Introduction to étale cohomology

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1 Cohomology theories

In this section we briefly review two classical cohomological theories for smooth complex projective varieties and try to highlight some of their key features. We start with **de Rham cohomology**, a cohomology theory for the constant sheaves $\mathbb{R}$ and $\mathbb{C}$.

1.1 de Rham cohomology

Attached to a smooth manifold X we have spaces $A^k(X,\mathbb{R})$ (resp. $A^k(X,\mathbb{C})$) of smooth differential forms with real (resp. complex) coefficients; we shall mostly be interested in the case when X is a smooth projective *complex* manifold, but the definition works equally well for a real manifold, so we give it in this generality (the two are closely related anyway, since $A^k(X,\mathbb{C}) = A^k(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$).

The various spaces of differential forms fit into a complex

$$0 \to \mathcal{C}^\infty(X,\mathbb{R}) \xrightarrow{d} A^1(X,\mathbb{R}) \xrightarrow{d} A^2(X,\mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} A^{\dim_{\mathbb{R}} X}(X,\mathbb{R}) \xrightarrow{d} 0,$$

where d is the so-called *exterior derivative*. In the interest of having a more uniform notation we also set $A^0(X,\mathbb{R}) = \mathcal{C}^\infty(X,\mathbb{R})$.

The k-th de Rham cohomology group is then defined to be

$$H^k(X,\mathbb{R}) = \frac{\ker d : A^k(X,\mathbb{R}) \to A^{k+1}(X,\mathbb{R})}{\operatorname{Im} d : A^{k-1}(X,\mathbb{C}) \to A^k(X,\mathbb{R})}.$$

An analogous definition can be given for the case of complex coefficients, and it's not hard to see that $H^k(X,\mathbb{C}) = H^k(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

Some notation: a k-differential form α lying in the kernel of d is said to be **closed**, while a k-differential form β which can be written as $d(\gamma)$ for some $\gamma \in A^{k-1}(X,\mathbb{R})$ is said to be **exact**. Finally, an element of the cohomology group $H^k(X,\mathbb{R})$ is called a **cohomology class (of degree** k**)**.

We now recall several fundamental facts about these cohomology groups, starting with the obvious fact that they are $\mathbb{R}$-vector spaces.

1.1.1 Functoriality

de Rham cohomology is functorial: if $f : X \to Y$ is a smooth morphism, then pulling back differential forms on Y through f induces linear maps

$$f^* : A^k(Y,\mathbb{R}) \to A^k(X,\mathbb{R}),$$

and since the equality $f^*(d\alpha) = df^*(\alpha)$ holds for every differential form α on Y we obtain easily that f^* as defined above induces maps $f^* : H^k(Y,\mathbb{R}) \to H^k(X,\mathbb{R})$. These are functorial in the sense that if $g : Y \to Z$ is a further smooth map we have $(g \circ f)^* = f^* \circ g^*$ (this is obvious at the level of differential forms, hence remains true for cohomology classes).

1.1.2 The Mayer-Vietoris exact sequence

Let U_0, U_1 be open submanifolds of X such that $X = U_0 \cup U_1$. Then for every k there is a short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & A^k(X,\mathbb{R}) & \to & A^k(U_0,\mathbb{R}) \oplus A^k(U_1,\mathbb{R}) & \to & A^k(U_0 \cap U_1,\mathbb{R}) & \to & 0 \\ & & \omega & \mapsto & (\omega|_{U_0}, \omega|_{U_1}) & & & & \\ & & & & (\alpha,\beta) & \mapsto & \alpha|_{U_0\cap U_1} - \beta|_{U_0\cap U_1}, & & \end{array}$$

which induces a long exact sequence in cohomology of the form

$$\cdots \to H^{k-1}(U_0 \cap U_1,\mathbb{R}) \to H^k(X,\mathbb{R}) \to H^k(U_0,\mathbb{R}) \oplus H^k(U_1,\mathbb{R}) \to H^k(U_0 \cap U_1,\mathbb{R}) \to H^{k+1}(X,\mathbb{R}) \to \cdots$$

1.1.3 Finite dimensionality

The de Rham cohomology groups are all **finite-dimensional** vector spaces. This is a nontrivial fact that can be shown for example by combining the computation of $H^0(X,\mathbb{R})$, the Mayer-Vietoris exact sequence, and the Künneth formula (see below).

1.1.4 $H^0(X,\mathbb{R})$

By definition, the 0-th de Rham cohomology group is the kernel of d acting on $\mathcal{C}^\infty(X,\mathbb{R})$, that is, it is the vector space of locally constant functions. Since a locally constant function on a connected space is constant, we see that $\dim_{\mathbb{R}} H^0(X,\mathbb{R})$ equals the number of connected components of X.

1.1.5 $H^d(X,\mathbb{R})$ and orientability

Let $d = \dim_{\mathbb{R}}(X)$ and consider a nonzero form $\omega \in H^d(X,\mathbb{R})$. For any choice of basis $v_1,\ldots,v_d$ of a tangent space T_xX we then obtain a number $\omega(v_1,\ldots,v_d)$, which (since ω is nonzero and $v_1,\ldots,v_d$ is a basis) is nonzero. It follows that it is either positive or negative, hence it allows us to decide whether any given basis of any tangent space T_xX is 'positively' or 'negatively' oriented: this provides (by definition) an *orientation* of the manifold X. If X is a *complex* manifold, then one can check that $H^d(X,\mathbb{R})$ is always nonzero, so that X always admits an orientation. More precisely, the complex structure induces a canonical orientation on X, that is, a nonzero cohomology class of top degree (up to positive scalars).

From now on we shall assume that X is a **complex projective** manifold, and we will denote by n its **complex** dimension.

1.1.6 Algebra structure

Consider the *wedge* product for differential forms,

$$\begin{array}{cccc} \wedge : & A^h(X,\mathbb{C}) \times A^k(X,\mathbb{C}) & \to & A^{h+k}(X,\mathbb{C}) \\ & (\alpha,\beta) & \mapsto & \alpha \wedge \beta. \end{array}$$

It is immediate to check that if α is a closed form h-form and β is an exact k-form, then $\alpha \wedge \beta$ is an exact form: indeed, if $\beta = d\gamma$ we have

$$\alpha \wedge \beta = \alpha \wedge d\gamma = d(\alpha \wedge (-1)^h \gamma)$$

since

$$d(\alpha \wedge \gamma) = d(\alpha) \wedge \gamma + (-1)^h \alpha \wedge d\gamma = 0 \wedge \gamma + (-1)^h \alpha \wedge \beta = (-1)^h \alpha \wedge \beta.$$

This implies that $\wedge$ induces a product $\wedge : H^h(X,\mathbb{C}) \times H^k(X,\mathbb{C}) \to H^{h+k}(X,\mathbb{C})$ which endows $H^\bullet(X,\mathbb{C}) := \bigoplus_{k=0}^{2n} H^k(X,\mathbb{C})$ with the structure of an associative $\mathbb{C}$-algebra.

1.1.7 Poincaré duality

There is an obvious $\mathbb{C}$-linear map

$$\begin{array}{cccc} \int_X : & A^{2n}(X,\mathbb{C}) & \to & \mathbb{C} \\ & \omega & \mapsto & \int_X \omega, \end{array}$$

and (if X has no boundary, which we are assuming) Stokes' theorem shows that

$$\int_X (\omega + d\alpha) = \int_X \omega + \int_{\partial X} \alpha = \int_X \omega$$

for every $\alpha \in A^{2n-1}(X,\mathbb{C})$, so that $\int_X$ induces a linear map (still denoted by $\int_X$)

$$\int_X : H^{2n}(X,\mathbb{C}) \to \mathbb{C}.$$

It is a theorem that when X is connected this natural map is an isomorphism, and furthermore we have:

Theorem 1.1 (Poincaré duality)**.** *Let X be smooth, connected and compact. Then the map*

$$\begin{array}{ccccc} H^k(X,\mathbb{C}) \times H^{2n-k}(X,\mathbb{C}) & \to & H^{2n}(X,\mathbb{C}) & \xrightarrow{\int_X} & \mathbb{C} \\ (\alpha,\beta) & \mapsto & \alpha \wedge \beta & \mapsto & \int_X \alpha \wedge \beta \end{array}$$

is a nondegenerate bilinear form, hence in particular it identifies $H^k(X,\mathbb{C})$ with the dual of $H^{2n-k}(X,\mathbb{C})$.

Corollary 1.2. *Under the same hypotheses,* $\dim_{\mathbb{C}} H^k(X,\mathbb{C}) = \dim_{\mathbb{C}} H^{2n-k}(X,\mathbb{C})$.

1.1.8 The Künneth Formula

Let now X, Y be smooth projective varieties. The cohomology of $X \times Y$ can be computed from the cohomologies of X, Y thanks to the following formula:

Theorem 1.3 (Künneth)**.** *Let $\pi_X, \pi_Y : X \times Y \to X, Y$ be the projections on the two factors. Then the maps (defined for all $a, b \in \mathbb{N}$)*

$$\begin{array}{ccc} H^a(X,\mathbb{C}) \times H^b(Y,\mathbb{C}) & \to & H^{a+b}(X \times Y,\mathbb{C}) \\ (\alpha,\beta) & \mapsto & \pi_X^*(\alpha) \wedge \pi_Y^*(\beta) \end{array}$$

induce an isomorphism of graded algebras

$$H^\bullet(X) \otimes H^\bullet(Y) \cong H^\bullet(X \times Y).$$

Notice that in this theorem the tensor product of two graded vector spaces $V^\bullet$ and $W^\bullet$ is defined to be the graded vector space whose degree-k piece is given by

$$(V^\bullet \otimes W^\bullet)^k = \bigoplus_{a+b=k} V^a \otimes W^b.$$

1.1.9 Reinterpretation of the algebra structure via the Künneth morphism

In order to construct the Künneth morphism we have used the algebra structure on $H^\bullet(X \times Y)$; however, given an isomorphism $\phi : H^\bullet(X,\mathbb{C}) \otimes H^\bullet(X,\mathbb{C}) \cong H^\bullet(X \times X)$, we can also reconstruct an algebra structure on $H^\bullet(X,\mathbb{C})$ as follows. Let $\Delta : X \hookrightarrow X \times X$ be the diagonal immersion of X in X^2; then we can define the product of two differential forms α, β on X by the formula

$$\alpha \wedge \beta := \Delta^* \left(\phi(\alpha \otimes \beta)\right).$$

It is a simple matter to check that if ϕ is the Künneth isomorphism then this definition really recovers the usual wedge product.

1.1.10 Dimension

Essentially by definition, $H^i(X,\mathbb{C})$ is zero for $i \notin [0,2n]$: indeed at each point $x \in X$ a k-differential form is an element of $\Lambda^k(T_xX)^\vee$, and since $\dim_{\mathbb{R}} T_xX = 2\dim_{\mathbb{C}} X$ we have that for $k > 2n = \dim_{\mathbb{R}} X$ a k-differential form vanishes at every point of x, hence is trivial.

1.1.11 The cohomology class of a submanifold

Let Y be a closed submanifold of X of (complex) dimension d_Y. We claim that with Y we can associate a cohomology class $[Y] \in H^{2n-2d_Y}(X,\mathbb{C})$ as follows.

Let $A^{2d_Y}(X,\mathbb{C})^{d=0}$ be the subspace of closed $2d_Y$-forms. There is an obvious linear map

$$\begin{array}{cccc} \int_Y : & A^{2d_Y}(X,\mathbb{R})^{d=0} & \to & \mathbb{C} \\ & \omega & \mapsto & \int_Y \omega \end{array}$$

whose kernel contains the subspace of exact forms (this follows from Stokes' theorem since Y has no boundary). The map $\int_Y$ therefore induces a linear map $\int_Y : H^{2d_Y}(X,\mathbb{R}) \to \mathbb{C}$, that is, an element of $H^{2d_Y}(X,\mathbb{R})^*$, a space which by Poincaré duality is naturally isomorphic to $H^{2n-2d_Y}(X,\mathbb{C})$. We denote by $[Y]$ the element of $H^{2n-2d_Y}(X,\mathbb{C})$ corresponding to $\int_Y$ under Poincaré duality. An equivalent description of $[Y]$ is as follows: there is a closed differential form ω_Y, well-defined up to exact forms, such that the equality

$$\int_X \omega \wedge \omega_Y = \int_Y \omega$$

holds for every closed $2d_Y$-form $\omega \in A^{2d_Y}(X,\mathbb{C})^{d=0}$, and $[Y]$ is the cohomology class of ω_Y for any such ω_Y.

The association $Y \mapsto [Y]$ has many nice properties, including the fact that whenever Y_1, Y_2 are closed submanifolds such that $Y_1 \cap Y_2$ is smooth we have $[Y_1 \cap Y_2] = [Y_1] \wedge [Y_2]$.

Remark 1.4. Let $f : X \to Z$ be a smooth morphism and let Y be a closed submanifold of Z. In general it is not true that $f^*[Y] = [f^{-1}Y]$: consider for example the case $X = Y = Z = \mathbb{C}^\times$ and $f : X \to X$ given by $x \mapsto x^2$. Then $f^{-1}(Y) = X$, but $f^*[Y] = 2[X]$: this shows for example that (in the special case of finite morphisms) one should also take into account 'multiplicities'. One can in fact introduce a suitable pullback operator f^* on subvarieties (or more precisely on Chow groups...), and for this operator we have $f^*[Y] = [f^*Y]$.

