{d}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n},$$

$$\text{upper asymptotic density } \ \overline{d}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n},$$

$$\text{upper Banach density } \ BD(A) = \lim_{n\to\infty} \sup_{k \geqslant 0} \frac{A(k, k+n-1)}{n}.$$

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