1.2 Čech cohomology

We now turn to a more general cohomology theory which can also handle non-constant sheaves: Čech cohomology. Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. We define $\check{H}^\bullet(\mathcal{U},\mathcal{F})$ as follows. For $p \geq 0$ set

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{I=(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}\left(\bigcap_{i\in I} U_i\right),$$

and consider the Čech (co)differential

$$\delta^p : C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$$

which associates to a collection $c_L \in C^p(\mathcal{U},\mathcal{F})$ (where L ranges over the $(p+1)$-tuples of elements of I) the collection $(\delta^p c)_J$ (where J ranges over the $(p+2)$-tuples of elements of I) given by

$$(\delta^p c)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j c_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

One can check that $\delta^{p+1} \circ \delta^p = 0$, so that the sequence

$$0 \xrightarrow{\delta^{-1}} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta_{n-1}} C^n(\mathcal{U}, \mathcal{F}) \to 0$$

is a complex of abelian groups. We define $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$ as the cohomology of this complex, that is, as the graded vector space whose graded pieces are

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{\ker \delta^p}{\operatorname{Im} \delta^{p-1}}.$$

Finally, we define the Čech cohomology of X with values in $\mathcal{F}$ as the limit

$$\check{H}^\bullet(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^\bullet(\mathcal{U}, \mathcal{F}),$$

where open covers $\mathcal{U}$ are ordered by *refinement*: $\mathcal{V} = (V_j)_{j\in J}$ is a refinement of $\mathcal{U} = (U_i)_{i\in I}$ if every open set V_j in $\mathcal{V}$ is a subset of an open set $U_{c(j)}$ in $\mathcal{U}$. Notice that if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a canonical morphism $\check{H}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{H}^\bullet(\mathcal{V}, \mathcal{F})$ constructed as follows. Choose a map $c : J \to I$ such that $V_j \subseteq U_{c(j)}$ and define

$$\begin{array}{rccc} \gamma : & C^p(\mathcal{U}, \mathcal{F}) & \to & C^p(\mathcal{V}, \mathcal{F}) \\ & s & \mapsto & \gamma(s), \end{array}$$

where

$$\gamma(s)_{j_0,\ldots,j_p} = s_{c(j_0),\ldots,c(j_p)}|_{V_{j_0}\cap\cdots\cap V_{j_p}}.$$

Of course γ depends on the choice of c, but different c's induce *homotopic*[1] maps γ, so that we obtain a well-defined morphism $\gamma : \check{H}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{H}^\bullet(\mathcal{V}, \mathcal{F})$.

Exercise 1.5 (if this is your first encounter with Čech cohomology)**.** Check that if $\mathcal{F}$ is the constant sheaf of group G and X is connected, then $\check{H}(X, \mathcal{F}) = G$.

Even though, at first glance, de Rham and Čech cohomology look very different, they agree for any manifold (provided that one only considers the constant sheaves $\mathbb{R}$ and $\mathbb{C}$):

Theorem 1.6 (de Rham)**.** *Let X be a smooth manifold. For every $p \geq 0$ we have*

$$H^p(X, \mathbb{R}) = \check{H}^p(X, \mathbb{R}) \text{ and } H^p(X, \mathbb{C}) = \check{H}^p(X, \mathbb{C}).$$

This result implies in particular that Čech cohomology reproduces all the good features of de Rham cohomology, without the need for a differential structure. Therefore, Čech cohomology looks like a natural candidate to be generalized to the algebraic setting. However, we will see in the next section that the coarseness of the Zariski topology precludes the possibility of such a straightforward generalization.

1.3 Insufficiency of the Zariski topology

We now make a first (naive) attempt to reproduce the features of (de Rham or Čech) cohomology in a more algebraic setting. Since we will eventually be interested in the cohomology of arbitrary sheaves (and not just the constant sheaf $\mathbb{C}$...) it seems more natural to try and generalize Čech

[1] Recall that two maps $f, g : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ of complexes of abelian groups are said to be *homotopic* if there exists a collection of maps $h^n : \mathcal{A}^n \to \mathcal{B}^{n-1}$ such that $f^n - g^n = d_{\mathcal{B}}^{n-1} \circ h^n + h^{n+1} \circ d_{\mathcal{A}}^n$ for all n. See the next paragraph for a discussion of why homotopic maps induce the same morphism in cohomology.

In our case, if $c, c' : J \to I$ are two maps as in the main text, the homotopy is given by

$$h^{p+1}(s)_{j_0,\ldots,j_p} = \sum_{a=0}^{p} (-1)^a s_{c(j_0),\ldots,c(j_a),c'(j_a),\ldots,c'(j_p)}$$

cohomology (as opposed to de Rham cohomology). However, this immediately runs into trouble: let X be any **irreducible** algebraic variety, and consider for example the cohomology of the constant sheaf $\mathcal{F} = \mathbb{Z}$. Fix any open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X (for the Zariski topology!) and compute $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$: this requires us to consider the complex

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to \cdots \to C^n(\mathcal{U}, \mathcal{F}) \to 0;$$

the crucial remark is that (since X is irreducible) any open set is dense, and therefore so is the intersection of finitely many open sets. It follows that (for any set I of indices) $\mathcal{F}\left(\bigcap_{i\in I} U_i\right) = \mathbb{Z}$, and the restriction

$$\mathbb{Z} = \mathcal{F}(X) \to \mathcal{F}\left(\bigcap_{i\in I} U_i\right) = \mathbb{Z}$$

is the identity. Thus $\mathcal{F}$ is flabby, hence Čech-acyclic. In our case of interest this can also be seen in a completely elementary way as follows.

Notice that the Čech complex is simply

$$0 \to \prod_{i_0\in I} \mathbb{Z} \to \prod_{(i_0,i_1)\in I^2} \mathbb{Z} \to \prod_{(i_0,i_1,i_2)\in I^3} \mathbb{Z} \to \cdots \quad ;$$

we want to show that the cohomology of this complex is trivial in positive degree. We do this by finding a *homotopy* from the identity to the zero map (in positive degree). Concretely, this means finding a collection of maps

$$h^p : C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,\dots,i_p)\in I^{p+1}} \mathbb{Z} \to \prod_{(i_0,\dots,i_{p-1})\in I^p} \mathbb{Z} = C^{p-1}(\mathcal{U}, \mathcal{F}),$$

one for each $p > 0$, with the property that

$$\mathrm{id}_{C^p(\mathcal{U},\mathcal{F})} = \delta^{p-1} \circ h^p + h^{p+1} \circ \delta^p.$$

If such a collection of maps exists, given a cohomology class $[s] \in \check{H}^p(\mathcal{U}, \mathcal{F})$ represented by a certain $s \in C^p(\mathcal{U}, \mathcal{F})$ such that $\delta^p(s) = 0$, we obtain

$$[s] = [\delta^{p-1} \circ h^p(s) + h^{p+1} \circ \delta^p(s)] = [\delta^{p-1}(h^p(s))] = [0]$$

as desired. To define our homotopy, we fix (arbitrarily) an index $\ell \in I$ and set, for $s \in C^p(\mathcal{U}, \mathcal{F})$,

$$h^p(s)_{i_0,\dots,i_{p-1}} = s_{\ell,i_0,\dots,i_{p-1}}$$

Let us check that this is indeed a homotopy: we compute

$$\begin{aligned}(\delta^{p-1} \circ h^p + h^{p+1} \circ \delta^p)(s)_{i_0,\dots,i_p} &= \left(\delta^{p-1}h^p(s)\right)_{i_0,\dots,i_p} + h^{p+1}(\delta^p(s))_{i_0,\dots,i_p} \\ &= \sum_{j=0}^{p} (-1)^j h^p(s)_{i_0,\dots,\widehat{i_j},\dots,i_p} + (\delta^p(s))_{\ell,i_0,\dots,i_p} \\ &= \sum_{j=0}^{p} (-1)^j h^p(s)_{i_0,\dots,\widehat{i_j},\dots,i_p} + s_{i_0,\dots,i_p} + \sum_{j=0}^{p} (-1)^{j+1} s_{\ell,i_0,\dots,\widehat{i_j},\dots,i_p}\end{aligned},$$

where in the last sum we have isolated the first term (that in which we remove the index ℓ) from the others (notice that when we remove index i_j we are in fact removing the $(j+1)$-th index from $(\ell, i_0, \dots, i_p)$). We thus obtain

$$\begin{aligned}(\delta^{p-1} \circ h^p + h^{p+1} \circ \delta^p)(s)_{i_0,\dots,i_p} &= \sum_{j=0}^{p} (-1)^j h^p(s)_{i_0,\dots,\widehat{i_j},\dots,i_p} + s_{i_0,\dots,i_p} + \sum_{j=0}^{p} (-1)^{j+1} s_{\ell,i_0,\dots,\widehat{i_j},\dots,i_p} \\ &= \sum_{j=0}^{p} (-1)^j s_{\ell,i_0,\dots,\widehat{i_j},\dots,i_p} + s_{i_0,\dots,i_p} + \sum_{j=0}^{p} (-1)^{j+1} s_{\ell,i_0,\dots,\widehat{i_j},\dots,i_p} \\ &= s_{i_0,\dots,i_p}\end{aligned}$$

as desired. This proves that $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for any open cover $\mathcal{U}$ and any $p > 0$; by passing to the limit, we also obtain $\check{H}^p(X, \mathcal{F}) = 0$ for any $p > 0$ – and this for any constant sheaf $\mathcal{F}$. Thus our naïve attempt to use Čech cohomology to detect topologically interesting information has failed rather miserably: the Zariski topology is way too coarse to capture any interesting phenomenon!

One of the key insights of Grothendieck was that one could build a good cohomology theory by allowing more general covers than just those with Zariski-open sets: in our setting, the good notion of cover will be that of *étale morphism*. As a first motivation as to why étale maps might be relevant, remember that Grothendieck had been able to formulate an algebraic theory of the fundamental group by noticing that the (usual) π_1 of a space X could be recovered from the knowledge of its universal cover of X; since étale maps are the algebraic version of topological covers, it followed that étale covers were enough to recover algebraically a good approximation of $\pi_1(X, x_0)$ (the formalisation of this vague idea is of course Grothendieck's theory of the étale fundamental group of a scheme). If one is ready to believe the analogy with topology, and recalling that (for reasonable topological spaces) the first homology group is nothing but the abelianisation of the fundamental group, then we might be led to suspect that étale covers should also be enough to capture information about the cohomology of X, or at least about its first cohomology group. As it turned out, looking at étale covers is actually enough to construct a full cohomology theory in all degrees! Making all this precise will have to wait...

2 Point-counting and the Weil conjectures

Today we start exploring the second main theme that led to the formulation (and eventual proof) of the Weil conjectures: counting points on schemes defined over a finite field. As a preliminary step, let's clarify what we mean by a *point*.

2.1 Schematic points

Let X be a scheme of finite type over a finite field $\mathbb{F}_q$. One possible notion of *point of* X is that of a (set-theoretic) point of the underlying topological space $|X|$: such a point will be called a *schematic point* of X.

Definition 2.1. Given a schematic point $x \in |X|$, consider its topological closure $Z(x) := \overline{\{x\}}$. We define

$$X^{(r)} := \{x \in |X| : \operatorname{codim}_X Z(x) = r\}, \quad X_{(r)} := \{x \in |X| : \dim Z(x) = r\}.$$

Here *dimension* and *codimension* are to be taken in the topological sense, namely, for a subscheme Y of X we have

$$\dim Y = \sup\{n : \exists Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq Y,\ Z_i \text{ non-empty closed irreducible}\}$$

and if Y is closed and irreducible

$$\operatorname{codim}_X Y = \sup\{n : \exists Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n,\ Z_i \text{ non-empty closed irreducible}\}$$

Exercise 2.2. Find an example of an irreducible scheme X and of a closed irreducible subscheme Y such that $\dim(X) \neq \dim(Y) + \operatorname{codim}_X(Y)$.

Remark 2.3. It is clear by definition that $|X| = \coprod_{r \geq 0} X^{(r)} = \coprod_{r \geq 0} X_{(r)}$.

In order to make the connection with the notion of *rational* points which we will need in what follows, it is useful to notice the following:

Lemma 2.4. *Let X be a scheme of finite type over $\mathbb{Z}$ and let $x \in |X|$ be a schematic point. The following are equivalent:*

1. *the residue field $k(x)$ is finite;*
2. *$x \in X_{(0)}$.*

Furthermore, if we define the ***norm*** *of a point $x \in X_{(0)}$ as $N(x) := \#k(x)$, then X possesses only finitely many points of any given norm.*

Sketch of proof. Fix an open cover by affine subschemes $U_i = \operatorname{Spec}(A_i)$ of X. If $x \in X_{(0)}$, then x is closed in X, hence a fortiori in every U_i. Seen as a point of U_i, x corresponds to a certain prime ideal $\mathfrak{p}_i$, which (since x is closed) is maximal in A_i. Conversely, suppose that $x \notin X_{(0)}$. Then x is not closed in at least one of the U_i[2], which means that there is at least an index i for which $\mathfrak{p}_i$ is not a maximal ideal in A_i.

Given that $k(x)$ is $A_i/\mathfrak{p}_i$, it suffices to show that if A is a ring of finite type over $\mathbb{Z}$ and $\mathfrak{p}$ is a prime ideal of A, then $A/\mathfrak{p}$ is finite if and only if $\mathfrak{p}$ is maximal. Both implications are well-known: since a finite integral ring is a field, if $A/\mathfrak{p}$ is finite then $\mathfrak{p}$ must be maximal; conversely, if $A/\mathfrak{p}$ is a field, then one can prove that it is of finite characteristic p, at which point (since $A/\mathfrak{p}$ is a finitely generated $\mathbb{F}_p$-algebra which is a field) the Nullstellensatz implies that $A/\mathfrak{p}$ is a finite extension of $\mathbb{F}_p$, that is, a finite field. To see that $A/\mathfrak{p}$ is of finite characteristic, suppose by contradiction that it is not: then it contains a copy of $\mathbb{Q}$. Since $A/\mathfrak{p}$ is finitely generated over $\mathbb{Z}$, say by $t_1, \ldots, t_k$,

[2] a subset C of a topological space is closed iff given an open cover U_i of X the intersection $C \cap U_i$ is closed in U_i for every i.

we can choose a sufficiently divisible integer N such that $t_1, \ldots, t_k$ are integral over $\mathbb{Z}[1/N]$. It follows that we have inclusions $\mathbb{Z}[1/N] \subseteq \mathbb{Q} \subseteq A/\mathfrak{p}$ with $A/\mathfrak{p}$ integral over $\mathbb{Z}[1/N]$. It follows that $\mathbb{Q}$ is integral over $\mathbb{Z}[1/N]$, contradiction.

For the second statement, one reduces immediately to the case that X is affine and integral. By Noether normalisation, it suffices to handle the case of $\mathbb{A}^n_{\mathbb{Z}}$, which is easy. $\square$

2.2 Scheme-valued points and rational points

An arithmetically more natural notion of *point* of a scheme X is that of a $\mathbb{F}_{q^n}$-rational point: morally, we would like to say that a $\mathbb{F}_{q^n}$-rational point is a solution to the equations that define X.

Definition 2.5. Let X be a scheme over S and let T be an S-scheme. We define the T-points of X as a scheme as

$$X(T) = \mathrm{Hom}_{\mathrm{Sch}}(T, X)$$

and the T-points of X as an S-scheme as

$$X(T)_S = \mathrm{Hom}_S(T, X).$$

When $S = \mathrm{Spec}(\mathbb{F}_q)$ we shall also write $X(T)_{\mathbb{F}_q}$ for $X(T)_{\mathrm{Spec}\,\mathbb{F}_q}$, and similarly when $T = \mathrm{Spec}(\mathbb{F}_{q^n})$ we shall also write $X(\mathbb{F}_{q^n})$ for $X(\mathrm{Spec}\,\mathbb{F}_{q^n})$.

Remark 2.6. Notice that when X is a scheme defined over the prime field $\mathbb{F}_p$ the number of $\mathbb{F}_{p^n}$-valued points $\#X(\mathbb{F}_{p^n})$ is precisely what we would call "the number of solutions to the equations defining X". However, in general (that is, when X is defined over a not-necessarily prime field $\mathbb{F}_q$) one really needs to consider $X(\mathbb{F}_{q^n})_{\mathbb{F}_q}$ in order to get the correct "number of solutions": for example, if $X = \mathrm{Spec}(\mathbb{F}_{p^n})$, then $|X(\mathbb{F}_{p^n})| = n$ while $|X(\mathbb{F}_{p^n})_{\mathbb{F}_{p^n}}| = 1$.

The following is a useful characterisation of the scheme-valued points $X(T)$ when T is the spectrum of a field:

Lemma 2.7. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then for every field extension K of $\mathbb{F}_q$ we have*

$$X(K) = \{(x, i) : x \in |X|, i : k(x) \to K \text{ field homomorphism}\}$$

and for $\mathbb{F}_q \subseteq K' \subseteq K$

$$X(K)_{K'} = \coprod_{x \in |X|} \mathrm{Hom}_{K'-alg}(k(x), K)$$

Proof. Let $s : \mathrm{Spec}(K) \to X$ be a K-valued point. Topologically, s is determined by its image $x \in |X|$. At the level of sheaves, once x is fixed the datum of a K-valued point is equivalent to the datum of the pullback morphism

$$x^\sharp : s^* \mathcal{O}_X \to \mathcal{O}_{\mathrm{Spec}\,K};$$

since these are sheaves over a point, this is in turn equivalent to a local map of rings $\mathcal{O}_{X,x} = \Gamma(\mathrm{Spec}\,K, s^*\mathcal{O}_K) \to \Gamma(\mathrm{Spec}\,K, \mathcal{O}_{\mathrm{Spec}\,K}) = K$; by locality, the maximal ideal of $\mathcal{O}_{X,x}$ is mapped to 0 in K, so this is in turn equivalent to a map of rings $\mathcal{O}_{X,x}/\mathfrak{M}_x = k(x) \to K$ as claimed. The proof for $X(K)_{K'}$ is essentially identical. $\square$

We can finally make the connection between schematic and scheme-valued points (for points with values in fields):

Lemma 2.8. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then*

$$\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} = \sum_{e \mid n} e \cdot \# \left\{ x \in X_{(0)} : [k(x) : \mathbb{F}_q] = e \right\}.$$

Proof. By Lemma 2.7 we have

$$X(\mathbb{F}_{q^n})_{\mathbb{F}_q} = \{(x,i) : x \in |X|, i : k(x) \to \mathbb{F}_{q^n} \text{ a } \mathbb{F}_q\text{-algebra morphism}\};$$

in particular, the field $k(x)$ needs to be finite, so that $x \in X_{(0)}$ by Lemma 2.4. Given $x \in X_{(0)}$, the number of $\mathbb{F}_q$-algebra homomorphisms $i : k(x) \to \mathbb{F}_{q^n}$ is clearly $[k(x) : \mathbb{F}_q]$ if $[k(x) : \mathbb{F}_q]$ divides n and is 0 otherwise, so the claimed formula follows. □

2.3 The ζ function of a scheme

Let X be a scheme of finite type over $\mathbb{Z}$. One defines

$$\zeta(X,s) = \prod_{x \in X_{(0)}} \frac{1}{1 - N(x)^{-s}}.$$

Here s is a complex variable. Note the following elementary facts:

Lemma 2.9. *1. The ζ function of X depends only on the underlying topological space $|X|$ and not on its schematic structure.*

2. If $X_{(0)} = \coprod_i (X_i)_{(0)}$ for a family (X_i) of subschemes of X, then

$$\zeta(X,s) = \prod_i \zeta(X_i, s).$$

3. In particular, if $X \to \operatorname{Spec} \mathbb{Z}$ is a scheme of finite type, we have

$$\zeta(X,s) = \prod_{p \text{ prime}} \zeta(X_p, s)$$

where X_p is the fiber of X over $\mathbb{F}_p$.

As for the convergence of $\zeta(X,s)$ we have the following general result:

Theorem 2.10. *Let X be a scheme of finite type over $\mathbb{Z}$. The product defining $\zeta(X,s)$ converges for $\Re s > \dim X$.*

Exercise 2.11. Prove Theorem 2.10.

Hint. Proceed by induction on the dimension of X. Show that one may assume that X is irreducible and affine. Show that if $f : X \to Y$ is finite, then the claim for Y implies the claim for X. Using Noether normalisation and the previous reductions, show that it's enough to handle the cases $X = \mathbb{A}^n_{\mathbb{Z}}$ and $X = \mathbb{A}^n_{\mathbb{F}_p}$.

2.4 The geometric Z function of a scheme over $\mathbb{F}_q$

When X is a scheme of finite type over a finite field $\mathbb{F}_q$, there is a more natural generating function that one can attach to $X/\mathbb{F}_q$, namely its *geometric zeta function*

$$Z(X/\mathbb{F}_q, t) = \exp\left(\sum_{n \geq 1} \#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} \frac{t^n}{n}\right).$$

2.4.1 The fundamental identity for schemes over $\mathbb{F}_q$

The following (essentially formal) identity is crucial to most manipulations of ζ functions of schemes:

Theorem 2.12. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then*

$$\zeta(X,s) = Z(X/\mathbb{F}_q, q^{-s}).$$

Proof. This follows from a direct computation:

$$\begin{aligned}
\log\zeta(X,s) &= \sum_{x\in X_{(0)}} -\log(1-N(x)^{-s}) = \sum_{x\in X_{(0)}}\sum_{n\geq 1}\frac{N(x)^{-ns}}{n} \\
&= \sum_{d}\sum_{\substack{x\in X_{(0)}\\ N(x)=q^d}}\sum_{n\geq 1}\frac{q^{-nds}}{n} = \sum_{k\geq 1}\sum_{d|k}\sum_{\substack{x\in X_{(0)}\\ N(x)=q^d}}\frac{q^{-ks}}{k/d} \\
&= \sum_{k\geq 1}\frac{q^{-ks}}{k}\sum_{d|k} d\sum_{\substack{x\in X_{(0)}\\ N(x)=q^d}} 1 = \sum_{k\geq 1}\frac{q^{-ks}}{k}\#X(\mathbb{F}_{q^k})_{\mathbb{F}_q} \\
&= \log Z(X/\mathbb{F}_q, q^{-s}).
\end{aligned}$$

□

2.4.2 Examples

Example 2.13 (The Riemann ζ function)**.** Take $X = \mathrm{Spec}(\mathbb{Z})$. Then

$$\zeta_X(s) = \prod_{x\in X_{(0)}}\frac{1}{1-N(x)^{-s}} = \prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$$

is the Riemann zeta function.

Example 2.14 (Zeta function of an $\mathbb{F}_q$-point)**.** Take $X = \mathrm{Spec}(\mathbb{F}_q)$: then by definition

$$\zeta(X,s) = \frac{1}{1-q^{-s}}.$$

Example 2.15 (Affine line over a scheme)**.** Consider the scheme $\mathbb{A}^1_X$, where X is a scheme of finite type over $\mathbb{F}_q$. Then

$$\zeta(\mathbb{A}^1_X, s) = \zeta(X, s-1).$$

Indeed, using the fact that the equality $\#\mathbb{A}^1(\mathbb{F}_{q^n})_{\mathbb{F}_q} = q^n$ holds for any finite field $\mathbb{F}_{q^n}$ we have

$$\begin{aligned}
\zeta(\mathbb{A}^1_X, s) &= \prod_{x\in X_{(0)}}\zeta(\mathbb{A}^1_x, s) = \prod_{x\in X_{(0)}} Z(\mathbb{A}^1_x/k(x), N(x)^{-s}) \\
&= \prod_{x\in X_{(0)}}\exp\left(\sum_{n\geq 1}\#\mathbb{A}^1_x(\mathbb{F}_{N(x)^n})_{k(x)}\frac{N(x)^{-ns}}{n}\right) \\
&= \prod_{x\in X_{(0)}}\exp\left(\sum_{n\geq 1}N(x)^n\frac{N(x)^{-ns}}{n}\right) \\
&= \prod_{x\in X_{(0)}}\exp\left(\sum_{n\geq 1}\frac{N(x)^{-n(s-1)}}{n}\right) \\
&= \prod_{x\in X_{(0)}}\frac{1}{1-N(x)^{1-s}} = \zeta(X, s-1).
\end{aligned}$$

In particular, by induction on d one gets

$$\zeta(\mathbb{A}^d_{\mathbb{F}_q}, s) = \zeta(\mathrm{Spec}(\mathbb{F}_q)/\mathbb{F}_q, s-d) = \frac{1}{1-q^{d-s}}$$

Example 2.16 (Projective space over $\mathbb{F}_q$)**.** Take $X = \mathbb{P}^n_{\mathbb{F}_q}$. There are several ways to compute the ζ function of X; one can for example use Theorem 2.12, or notice that $(\mathbb{P}^n_{\mathbb{F}_q})_{(0)} = \coprod_{k=0}^n (\mathbb{A}^k_{\mathbb{F}_q})_{(0)}$, so that

$$\zeta(\mathbb{P}^n_{\mathbb{F}_q}, s) = \prod_{k=0}^{n} \frac{1}{1-q^{k-s}}.$$

With a view towards the Weil conjectures (which we will state in a moment), we rewrite this ζ function as $\zeta(\mathbb{P}^n_{\mathbb{F}_q}, s) = \prod_{k=0}^n \frac{1}{P_{2k}(q^{-s})}$, where $P_{2k}(t) = 1 - q^k t$ is a polynomial of degree $1 = \dim H^{2k}(\mathbb{P}^n)$ with roots of absolute value q^{-k}.

Example 2.17 (Curves)**.** It is a theorem of Hasse-Weil that for a smooth projective curve $\mathcal{C}/\mathbb{F}_q$ of genus g we have

$$\zeta(\mathcal{C}, s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})},$$

where $P_0(t) = 1-t$, $P_2(t) = 1-qt$, and $P_1(t) \in 1+t\mathbb{Z}[t]$ is a polynomial of degree $2g = \dim H^1(\mathcal{C})$ all of whose roots have absolute value $q^{-1/2}$. In particular, if $\mathcal{C}$ is a curve of genus 1, its ζ function takes the form

$$\frac{1-at+qt^2}{(1-t)(1-qt)}$$

for some $a \in \mathbb{Z}$.

Exercise 2.18. Let $E/\mathbb{F}_5$ be the elliptic curve defined in $\mathbb{P}^2_{\mathbb{F}_5}$ by the homogeneous equation $y^2z = x^3 + z^3$. Find an explicit formula for $|E(\mathbb{F}_{5^n})|$.

2.5 The Weil conjectures

We are now in a position to notice (at least in our very special cases!) that the following properties seem to hold for all Z- (or equivalently ζ-)functions of **smooth projective** schemes of finite type over finite fields:

1. Rationality: $Z(X/\mathbb{F}_q, t)$ is a rational function with rational (and in fact even integral) coefficients. More precisely, one has

$$Z(X/\mathbb{F}_q, t) = \frac{P_1(t)\cdots P_{2n-1}(t)}{P_0(t)\cdots P_{2n}(t)},$$

   where $n = \dim X$ and $P_i(t)$ is a polynomial in $\mathbb{Z}[t]$ of the form $P_i(t) = 1 + tq_i(t)$ with $q_i(t) \in \mathbb{Z}[t]$.

2. Functional equation: this is best seen from the case of projective space. Notice that

$$\zeta(\mathbb{P}^n_{\mathbb{F}_q}/\mathbb{F}_q, s) = \prod_{k=0}^{n} \frac{1}{1-q^{k-s}}$$

   is *almost* symmetric under the transformation $s \mapsto n-s$: more precisely,

$$\begin{aligned}\zeta(\mathbb{P}^n_{\mathbb{F}_q}/\mathbb{F}_q, n-s) &= \prod_{k=0}^{n} \frac{1}{1-q^{s-(n-k)}} = \prod_{k=0}^{n} \frac{1}{1-q^{s-k}} \\ &= \prod_{k=0}^{n} \frac{-1}{q^{s-k}} \frac{1}{1-q^{k-s}} = (-1)^{n+1} q^{(n+1)(n/2-s)} \zeta(\mathbb{P}^n_{\mathbb{F}_q}/\mathbb{F}_q, s),\end{aligned}$$

which in hindsight should be written as

$$\zeta(\mathbb{P}^n_{\mathbb{F}_q}/\mathbb{F}_q, n-s) = \pm q^{\chi(\mathbb{P}^n)(n/2-s)}\zeta(\mathbb{P}^n_{\mathbb{F}_q}/\mathbb{F}_q, s),$$

where $n = \dim \mathbb{P}^n$ and χ denotes the Euler characteristic. In terms of the geometric zeta function, this reads

$$Z\left(X/\mathbb{F}_q, \frac{1}{q^n t}\right) = \pm q^{\frac{n\chi}{2}} t^{\chi} Z(X/\mathbb{F}_q, t).$$

3. Riemann hypothesis and purity: all the roots of $P_i(t)$ are have absolute value $q^{-i/2}$, and their inverses are algebraic integers. Moreover,

$$P_{2n-i}(t) = C_i t^{\deg P_i} P_i\left(\frac{1}{q^n t}\right)$$

with $C_i \in \mathbb{Z}$.

4. Link with topology/Betti numbers: one would like to say that, in the previous notation, $\deg P_i(t) = \dim H^i(X)$.

   More precisely, this can be formalised as follows. Suppose we have a smooth and proper scheme $\mathcal{X}$ defined over a finitely generated $\mathbb{Z}$-algebra R such that $R \twoheadrightarrow \mathbb{F}_q$. Suppose furthermore that X is the (smooth and proper) fiber over $\mathbb{F}_q$ of this $\mathcal{X}$. Then we can embed R into $\mathbb{C}$, thus obtaining a complex variety $\mathcal{X}_{\mathbb{C}}$: the precise conjecture is that $\deg P_i(t) = \dim H^i\left((\mathcal{X}_{\mathbb{C}})^{\mathrm{an}}, \mathbb{C}\right)$, where $(\mathcal{X}_{\mathbb{C}})^{\mathrm{an}}$ denotes the analytic complex manifold obtained from $\mathcal{X}_{\mathbb{C}}$.

These are the famous Weil conjectures!

Definition 2.19. The **Betti numbers** of a variety X are $b_i = \dim H^i(X)$.

An immediate consequence of these statements is the following bound for the number of $\mathbb{F}_{q^n}$-rational points of a scheme defined over $\mathbb{F}_q$:

Corollary 2.20. *Let X be a smooth, proper, geometrically irreducible scheme of finite type over* $\operatorname{Spec}\mathbb{F}_q$*. Suppose that X is of pure dimension d and that is the special fiber of a smooth proper scheme $\mathcal{X}/R$. Then $|X(\mathbb{F}_{q^n})| = q^{nd} + O_X(q^{nd/2})$, where the implicit constant depends only on the Betti numbers of X in the sense specified above.*

Exercise 2.21. Prove Corollary 2.20.

Exercise 2.22 (Grassmannians)**.** For any $1 \le k \le n-1$, let $G(k,n)$ be the Grassmannian defined over $\operatorname{Spec}\mathbb{Z}$; for each field K, its K-valued points are the k-dimensional linear subspaces in K^n.

1. Show that $\mathrm{GL}_n(\mathbb{F}_q)$ acts transitively on $G(k,n)(\mathbb{F}_q)$, and the stabilizer of each point is isomorphic to $\mathrm{GL}_k(\mathbb{F}_q) \times \mathrm{GL}_{n-k}(\mathbb{F}_q) \times \mathrm{M}_{k,n-k}(\mathbb{F}_q)$.

2. Show that for each $k \ge 1$ one has

$$|\operatorname{GL}_k(\mathbb{F}_q)| = q^{\frac{k(k-1)}{2}}(q^k-1)(q^{k-1}-1)\cdots(q-1).$$

3. Use the previous parts to show that

$$|G(k,n)(\mathbb{F}_q)| = \frac{(q^n-1)\cdots(q^{n-k+1}-1)}{(q^k-1)\cdots(q-1)} =: \binom{n}{k}_q$$

4. Show that

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

   and use this to deduce that $\binom{n}{k}_q = \sum_{i=0}^{k(n-k)} \lambda_{n,k}(i) q^i$, where $\lambda_{n,k}(i)$ is the number of partitions of i into at most $n-k$ parts, each of size at most k.

5. With the notation in (4), deduce that

$$Z(G(k,n)/\mathbb{F}_q, t) = \prod_{i=0}^{k(n-k)} \frac{1}{(1-q^i t)^{\lambda_{n,k}(i)}}.$$

6. Deduce that the Betti numbers of the complex Grassmannian are

$$b_{2i+1}(G(k,n)) = 0 \text{ for all } i, \text{ and } b_{2i}(G(k,n)) = \lambda_{n,k}(i) \text{ for } 1 \leq i \leq k(n-k).$$

3 Weil cohomologies

3.1 Weil cohomologies

In the light of our discussion in the first lecture the following definition should not be surprising:

Definition 3.1. Let k be a field and $\mathrm{SmProj}(k)$ be the category of smooth projective k-varieties. Also fix a "field of coefficients" K with $\mathrm{char}(K) = 0$ and let $\mathrm{GrAlg}(K)$ be the category of $\mathbb{Z}$-graded commutative K-algebras. A (pure) **Weil cohomology** for smooth projective k-varieties with K-coefficients is given by the following set of data:

(D1) **Functor**. A contravariant functor $H^\bullet(-, K) : \mathrm{SmProj}(k) \to \mathrm{GrAlg}(K)$. Explicitly, this means that we are given the following:

- For every $X \in \mathrm{SmProj}(k)$, a graded commutative K-algebra $H^\bullet(X, K)$, on which the grading is indexed by the integers: $H^\bullet(X, K) = \bigoplus_{n\in\mathbb{Z}} H^n(X, K)$. The multiplication $H^\bullet(X, K) \times H^\bullet(X, K)$ is denoted by $(\alpha, \beta) \mapsto \alpha \cup \beta$ and is K-bilinear, hence induces a map $H^\bullet(X, K) \otimes H^\bullet(X, K) \to H^\bullet(X, K)$. Recall that *graded commutative* means $\alpha \cup \beta = (-1)^{\deg\alpha\cdot\deg\beta}\beta \cup \alpha$ for homogeneous elements α, β.
- For every morphism of smooth projective varieties $f : X \to Y$, a *pullback* map $f^* : H^\bullet(Y, K) \to H^\bullet(X, K)$ which is a K-algebra map preserving the grading. Furthermore, if $f : X \to Y$ and $g : Y \to Z$ are two morphisms of smooth projective varieties, then $(g \circ f)^* = f^* \circ g^*$.

(D2) **Tate twist**. A (graded) 1-dimensional K-vector space $K(1)$, with grading concentrated in degree -2. For any (graded) K-vector space V we will then set $V(n) := V \otimes_K K(1)^{\otimes n}$, where by definition $K(1)^{\otimes n} = \mathrm{Hom}_K\big(K(1)^{\otimes(-n)}, K\big)$ when $n < 0$.

(D3) **Trace map**. For every $X \in \mathrm{SmProj}(k)$, a K-linear *trace map*

$$\mathrm{Tr}_X : H^{2\dim X}(X, K) \to K(-\dim X).$$

(D4) **Cycle class map**. For every $X \in \mathrm{SmProj}(k)$ and every closed irreducible subvariety $Z \subset X$ of codimension c, a cohomology class $\gamma_X(Z) \in H^{2c}(X, K)(c)$.

These data should satisfy the following axioms:

1. (finite dimensionality) Every $H^i(X, K)$ is a finite-dimensional K-vector space.
2. (dimension) If X is of dimension d_X, then $H^i(X, K) = 0$ for $i \notin [0, 2d_X]$
3. (orientability) $\dim_K H^2(\mathbb{P}^1_k, K) = 1$, and more precisely $H^2(\mathbb{P}^1_k, K) \cong K(-1)$.
4. (additivity) For every $X, Y \in \mathrm{SmProj}(k)$ the canonical morphism

$$H^\bullet(X \coprod Y, K) \to H^\bullet(X, K) \oplus H^\bullet(Y, K)$$

induced by the inclusions $X \hookrightarrow X \coprod Y$ and $Y \hookrightarrow X \coprod Y$ is an isomorphism.

5. (Künneth formula) For every $X, Y \in \mathrm{SmProj}(k)$ the natural map

$$\begin{array}{rccc} \kappa_{X\times Y} : & H^\bullet(X, K) \otimes H^\bullet(Y, K) & \cong & H^\bullet(X \times_k Y, K) \\ & \alpha \otimes \beta & \mapsto & \pi_X^*\alpha \cup \pi_Y^*\beta \end{array}$$

is an isomorphism, where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the canonical projections. This implies in particular that $H^\bullet(\mathrm{Spec}(k), K) = K$, concentrated in degree 0.

6. (trace and Poincaré duality) for every $X \in \mathrm{SmProj}(k)$ of pure dimension d_X the trace map

$$\mathrm{Tr}_X : H^{2d_X}(X, K) \to K(-d_X)$$

is an isomorphism if X is **geometrically** connected. Furthermore, if X is of pure dimension d (but not necessarily geometrically connected), the pairing

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle_X : & H^i(X, K) \otimes H^{2d_X - i}(X, K) & \to & K(-d_X) \\ & \alpha \otimes \beta & \mapsto & \mathrm{Tr}_X(\alpha \cup \beta) \end{array}$$

is perfect.

7. (trace maps and products) Let $X, Y \in \mathrm{SmProj}(k)$ be geometrically connected. Then for classes $\alpha \in H^{2 \dim X}(X, K)$ and $\beta \in H^{2 \dim Y}(Y, K)$ we have

$$\mathrm{Tr}_{X \times Y}(\pi_X^* \alpha \cup \pi_Y^* \beta) = \mathrm{Tr}_X(\alpha) \, \mathrm{Tr}_Y(\beta).$$

8. (properties of the cycle class map) For every $X \in \mathrm{SmProj}(k)$ the cycle class map induces a homomorphism

$$\gamma_X : \mathrm{CH}^i(X) \to H^{2i}(X, K)(i) := \mathrm{Hom}(K(-i), H^{2i}(X, K)),$$

with the following properties:

   (a) for any morphism $f : X \to Z$ of smooth projective varieties, the two homomorphisms $f^* \circ \gamma_Z$ and $\gamma_X \circ f^*$ coincide.

   (b) for any $Y_1 \in \mathrm{CH}^{i_1}(X_1), Y_2 \in \mathrm{CH}^{i_2}(X_2)$ we have

$$\gamma_{X_1 \times_k X_2}(Y_1 \times Y_2) = \gamma_{X_1}(Y_1) \otimes \gamma_{X_2}(Y_2),$$

   where we interpret $\gamma_{X_1}(Y_1) \otimes \gamma_{X_2}(Y_2) \in H^\bullet(X_1) \otimes H^\bullet(X_2)$ as an element of $H^\bullet(X_1 \times X_2)$ via the Künneth isomorphism.

   (c) let $f : X \to Y$ be a morphism of smooth varieties and denote by m the degree of the induced morphism $Z \to f(Z)$. Then

$$\mathrm{Tr}_X(\gamma_X(Z) \cup f^* \alpha) = m \, \mathrm{Tr}_Y(\gamma_Y(f(Z)) \cup \alpha)$$

   for all $\alpha \in H^{2 \dim Z}(Y)(\dim Z)$.

   (d) if X is geometrically connected, then for any $\alpha \in \mathrm{CH}^{d_X}(X)$ we have

$$\langle 1, \alpha \rangle_X = \deg \alpha$$

   Remark 3.2. In order to make sense of this last property one needs to know about Chow groups. Morally, what we're asking is that for a single point $x \in |X|$ with residue field of degree r over $\mathbb{F}_q$ we have $\langle 1, \gamma_X(x) \rangle_X = r$.

9. (normalisation for the point) If $X = \mathrm{Spec}(k)$, then $\gamma_X(X) = 1$ and $\mathrm{Tr}_X(1) = 1$.

Remark 3.3. The innocent-looking properties 8 (a) and (b) have the following interesting consequence: take closed subvarieties V and W of a smooth projective variety X. Then (if V and W intersect properly, and modulo some intersection theory) $V \cap W$ can be identified with $\Delta_X^*(V \times W)$. It follows that

$$\gamma_X(\Delta_X^*(V \times W)) = \Delta_X^*(\gamma_{X \times X}(V \times W)) = \Delta_X^*(\gamma_X(V) \otimes \gamma_X(W)) = \gamma_X(V) \cup \gamma_X(W) :$$

in other words, the class of an intersection is the product of the classes of the subvarieties being intersected.

3.2 The field of coefficients

The following remark, due to Weil and Serre, shows that in order to build a Weil cohomology that gives topologically sensible answers one cannot take as field of coefficients any of the fields $\mathbb{Q}, \mathbb{R}$, or $\mathbb{Q}_p$ (if we are working in characteristic p). More precisely, we have:

Theorem 3.4. *For $K = \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$ there is no Weil cohomology theory for smooth projective varieties over $\mathbb{F}_{p^2}$ such that $\dim_K H^1(C, K) = 2$ for every curve of genus 1.*

Proof. We only give the proof for $K = \mathbb{Q}$. Recall that an elliptic curve E is a curve of genus 1, and an elliptic curve over a field of positive characteristic p is called **supersingular** if it has no p-torsion points. In this case, $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a quaternion algebra, and in particular, $\dim_{\mathbb{Q}} \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} = 4$, and all non-zero elements are invertible. Supersingular elliptic curves E exist over every field $\mathbb{F}_{p^2}$ (and all the elements of $\mathrm{End}(E)$ can be defined over $\mathbb{F}_{p^2}$).

Since all elements of $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ are invertible, every nonzero homomorphism to a (nonzero) ring must be injective. On the other hand, since $H^1(-, \mathbb{Q})$ is functorial, we have a canonical morphism $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^1(E, \mathbb{Q})$. But if $\dim_{\mathbb{Q}} H^1(E, \mathbb{Q}) = 2$, then $\mathrm{End}(H^1(E, \mathbb{Q})) \cong M_{2\times 2}(\mathbb{Q})$. Since $\dim_{\mathbb{Q}} M_{2\times 2}(\mathbb{Q}) = 4 = \dim_{\mathbb{Q}} \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$, the morphism must be an isomorphism, which is a contradiction because there are nonzero noninvertible elements of $M_{2\times 2}(\mathbb{Q})$. □

Exercise 3.5. Complete the proof for $K = \mathbb{R}, \mathbb{Q}_p$ (for this you will need to know that if E is a supersingular elliptic curve over $\mathbb{F}_{p^2}$, then $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the unique quaternion algebra that is nonsplit precisely at p and at ∞. This is a theorem of Deuring).

At this point we can state a theorem which, while currently devoid of any mathematical content (since we don't know what étale cohomology *is* yet!), will serve as a motivation for the rest of the course:

Theorem 3.6. *Let k be a finite field of characteristic p and let ℓ be a prime different from p. Étale cohomology $H^\bullet_{ét}(-, \mathbb{Q}_\ell)$ is a Weil cohomology on* $\mathrm{SmProj}(k)$.

3.3 Some formal properties of Weil cohomologies

3.3.1 Pushing and pulling

Let k be a field and let $H^\bullet(-, K)$ be a Weil cohomology on $\mathrm{SmProj}(k)$.

Let $f : X \to Y$ be a morphism in $\mathrm{SmProj}(k)$. Assume that X and Y are geometrically connected. By functoriality we obtain a pullback morphism $f^* : H^\bullet(Y, K) \to H^\bullet(X, K)$, and we define the pushforward operator corresponding to f as the dual to f^* under Poincaré duality. More precisely, for every $p \geq 0$ the pullback f^* induces $(f^*)^\vee : H^p(X, K)^\vee \to H^p(Y, K)^\vee$, and by Poincaré duality these spaces are identified with $H^{2d_X - p}(X, K)(d_X)$ and $H^{2d_Y - p}(Y, K)(d_Y)$ respectively. We thus obtain a map

$$f_* : H^{2d_X - p}(X, K)(d_X) \to H^{2d_Y - p}(Y, K)(d_Y),$$

whence, choosing $p = 2d_X - q$, we have

$$f_* : H^q(X, K)(d_X) \to H^{q+2(d_Y - d_X)}(Y, K)(d_Y),$$

and twisting yet again by $K(-d_X)$ we finally obtain

$$f_* : H^q(X, K) \to H^{q+2(d_Y - d_X)}(Y, K)(d_Y - d_X),$$

which we call the **pushforward** or **direct image** morphism induced by f.

Lemma 3.7. *The following hold:*

1. *Let $X \in \mathrm{SmProj}(k)$ and let $\alpha : X \to \mathrm{Spec}(k)$ be the structure morphism. Then*
$$\alpha_* : H^{2d_X}(X, K) \to H^0(\mathrm{Spec}\, k, K)(-d_X)$$
is the (Poincaré) trace map.
2. *For every choice of forms α, β, γ in $H^\bullet(X, K)$ we have*
$$\langle \alpha \cup \beta, \gamma \rangle_X = \langle \alpha, \beta \cup \gamma \rangle_X.$$
3. *For every pair of forms $\alpha, \beta \in H^\bullet(Y, K)$ we have $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$.*
4. *The following **projection formula** holds:*
$$f_*(\alpha \cup f^*\beta) = f_*(\alpha) \cup \beta. \tag{1}$$

Proof. 1. Since $H^\bullet(\mathrm{Spec}(k), K) = K$, the pullback $\alpha^* : K \to H^\bullet(X, K)$ is simply the structure map of $H^\bullet(X, K)$ as a K-algebra. It follows that for every $\omega \in H^{2\dim X}(X, K)$ we have
$$\alpha_*(\omega) = \langle \alpha_*(\omega), 1 \rangle_{\mathrm{Spec}(k)} = \langle \omega, \alpha^*(1) \rangle_X = \langle \omega, 1 \rangle_X = \mathrm{Tr}_X(\omega \cup 1) = \mathrm{Tr}_X(\omega).$$

2. By definition one has
$$\langle \alpha \cup \beta, \gamma \rangle_X = \mathrm{Tr}_X((\alpha \cup \beta) \cup \gamma) = \mathrm{Tr}_X(\alpha \cup (\beta \cup \gamma)) = \langle \alpha, \beta \cup \gamma \rangle_X.$$

3. This is part of the axioms.

4. As the Poincaré pairing is nondegenerate, it suffices to to show that, for every $\gamma \in H^\bullet(Y, K)$, we have
$$\langle f_*(\alpha \cup f^*\beta), \gamma \rangle_Y = \langle f_*\alpha \cup \beta, \gamma \rangle_Y. \tag{2}$$
Since f_* is by definition the dual of f^* with respect to the Poincaré pairing, and using parts 2 and 3, the left hand side of this expression is
$$\langle \alpha \cup f^*\beta, f^*\gamma \rangle_Y = \langle \alpha, f^*\beta \cup f^*\gamma \rangle_Y = \langle \alpha, f^*(\beta \cup \gamma) \rangle_Y$$
while the right hand side is
$$\langle f_*\alpha, \beta \cup \gamma \rangle_Y = \langle \alpha, f^*(\beta \cup \gamma) \rangle_Y$$
so that (1) holds as claimed.

□

Now that we have defined the pushforward for cohomology classes, it's time to point out that one of the data in a Weil cohomology theory – the cycle class map – is in fact determined by the others:

Remark 3.8. The cycle class map is not independent from the other data (just like in the de Rham case, where it's completely determined by Poincaré duality). One can show that if $Z \subseteq X$ is a *smooth* irreducible subvariety, then $\gamma_X(Z)$ is simply $\iota_*(1)$, where $\iota : Z \hookrightarrow X$ is the inclusion of Z in X. To see this, we combine axioms 8 (a), (d) and (c). Let Z be any smooth projective variety and let $f : Z \to \mathrm{Spec}(k)$ be the structure map. Applying 8(a) we obtain $f^*(\gamma_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k)) = \gamma_Z(Z)$, so that $\gamma_Z(Z) = f^*(1) = 1$. Now we apply 8(c), taking f to be the inclusion of Z in X. We obtain
$$\mathrm{Tr}_Z(f^*\alpha) = \mathrm{Tr}_Z(\gamma_Z(Z) \cup f^*\alpha) = \mathrm{Tr}_X(\gamma_X(Z) \cup \alpha),$$
which – since the Poincaré pairing is nondegenerate and the left hand side does not involve the cycle class map – shows that $\gamma_X(Z)$, if it exists, is uniquely determined by the Poincaré pairing. More precisely, we have obtained
$$\langle f_*(1), \alpha \rangle_X = \langle 1, f^*\alpha \rangle_Z = \langle \gamma_X(Z), \alpha \rangle_X,$$

and the nondegeneracy of the Poincaré pairing yields $\gamma_X(Z) = f_*(1)$. More generally, for any closed subvariety Z, choose a nonsingular alteration $\varphi : Z' \to Z$ and let ι be the inclusion of Z in X. Then

$$\gamma_X(Z) = \frac{1}{\deg \varphi}(\iota \circ \varphi)_*(1); \tag{3}$$

to see this, simply let $f : Z' \to X$ be the composition $\iota \circ \phi$, and observe that this is now a map in the category of smooth projective varieties. We can then apply 8(c) again, and since the induced map $Z' \to f(Z')$ is by definition of degree $\deg \varphi$ we obtain that

$$\langle \frac{1}{\deg \varphi} f_*(1), \alpha \rangle_X = \frac{1}{\deg \varphi} \langle 1, f^*\alpha \rangle_{Z'} = \frac{1}{\deg \varphi} \operatorname{Tr}_{Z'}(f^*\alpha) \overset{8(c)}{=} \operatorname{Tr}_X(\gamma_X(Z) \cup \alpha) = \langle \gamma_X(Z), \alpha \rangle_X$$

holds for every $\alpha \in H^{2\dim Z}(X, K)(\dim Z)$. As before, this shows that $\gamma_X(Z)$ is determined by the other data of the cohomology theory, and that in fact it must be given by (3).

We leave one final property of the pullback/pushforward operators as an exercise:

Exercise 3.9. Let $f : X \to Y$ be a morphism of smooth varieties of the same dimension d. Define $\deg(f)$ to be 0 if f is not dominant, and to be the degree $[k(X) : f^*k(Y)]$ otherwise. Prove that $f_*(1) = \deg(f) \cdot 1$, or equivalently that

$$\operatorname{Tr}_X(f^*\alpha) = \deg(f) \operatorname{Tr}_Y(\alpha)$$

for all $\alpha \in H^{2d}(Y, K)(d)$. Deduce that $f_*f^*\alpha = \deg(f)\alpha$ for all $\alpha \in H^\bullet(Y, K)$.

3.3.2 Correspondences as morphisms in cohomology

Lemma 3.10. *Let $H^\bullet(-, K)$ be a Weil cohomology. For every $r \geq 0$ there is an isomorphism*

$$\operatorname{Hom}^r(H^\bullet(X, K), H^\bullet(Y, K)) = H^{2d_X+r}(X \times Y, K)(d_X),$$

where for two graded K-vector spaces V, W we denote by $\operatorname{Hom}^r(V, W)$ the space of K-linear maps that shift the grading by r.

Proof. This follows immediately from the axioms of a Weil cohomology:

$$\begin{aligned}
\operatorname{Hom}^r(H^\bullet(X, K), H^\bullet(Y, K)) &= \prod_{n \geq 0} \operatorname{Hom}\left(H^n(X, K), H^{n+r}(Y, K)\right) \\
&\cong \prod_{n \geq 0} H^n(X, K)^\vee \otimes H^{n+r}(Y, K) \\
&\cong \prod_{n \geq 0} H^{2d_X-n}(X, K)(d_X) \otimes H^{n+r}(Y, K) \qquad \text{(Poincaré)} \\
&\cong H^{2d_X+r}(X \times Y, K)(d_X) \qquad \text{(Künneth)}
\end{aligned}$$

□

Remark 3.11. Let $\alpha \in H^{2d_X+r}(X \times Y, K)(d_X)$. Chasing through the previous isomorphisms, one finds that the degree-r homomorphism $H^\bullet(X, K) \to H^\bullet(Y, K)$ corresponding to α can be described as follows. Let π_X, π_Y be the natural projections $X \times Y \to X, Y$ respectively: the homomorphism corresponding to α is

$$\beta \in H^\bullet(X, K) \mapsto (\pi_Y)_* (\pi_X^*\beta \cup \alpha) \in H^\bullet(Y, K). \tag{4}$$

Notice that if β has degree p then $\pi_X^*\beta$ is an element of $H^p(X \times Y, K)$, $\pi_X^*\beta \cup \alpha$ is an element of $H^{2d_X+r+p}(X \times Y, K)(d_X)$, and $(\pi_Y)_* (\pi_Y^*\beta \cup \alpha)$ is in

$$H^{2d_X+r+p+2(d_Y-d_{X\times Y})}(X, K)(d_X + d_Y - d_{X\times Y}) = H^{p+r}(X, K)$$

as desired. Formula (4) has a very nice geometric interpretation if one thinks of cohomology classes as duals of subvarieties (via the cycle class map). The homomorphism corresponding to a class α on $X \times Y$ can then be described as follows: we start from a class in X, pull it back to $X \times Y$, intersect it with the fixed class α, and finally pushforward the result of this intersection to Y.

For the reader's convenience we also make explicit all the isomorphisms implied by the previous formulas. Let $\alpha = \pi_X^* \beta_{2d_X - n}(d_X) \cup \pi_Y^* \gamma_{n+r}$ be a class in $H^{2d_X+r}(X \times Y, K)(d_X)$, where $\beta_i \in H^i(X, K)$ and $\gamma_j \in H^j(Y, K)$. By linearity and the Künneth isomorphism, it suffices to consider α of this form. The inverse of the Künneth isomorphism carries α to $\beta_{2d_X-n}(d_X) \otimes \gamma_{n+r} \in H^{2d_X-n}(X, K)(d_X) \otimes H^{n+r}(Y, K)$. By Poincaré duality, the class $\beta_{2d_X-n}(d_X)$ corresponds to the linear form $\alpha \mapsto \langle \alpha, \beta_{2d_X-n} \rangle_X (d_X)$ (which takes values in $K(-d_X + d_X) = K$, in degree 0), and therefore $\pi_X^* \beta_{2d_X-n} \cup \pi_Y^* \gamma_{n+r}$ corresponds to the homomorphism $H^n(X, K) \to H^{n+r}(Y, K)$ given by

$$\delta \mapsto \langle \delta, \beta_{2d_X-n} \rangle_X \ \gamma_{n+r},$$

where we drop (d_X) for readability. Our claim is that this coincides with

$$\delta \mapsto (\pi_Y)_* \left(\pi_X^* \delta \cup \alpha \right) = (\pi_Y)_* \left(\pi_X^* \delta \cup \pi_X^* \beta_{2d_X-n} \cup \pi_Y^* \gamma_{n+r} \right)$$

Since the Poincaré pairing is nondegenerate it suffices to prove that for every $\omega \in H^\bullet(Y, K)$ we have

$$\langle \langle \delta, \beta_{2d_X-n} \rangle_X \gamma_{n+r}, \omega \rangle_Y = \langle (\pi_Y)_* \left(\pi_X^* \delta \cup \pi_X^* \beta_{2d_X-n} \cup \pi_Y^* \gamma_{n+r} \right), \omega \rangle_Y .$$

This equality is not hard to show: indeed, the left hand side is simply given by

$$\langle \delta, \beta_{2d_X-n} \rangle_X \langle \gamma_{n+r}, \omega \rangle_Y ,$$

while the right hand side is equal to

$$\begin{aligned}
\langle (\pi_Y)_* \left(\pi_X^* \delta \cup \pi_X^* \beta_{2d_X-n} \cup \pi_Y^* \gamma_{n+r} \right), \omega \rangle_Y &= \langle \pi_X^* \delta \cup \pi_X^* \beta_{2d_X-n} \cup \pi_Y^* \gamma_{n+r}, \pi_Y^* \omega \rangle_{X \times Y} \\
&\overset{\text{(i)}}{=} \langle \pi_X^* \delta \cup \pi_X^* \beta_{2d_X-n}, \pi_Y^* \gamma_{n+r} \cup \pi_Y^* \omega \rangle_{X \times Y} \\
&\overset{\text{(ii)}}{=} \langle \pi_X^* (\delta \cup \beta_{2d_X-n}), \pi_Y^* (\gamma_{n+r} \cup \omega) \rangle_{X \times Y} \\
&= \mathrm{Tr}_{X \times Y} (\pi_X^* (\delta \cup \beta_{2d_X-n}) \cup \pi_Y^* (\gamma_{n+r} \cup \omega)) \\
&\overset{\text{(iii)}}{=} \mathrm{Tr}_X (\delta \cup \beta_{2d_X-n}) \, \mathrm{Tr}_Y (\gamma_{n+r} \cup \omega) \\
&= \langle \delta, \beta_{2d_X-n} \rangle_X \langle \gamma_{n+r}, \omega \rangle_Y ,
\end{aligned}$$

where (i) and (ii) hold by Lemma 3.7 (2) and (3) respectively, and (iii) is Axiom 7 in Definition 3.1.

3.3.3 Transposition

When stating and proving the Lefschetz Trace Formula we shall need one last operator on cohomology, namely transposition. This can be introduced in two equivalent ways:

1. let $X, Y \in \mathrm{SmProj}(k)$ and let $\sigma : Y \times X \to X \times Y$ be the map that exchanges the two factors. Then, for every $r \in \mathbb{Z}$ we have a transposition map

$$H^{2d_X+r}(X \times Y, K)(d_X) \to H^{2d_X+r}(Y \times X, K)(d_X)$$

given by σ^*.

2. By Lemma 3.10 we have

$$H^{2d_X+r}(X \times Y, K)(d_X) \cong \mathrm{Hom}^r \left(H^\bullet(X, K), H^\bullet(Y, K) \right).$$

Given a homomorphism $\varphi \in \mathrm{Hom}^r\left(H^\bullet(X,K), H^\bullet(Y,K)\right)$, that is, a collection of homomorphisms $H^n(X,K) \to H^{n+r}(Y,K)$, transposition (of linear maps) induces a collection of homomorphisms

$$H^{n+r}(Y,K)^\vee \to H^n(X,K)^\vee$$

which by Poincaré duality amounts to a collection of homomorphisms

$$H^{2d_Y-(n+r)}(Y,K)(d_Y) \to H^{2d_X-n}(X,K)(d_X),$$

that is, an element of $\mathrm{Hom}^{2(d_X-d_Y)+r}\left(H^\bullet(Y,K)(d_Y), H^\bullet(X,K)(d_X)\right)$, or equivalently of $\mathrm{Hom}^{2(d_X-d_Y)+r}\left(H^\bullet(Y,K), H^\bullet(X,K)\right)(d_X-d_Y)$. Using Lemma 3.10 again, this space can be identified with $H^{2d_X+r}(Y\times X)(d_X)$, and we have thus constructed a homomorphism $H^{2d_X+r}(X\times Y)(d_X) \to H^{2d_X+r}(Y\times X)(d_X)$ which agrees with σ^* as previously introduced.

We shall denote transposition by $\alpha \mapsto {}^t\alpha$. Unwinding the (second) definition shows that, for decomposable elements $\alpha\otimes\beta \in H^i(X)\otimes H^j(Y)$, ${}^t(\alpha\otimes\beta) = (-1)^{ij}\beta\otimes\alpha$.

3.3.4 The class of a graph

Given a morphism $f : X \to Y$ of smooth projective variety, we obtain two canonically defined classes in the cohomology of $Y\times X$:

1. on the one hand, the cycle class map allows us to consider $\gamma_{Y\times X}(\Gamma_f^t)$, where Γ_f^t is the transpose of the graph of γ (that is, the subscheme defined by $\{(y,x)\in Y\times X : y = f(x)\}$). Since Γ_f^t is of codimension d_Y, this is a class in $H^{2d_Y}(Y\times X,K)(d_Y)$.
2. On the other hand, functoriality of cohomology provides us with a graded homomorphism $f^* : H^\bullet(Y,K) \to H^\bullet(X,K)$ which, by lemma 3.10, is nothing but a cohomology class in $H^{2d_Y}(Y\times X,K)(d_Y)$

These two classes are in fact the same; more precisely,

Lemma 3.12. *The class $\gamma_{Y\times X}(\Gamma_f^t)$, seen as an element of* $\mathrm{Hom}^0(H^\bullet(Y,K), H^\bullet(X,K))$*, is equal to $\sum_j f^{*,j}$, where $f^{*,j}$ is the action of f^* on $H^j(Y,K)$.*

Proof. Exercise.

Hint. Denote by α the cohomology class of Γ_f^t and by φ_α the corresponding element of $\mathrm{Hom}^0(H^\bullet(Y), H^\bullet(X))$. The following might help:

1. Express α as $\sum_j \pi_Y^*\beta_j \cup \pi_X^*\gamma_j$.
2. It suffices to show that $\langle\varphi_\alpha(\delta),\zeta\rangle_X = \langle f^*\delta,\zeta\rangle_X$ holds for every $\zeta\in H^\bullet(X), \delta\in H^\bullet(Y)$, where we set $\mathrm{Tr}_X(\eta)=0$ if η is pure of degree not equal to $2\dim X$.
3. Use the characterisation of cohomology classes given in Remark 3.8 and the obvious nonsingular alteration $\varphi : X \to \Gamma_f^t$.

□

Finally, we have

Lemma 3.13. *Let $f : X \to Y$ be a morphism of smooth projective varieties. Then ${}^tf^* = f_*$, where we consider f^* and f_* as elements of $H^{2d_Y}(Y\times X,K)(d_Y)$ and $H^{2d_Y}(X\times Y,K)(d_Y)$.*

Proof. Let f_* correspond to the class $[\alpha]\in H^{2d_Y}(X\times Y,K)(d_Y)$: by the identification given in Remark 3.11, this means that for every $\beta\in H^\bullet(X,K)$ we have

$$f_*(\beta) = (\pi_Y)_*\left(\pi_X^*\beta\cup\alpha\right), \tag{5}$$

where π_X, π_Y are the natural projections $X \times Y \to X, Y$ respectively. Our claim is equivalent to ${}^t f_* = f^*$, that is, we want to show that $\sigma^*\alpha$ is the class in $H^{2d_Y}(Y \times X, K)(d_Y)$ that corresponds to the homomorphism $f^* : H^\bullet(Y, K) \to H^\bullet(X, K)$. By Remark 3.11 again, this is equivalent to the fact that the equality

$$f^*\beta = (\pi_X \circ \sigma)_* \left((\pi_Y \circ \sigma)^* \beta \cup \sigma^*\alpha \right)$$

holds for every $\beta \in H^\bullet(Y, K)$. Notice that the σ's appearing in the previous formula have two different origins: $\sigma^*\alpha$ is the class we want to consider, while $\pi_X \circ \sigma, \pi_Y \circ \sigma$ are the natural projections $Y \times X \to X, Y$ (notice the exchanged factors $Y \times X$). By the nondegeneracy of the Poincaré pairing, it suffices to show that for every class $\gamma \in H^\bullet(X, K)$ we have

$$\langle (\pi_X \circ \sigma)_* \left((\pi_Y \circ \sigma)^* \beta \cup \sigma^*\alpha \right), \gamma \rangle_X = \langle f^*\beta, \gamma \rangle_X.$$

We have

$$\langle (\pi_X \circ \sigma)_* \left((\pi_Y \circ \sigma)^* \beta \cup \sigma^*\alpha \right), \gamma \rangle_X = \langle (\pi_X)_* \circ \sigma_* \sigma^* \left(\pi_Y^* \beta \cup \alpha \right), \gamma \rangle_X,$$

and using Exercise 3.9 and the obvious fact that $\deg \sigma = 1$ we get

$$\langle (\pi_X \circ \sigma)_* \left((\pi_Y \circ \sigma)^* \beta \cup \sigma^*\alpha \right), \gamma \rangle_X = \langle (\pi_X)_* \left(\pi_Y^* \beta \cup \alpha \right), \gamma \rangle_X.$$

Now $(\pi_X)_*$ is the adjoint of π_X^* with respect to the Poincaré pairing, so the expression above is equal to

$$\langle \pi_Y^*\beta \cup \alpha, \pi_X^*\gamma \rangle_{X \times Y} = \langle \pi_Y^*\beta, \alpha \cup \pi_X^*\gamma \rangle_{X \times Y},$$

where we have used Lemma 3.7 (2). Finally, notice that α is a class of even degree, so $\alpha \cup \pi_X^*\gamma = \pi_X^*\gamma \cup \alpha$, and using Equation (5) (with β replaced by γ) we get

$$\begin{aligned}
\langle \pi_Y^*\beta, \alpha \cup \pi_X^*\gamma \rangle_{X \times Y} &= \langle \pi_Y^*\beta, \pi_X^*\gamma \cup \alpha \rangle_{X \times Y} \\
&= \langle \beta, (\pi_Y)_* \left(\pi_X^*\gamma \cup \alpha \right) \rangle_Y \\
&= \langle \beta, f_*\gamma \rangle_Y \\
&= \langle f^*\beta, \gamma \rangle_X,
\end{aligned}$$

which is what we wanted to show.

□

3.3.5 Endomorphisms of cohomology

As a consequence of the formal constructions above we obtain the following: any choice of classes $\phi \in H^{2d_X+r}(X \times Y)(d_X)$ and $\psi \in H^{2d_Y-r}(Y \times X)(d_Y)$ induces homomorphisms (still denoted by ϕ, ψ)

$$\phi \in \mathrm{Hom}^r \left(H^\bullet(X, K), H^\bullet(Y, K) \right), \quad \psi \in \mathrm{Hom}^{-r} \left(H^\bullet(Y, K), H^\bullet(X, K) \right)$$

hence in particular a degree-0 endomorphism $\psi \circ \phi \in \mathrm{End}\, H^\bullet(X, K)$.

4 The existence of a Weil cohomology implies most of the Weil conjectures

4.1 The Lefschetz trace formula

We are now in a position to deduce from the formal properties of a Weil cohomology the validity of the so-called (Grothendieck-)Lefschetz trace formula, namely, the following identity:

Theorem 4.1 (Lefschetz trace formula). *Let k be a field and let $H^\bullet(-,K)$ be a Weil cohomology theory on* $\mathrm{SmProj}(k)$*. Let X, Y be smooth projective k-varieties of pure dimension d_X, d_Y respectively. Let $\phi \in H^{2d_X+r}(X \times_k Y, K)(d_X)$, $\psi \in H^{2d_Y-r}(Y \times_k X, K)(d_Y)$. Then*

$$\langle \phi, {}^t\psi \rangle_{X\times Y} = \sum_{j=0}^{2d_X} (-1)^j \operatorname{tr}\left(\psi \circ \phi \mid H^j(X,K)\right)$$

Proof. This is essentially formal. Since both sides of the Lefschetz formula are bilinear in ϕ and ψ, it suffices to treat the case of $\phi = v \otimes w$ and $\psi = w' \otimes v'$ with $v \in H^{2d_X-i}(X,K)(d_X), w \in H^{i+r}(Y,K)$, $v' \in H^j(X,K), w' \in H^{2d_Y-j-r}(Y,K)(d_Y)$. In this case the left hand side of the trace formula is

$$\begin{aligned}
\langle \phi, {}^t\psi \rangle_{X\times Y} &= \langle v \otimes w, (-1)^{j(j+r)} v' \otimes w' \rangle_{X\times Y} \\
&= (-1)^{j(j+r)} \operatorname{Tr}_{X\times Y}\left((v \otimes w) \cup (v' \otimes w')\right) \\
&= (-1)^{j(j+r)}(-1)^{j(i+r)} \operatorname{Tr}_{X\times Y}\left((v \cup v') \otimes (w \cup w')\right) \\
&= (-1)^{j(i+j)} \operatorname{Tr}_{X\times Y}\left((v \cup v') \otimes (w \cup w')\right) \\
&= \delta_{ij} \langle v, v' \rangle_X \langle w, w' \rangle_Y.
\end{aligned}$$

In order to study the right hand side we notice that the morphism in cohomology given by $\phi = v \otimes w$ is $x \mapsto \langle x, v \rangle_X \cdot w$ if $x \in H^i(Y,K)$ and 0 if x is (pure) of degree different from i. Similarly, $\psi(y)$ is $\langle y, w' \rangle_X \cdot v'$ if y is in degree $j+r$, and is zero on the other graded pieces. It follows that $\psi \circ \phi$ is zero unless $i = j$, while if $i = j$ we have

$$\psi \circ \phi = \begin{cases} 0 \text{ on } H^k(X,K), k \neq i \\ x \mapsto \langle x, v \rangle_X \langle w, w' \rangle_Y \cdot v' \text{ on } H^i(X,K) \end{cases}$$

It is then immediate to check that the trace of $\psi \circ \phi$ is

$$\delta_{ij} \langle v', v \rangle_X \langle w, w' \rangle_Y = \delta_{ij} (-1)^{ij} \langle v, v' \rangle_X \langle w, w' \rangle_Y = (-1)^i \langle \phi, {}^t\psi \rangle_{X\times Y}$$

as claimed. $\square$

4.2 Frobenius

Let $X \in \mathrm{SmProj}(\mathbb{F}_q)$ be a smooth projective variety defined over a finite field $\mathbb{F}_q$, and denote by $\overline{X}$ the basechange of X to $\overline{\mathbb{F}_q}$, that is,

$$\overline{X} = X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}\left(\overline{\mathbb{F}_q}\right).$$

We now introduce one of our main players: the **(absolute) Frobenius**. This is the morphism of locally ringed spaces

$$\mathrm{Fr}_{X,q} : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$$

given by $(\mathrm{id}, \mathrm{Fr}^\sharp_{X,q})$, where $\mathrm{Fr}^\sharp_{X,q}$ acts on sections of $\mathcal{O}_X$ by sending $f \in \mathcal{O}_X(U)$ to $f^q \in \mathcal{O}_X(U)$. By basechange we also obtain

$$\mathrm{Fr}_{\overline{X},q} = \mathrm{Fr}_{X,q} \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{id}_{\mathrm{Spec}(\overline{\mathbb{F}_q})},$$

called the **relative (or $\overline{\mathbb{F}_q}$-linear) Frobenius**. As with any endomorphism of $\overline{X}$, the relative Frobenius $\mathrm{Fr}_{\overline{X},q}$ acts on $\overline{X}(\overline{\mathbb{F}_q})$ by sending a point $x : \mathrm{Spec}(\overline{\mathbb{F}_q}) \to \overline{X}$ to the point

$$\mathrm{Fr}_{\overline{X},q}(x) = \mathrm{Fr}_{\overline{X},q} \circ x : \mathrm{Spec}(\overline{\mathbb{F}_q}) \to \overline{X}.$$

Notice that by construction $\mathrm{Fr}_{\overline{X},q}$ also acts on the subset $\overline{X}(\overline{\mathbb{F}_q})_{\overline{\mathbb{F}_q}}$ of $\overline{X}(\overline{\mathbb{F}_q})$, and that $\overline{X}(\overline{\mathbb{F}_q})_{\overline{\mathbb{F}_q}} = X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$ (for the latter statement, notice that the universal property of the product implies that giving a $(\mathrm{Spec}\,\overline{\mathbb{F}_q})$-point of $\overline{X}$ is the same as giving a $(\mathrm{Spec}\,\overline{\mathbb{F}_q})$-point of X, since the map $\mathrm{Spec}\,\overline{\mathbb{F}_q} \to \overline{X} \to \mathrm{Spec}\,\overline{\mathbb{F}_q}$ is fixed).

There is also a second natural way to define an action of Frobenius on the set $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$: let Fr_q be the element of $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)$ given by $x \mapsto x^q$. Then there is a natural action of Fr_q on $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q} = \mathrm{Hom}_{\mathbb{F}_q}(\mathrm{Spec}\,\overline{\mathbb{F}_q}, X)$ via its action on $\mathrm{Spec}\,\overline{\mathbb{F}_q}$. Chasing through the definitions we obtain:

Lemma 4.2. *The actions of* $\mathrm{Fr}_{\overline{X},q}$ *on* $\overline{X}(\overline{\mathbb{F}_q})_{\overline{\mathbb{F}_q}} = X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$ *and of* Fr_q *on* $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$ *coincide.*

Proof. Given our identification of $\overline{X}(\overline{\mathbb{F}_q})_{\overline{\mathbb{F}_q}}$ with $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$, we need to show that, given a point $s : \mathrm{Spec}(\overline{\mathbb{F}_q}) \to \overline{X}$ such that $\mathrm{Spec}(\overline{\mathbb{F}_q}) \xrightarrow{s} \overline{X} \to \mathrm{Spec}(\overline{\mathbb{F}_q})$ is the identity, the composition

$$\mathrm{Spec}(\overline{\mathbb{F}_q}) \xrightarrow{s} \overline{X} \xrightarrow{\mathrm{Fr}_{\overline{X},q}} \overline{X} \xrightarrow{\pi_X} X$$

equals the composition

$$\mathrm{Spec}(\overline{\mathbb{F}_q}) \xrightarrow{\mathrm{Fr}_q} \mathrm{Spec}(\overline{\mathbb{F}_q}) \xrightarrow{s} \overline{X} \xrightarrow{\pi_X} X.$$

First we look at the topological image, which is the same in both cases since $\pi_X \circ \mathrm{Fr}_{\overline{X},q} = \mathrm{Fr}_{X,q}$, which is the identity on $|X|$. Secondly we look at the action on the level of sheaves: for $f \in \mathcal{O}_{X,x}$ we have

$$\begin{aligned}\left(\pi_X \circ \mathrm{Fr}_{\overline{X},q} \circ s\right)^\sharp (f) &= s^\sharp \left(\mathrm{Fr}^\sharp_{X,q}\, \pi^\sharp_X f\right)\\ &= s^\sharp((\pi^\sharp_X f)^q) = (s^\sharp(\pi^\sharp_X f))^q\\ &= \mathrm{Fr}^\sharp_q \left(s^\sharp \pi^\sharp_X f\right)\\ &= (\pi_X \circ s \circ \mathrm{Fr}_q)^\sharp(f),\end{aligned}$$

which is what we needed to show. □

Given that the fixed points of Fr^n_q on $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$ coincide with $X(\mathbb{F}_{q^n})_{\mathbb{F}_q}$ (this is clear from the description of Lemma 2.7) we get

Lemma 4.3. *Let* $\Delta_{\overline{X}}$ *be the diagonal in* $\overline{X} \times \overline{X}$ *and let* $\Gamma_{\mathrm{Fr}^n_{\overline{X},q}} \subset \overline{X} \times \overline{X}$ *be the graph of* $\mathrm{Fr}^n_{\overline{X},q}$. *The set* $X(\mathbb{F}_{q^n})_{\mathbb{F}_q}$ *is in natural bijection with the closed points of* $\Delta_{\overline{X}} \cap \Gamma_{\mathrm{Fr}^n_{\overline{X},q}}$.

Remark 4.4. It is important to note that while $\mathrm{Fr}_{X,q}$ is by definition the identity on $|X|$, the relative Frobenius $\mathrm{Fr}_{\overline{X},q}$ is (in general) **not** the identity on the underlying topological space $|\overline{X}|$.

Exercise 4.5. Let $X = \mathrm{Spec}\,\mathbb{F}_{p^n}$, seen as a scheme over $\mathbb{F}_p$. Describe the topological space $\overline{X}$ and the action of the relative Frobenius $\mathrm{Fr}_{\overline{X},p}$ on it.

Remark 4.6. There are several actions of Frobenius on $\overline{X}$ that it is natural to consider; even though we won't need them, I think it's useful to introduce the terminology:

1. the **absolute Frobenius** F: the same construction that works for X works equally well for $\overline{X}$, and gives the absolute Frobenius of $\overline{X}$, which by definition is the identity on the underlying topological space $|X|$;

2. the **relative Frobenius** F_r, which we have already defined;

3. the **arithmetic Frobenius** F_a, namely $\mathrm{id}_X \times_{\operatorname{Spec}\mathbb{F}_q} \mathrm{Fr}_q$, where Fr_q is the Frobenius of $\operatorname{Spec}(\mathbb{F}_q)$;

4. the **geometric Frobenius** F_g, namely $\mathrm{id}_X \times_{\operatorname{Spec}\mathbb{F}_q} \mathrm{Fr}_q^{-1}$.

It is a fact that $F_a \circ F_r = F_r \circ F_a = F$; this has interesting consequences in étale cohomology, because one can show that F acts trivially on cohomology (the "baffling theorem"), hence that the actions of F_a and F_r are inverse to each other.

Definition 4.7. We denote by Γ_m the graph $\Gamma_{\mathrm{Fr}^m_{\overline{X},q}} \subseteq \overline{X} \times \overline{X}$ of the m-th iterate of the relative Frobenius of $\overline{X}$.

Lemma 4.8. *If X is smooth (in fact, separated suffices) over $\mathbb{F}_q$, then the intersection $\Delta_{\overline{X}} \cap \Gamma_m$ is transverse at every point, so that $\Delta_{\overline{X}} \cap \Gamma_m$ consists of a reduced set of points.*

Proof. The statement is local; take an open affine neighbourhood $\overline{U} = \operatorname{Spec}\left(\operatorname{Spec}\overline{\mathbb{F}_q}[x_1, \dots, x_n]/I\right)$ of x in X. Then $\overline{U} \times \overline{U} = \operatorname{Spec}\left(\frac{\overline{\mathbb{F}_q}[x_1,\dots,x_n,y_1,\dots,y_n]}{(I_x,I_y)}\right)$ (where I_y is obtained from $I_x = I$ by replacing every x_i with the corresponding y_i) is a neighbourhood of (x,x) in $X \times X$, and we need to consider Δ, defined by the ideal $(x_i - y_i)$, and Γ_m, defined by $y_i - x_i^{q^m}$. The quotient

$$\frac{\overline{\mathbb{F}_q}[x_1, \dots, x_n, y_1, \dots, y_n]}{(I_x, I_y, x_i - y_i, y_i - x_i^{q^m} | i = 1, \dots, n)}$$

is a quotient of

$$\frac{\overline{\mathbb{F}_q}[x_1, \dots, x_n]}{(I_x, x_i - x_i^{q^m} | i = 1, \dots, n)},$$

itself simply a product of copies of $\overline{\mathbb{F}_q}$. Any quotient will thus have the same form, so that $\Delta \cap \Gamma_m$ is covered by (finitely many) collections of closed points. □

4.3 Cohomological approach to the Weil conjectures

The following fundamental theorem establishes a close link between the existence of a Weil cohomology and the truth of (some of) the Weil conjectures:

Theorem 4.9. *Suppose that there exists a Weil cohomology*[3] $H^\bullet : \mathrm{SmProj}(\overline{\mathbb{F}_q}) \to \mathrm{GrAlg}(K)$. *Then for any $X \in \mathrm{SmProj}(\mathbb{F}_q)$ one has:*

$$Z(X,t) = \prod_{i=0}^{2\dim X} \det\left(1 - t \cdot \mathrm{Fr}^*_{\overline{X},q} | H^i(\overline{X})\right)^{(-1)^{i+1}},$$

where $\overline{X} = X \times_{\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}_q}$. In particular $Z(X,t)$ is a rational function and satisfies the expected functional equation.

The proof is an almost immediate consequence of the following fact (in turn an avatar of the Lefschetz trace formula):

Lemma 4.10. *With the same notation, we have*

$$\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{tr}\left((\mathrm{Fr}^n_{\overline{X},q})^* \mid H^i(\overline{X})\right)$$

[3]for which we drop the explicit mention of K in $H^i(\overline{X}, K)$

Proof. We know from Lemma 4.3 that $X(\mathbb{F}_{q^n})_{\mathbb{F}_q}$ is in bijection with the closed points of $\Delta_{\overline{X}} \cap \Gamma_{\mathrm{Fr}^n_{\overline{X},q}}$, whence

$$\begin{aligned}
\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} &= \deg(\Delta_{\overline{X}} \cap \Gamma_{\mathrm{Fr}^n_{\overline{X},q}}) \\
&\overset{(1)}{=} \langle 1, \gamma_{\overline{X}\times\overline{X}}\left(\Delta_{\overline{X}} \cap \Gamma_{\mathrm{Fr}^n_{\overline{X},q}}\right)\rangle_{\overline{X}\times\overline{X}} \\
&\overset{(2)}{=} \langle 1, \gamma_{\overline{X}\times\overline{X}}(\Delta_{\overline{X}}) \cup \gamma_{\overline{X}\times\overline{X}}\left(\Gamma_{\mathrm{Fr}^n_{\overline{X},q}}\right)\rangle_{\overline{X}\times\overline{X}} \\
&\overset{(3)}{=} \langle 1, (\Gamma_{\mathrm{id}_{\overline{X}}})_* \cup (\Gamma_{\mathrm{Fr}^n_{\overline{X},q}})_*\rangle_{\overline{X}\times\overline{X}} \\
&\overset{(4)}{=} \langle 1, (\Gamma_{\mathrm{id}_{\overline{X}}})_* \cup {}^t(\Gamma_{\mathrm{Fr}^n_{\overline{X},q}})^*\rangle_{\overline{X}\times\overline{X}} \\
&= \langle (\Gamma_{\mathrm{id}_{\overline{X}}})_*, {}^t(\Gamma_{\mathrm{Fr}^n_{\overline{X},q}})^*\rangle_{\overline{X}\times\overline{X}} \\
&= \langle (\Gamma_{\mathrm{id}_{\overline{X}}})^*, {}^t(\Gamma_{\mathrm{Fr}^n_{\overline{X},q}})^*\rangle_{\overline{X}\times\overline{X}} \\
&\overset{(5)}{=} \sum_{i=0}^{2\dim X} (-1)^i \operatorname{tr}\left((\mathrm{Fr}^n_{\overline{X},q})^* \circ \mathrm{id}^*_{\overline{X}} \mid H^i(\overline{X})\right) \\
&= \sum_{i=0}^{2\dim X} (-1)^i \operatorname{tr}\left((\mathrm{Fr}^n_{\overline{X},q})^* \mid H^i(\overline{X})\right)
\end{aligned},$$

where we have used:

- for (1), property 8 (d) of a Weil cohomology;
- for (2), the properties of the cycle class map with respect to intersection established in Remark 3.3 (the intersection is proper – or *transverse* – by Lemma 4.8);
- for (3), the characterisation of the cohomology class of a graph given in Lemma 3.12;
- for (4), the fact that the transpose of f^* is f_*, see Lemma 3.13;
- for (5), the Lefschetz Trace Formula (Theorem 4.1).

□

Before proving Theorem 4.9 we need one last linear algebra lemma:

Lemma 4.11. *Let $\phi : V \times W \to K$ be a perfect pairing of vector spaces of dimension r over a field K. Let $f \in \mathrm{End}_K(V), g \in \mathrm{End}_K(W)$, and $\lambda \in K^\times$ be such that*

$$\phi(f(v), g(w)) = \lambda\phi(v, w)$$

for all $v \in V$ and $w \in W$. Then

$$\det(1 - tg) = \frac{(-1)^r\lambda^r t^r}{\det f}\det(1 - \lambda^{-1}t^{-1}f)$$

and

$$\det(g) = \frac{\lambda^r}{\det(f)}.$$

The proof of this lemma is immediate and left to the reader.

Proof of Theorem 4.9. **Rationality**. By definition and Lemma 4.10 we have

$$\begin{aligned}
\log Z(X, t) &= \sum_{n\geq 1} \#X(\mathbb{F}_{q^n})_{\mathbb{F}_q}\frac{t^n}{n} \\
&\overset{(4.10)}{=} \sum_{n\geq 1}\frac{t^n}{n}\sum_{i=0}^{2\dim X}(-1)^i \operatorname{tr}\left((\mathrm{Fr}^n_{\overline{X},q})^* \mid H^i(\overline{X})\right);
\end{aligned}$$

let $\lambda_{i,j}$, for $i = 0, \dots, 2\dim X$ and $j = 1, \dots, \dim H^i(\overline{X})$, be the eigenvalues of $\mathrm{Fr}_{\overline{X},q}$ on $H^i(\overline{X})$. Then, since $(\mathrm{Fr}^n_{\overline{X},q})^* = (\mathrm{Fr}^*_{\overline{X},q})^n$, the eigenvalues of $\mathrm{Fr}^n_{\overline{X},q}$ on $H^i(\overline{X})$ are the $\lambda^n_{i,j}$. Hence

$$\begin{aligned}
\log Z(X,t) &= \sum_{n\geq 1} \frac{t^n}{n} \sum_{i=0}^{2\dim X} (-1)^i \sum_{j=1}^{\dim H^i(\overline{X})} \lambda^n_{i,j} \\
&= \sum_{i=0}^{2\dim X} (-1)^i \sum_{j=1}^{\dim H^i(\overline{X})} \sum_{n\geq 1} \frac{(\lambda_{i,j}t)^n}{n} \\
&= \sum_{i=0}^{2\dim X} (-1)^i \sum_{j=1}^{\dim H^i(\overline{X})} -\log(1-\lambda_{i,j}t) \\
&= \sum_{i=0}^{2\dim X} (-1)^{i+1} \log\left(\prod_{j=1}^{\dim H^i(\overline{X})} (1-\lambda_{i,j}t) \right) \\
&= \sum_{i=0}^{2\dim X} (-1)^{i+1} \log\left(\det(1-\mathrm{Fr}^*_{\overline{X},q}\, t \mid H^i(\overline{X})) \right),
\end{aligned}$$

from which we get

$$Z(X,t) = \frac{P_1(t)\cdots P_{2\dim X-1}(t)}{P_0(t)\cdots P_{2\dim X}(t)}$$

with

$$P_i(t) = \det\left(1-\mathrm{Fr}^*_{\overline{X},q}\, t \mid H^i(\overline{X})\right).$$

Functional equation. Consider the perfect pairing

$$\langle\cdot,\cdot\rangle : H^i(\overline{X}) \times H^{2d-i}(\overline{X}) \to K$$

and denote by F the relative Frobenius $\mathrm{Fr}_{\overline{X},q}$. For every $\alpha \in H^i(\overline{X})$, $\beta \in H^{2d-i}(\overline{X})$ we have

$$\langle F^*\alpha, F^*\beta\rangle_{\overline{X}} = \langle \alpha, F_*F^*\beta\rangle_{\overline{X}} = \langle \alpha, \deg F \cdot \beta\rangle_{\overline{X}} = q^d\langle\alpha,\beta\rangle_{\overline{X}},$$

so that Lemma 4.11 applies with $\lambda = q^d$. Denote by $b_i = \dim H^i(\overline{X})$; by Poincaré duality we have $b_i = b_{2d-i}$. We obtain

$$\begin{aligned}
P_{2d-i}(t) &= \det(1-tF^* \mid H^{2d-i}(\overline{X})) \\
&= \frac{(-1)^{b_i} q^{db_i} t^{b_i}}{\det(F^* \mid H^i(\overline{X}))} \det(1-q^{-d}t^{-1}F^* \mid H^i(\overline{X})) \\
&= \frac{(-1)^{b_i} q^{db_i} t^{b_i}}{\det(F^* \mid H^i(\overline{X}))} P_i\left(\frac{1}{q^d t}\right)
\end{aligned}$$

and

$$\det(F^* \mid H^i(\overline{X}))\det(F^* \mid H^{2d-i}(\overline{X})) = q^{db_i},$$

so

$$\prod_{i=0}^{2d} \det(F^* \mid H^i(\overline{X}))^{(-1)^i} = q^{\frac{1}{2}d\chi}.$$

Putting everything together we obtain

$$\begin{aligned}
Z\left(X/\mathbb{F}_q, \frac{1}{q^d t}\right) &= \prod_{i=0}^{2d} P_i\left(\frac{1}{q^d t}\right)^{(-1)^{i+1}} \\
&= \prod_{i=0}^{2d} P_{2d-i}(t)^{(-1)^{i+1}} \left(\frac{(-1)^{b_i} q^{db_i} t^{b_i}}{\det(F^* \mid H^i(\overline{X}))}\right)^{(-1)^i} \\
&= \pm Z(X/\mathbb{F}_q, t) \cdot \prod_{i=0}^{2d} \frac{q^{(-1)^i db_i} t^{(-1)^i b_i}}{\det\left(F^* \mid H^i(\overline{X})\right)} \\
&= \pm Z(X/\mathbb{F}_q, t) \cdot t^{\chi} q^{d\chi} \prod_{i=0}^{2d} \frac{1}{\det\left(F^* \mid H^i(\overline{X})\right)^{(-1)^i}} \\
&= \pm \cdot t^{\chi} q^{\frac{1}{2} d\chi} Z(X/\mathbb{F}_q, t).
\end{aligned}$$

□

Remark 4.12. From the proof it follows that the sign in the functional equation can be determined from the knowledge of $\det(F^* \mid H^d(\overline{X}))$.

Exercise 4.13. Assuming the **full** Weil conjectures for $\overline{X}$ (that is, including the Riemann hypothesis), prove that if we set (as above) $P_i(t) = \det\left(1 - \mathrm{Fr}^*_{\overline{X},q}\, t \mid H^i(\overline{X})\right)$ then $P_i(t)$ does **not** depend on the specific Weil cohomology theory $H^\bullet(-)$ chosen to compute $P_i(t)$.

5 Weil conjectures for curves and Kloosterman sums

5.1 Proof of the Riemann hypothesis for curves

Let X be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ and let $Z(t) = Z(X/\mathbb{F}_q, t)$ be its geometric Zeta function. Assuming the existence of a Weil cohomology theory for (smooth irreducible projective) varieties over $\overline{\mathbb{F}_q}$, we have established the rationality of $Z(t)$ and the functional equation linking $Z(t)$ and $Z(q/t)$, so that we know that $Z(t)$ takes the form

$$Z(t) = \frac{P_1(t)}{P_0(t)P_2(t)}.$$

We have also shown that $Z(t)$ satisfies a functional equation of the form

$$Z\left(\frac{1}{qt}\right) = \pm t^{2-2g} q^{1-g} Z(t),$$

where $2 - 2g$ is the Euler characteristic of X (and g is its genus). Starting from these result, we will now prove the following theorem, which completes the proof of the Weil conjectures in the special case of curves:

Theorem 5.1 (Weil conjectures for curves)**.** *Let $Z(t) = \frac{P_1(t)}{P_0(t)P_2(t)}$ be the geometric zeta function of a smooth projective geometrically connected curve over the finite field $\mathbb{F}_q$. Then the following hold:*

1. *We have $P_0(t) = 1 - t$ and $P_2(t) = 1 - qt$.*
2. *$P_1(t)$ has integer coefficients.*
3. *Write $P_1(t) = \prod_{j=1}^{2g}(1 - \alpha_j t)$ for some $\alpha_j \in \overline{\mathbb{Q}}$: then $|\alpha_j| = q^{1/2}$ for all j.*

5.1.1 Step 1: $P_0(t)$ and $P_2(t)$

It will be clear from the definition of étale cohomology that $H^0(\overline{X}, \mathbb{Q}_\ell)$ is 1-dimensional with trivial Galois action; it follows that $P_0(t) = \det\left(F^* \mid H^0(\overline{X}, \mathbb{Q}_\ell)\right) = 1 - t$. By Lemma 4.11 we then obtain

$$P_2(t) = \det(1 - tF^* \mid H^2(\overline{X})) = -qt \det\left(1 - \frac{1}{qt} F^* \mid H^0(\overline{X})\right) = 1 - qt.$$

5.1.2 Step 2: $P_1(t)$ has integer coefficients

From the previous paragraph we know that

$$P_1(t) = P_0(t)P_2(t)Z(t) = (1-t)(1-qt)Z(t);$$

it thus suffices to show that $Z(t) \in \mathbb{Z}[[t]]$. In order to see this, we show that

$$Z(t) = \sum_{x \in Z_0(X)^+} t^{\deg x},$$

where $Z_0(X)^+$ is the monoid of effective 0-cycles on X (formal combinations of closed points with non-negative coefficients) and the degree of a 0-cycle $\sum_i n_i x_i$ is by definition $\sum_i n_i [k(x_i) : \mathbb{F}_q]$. As $Z(q^{-s})$ converges for $\Re s$ large enough, it suffices to show that

$$Z(q^{-s}) = \sum_{x \in Z(X_0)^+} q^{-s \deg x}, \tag{6}$$

and by the fundamental identity for zeta functions the left hand side of this expression is simply

$$\prod_{x\in X_{(0)}} \frac{1}{1-N(x)^{-s}} = \prod_{x\in X_{(0)}} \frac{1}{1-q^{-s\deg x}}.$$

We now notice that by definition we have $Z_0(X)^+ \cong \bigoplus_{x\in X_{(0)}} \mathbb{N}\cdot x$, and on the other hand

$$\prod_{x\in X_{(0)}} \frac{1}{1-N(x)^{-s}} = \prod_{x\in X_{(0)}} \frac{1}{1-q^{-s\deg x}} = \prod_{x\in X_{(0)}} \sum_{n_x\geq 0} q^{-sn_x \deg x}.$$

We may now develop the infinite product appearing in the last expression as follows. The general term in the development is obtained by multiplying together a finite number of terms $q^{-sn_1\deg x_1}, \dots, q^{-sn_k \deg x_k}$, where the x_i are in $X_{(0)}$ and are all distinct. It follows that these terms are in bijection with elements of $\bigoplus_{x\in X_{(0)}} \mathbb{N}\cdot x$, with $\sum n_x x \in \bigoplus_{x\in X_{(0)}} \mathbb{N}\cdot x$ corresponding to $q^{-s\sum n_x \deg x} = q^{-s\deg(\sum n_x x)}$. The equality claimed in eqation (6) follows.

Remark 5.2. Writing $P_1(t) = \prod_{j=1}^{2g}(1-\alpha_j t)$ as in the statement of Theorem 5.1, the integrality of the coefficients of $P_1(t)$ implies that the α_j are algebraic integers.

5.1.3 Step 3: estimates on $|\alpha_j|$ are equivalent to estimates on $|\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q}|$

We now relate part (3) of theorem 5.1 to a more elementary statement concerning the number of rational points of X:

Lemma 5.3. *The following are equivalent:*

1. $|\alpha_j| = q^{1/2}$ *for* $j = 1, \dots, 2g$;
2. $\left|\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} - (q^n+1)\right| \leq 2gq^{n/2}$ *for all* $n \geq 1$.

Proof. We start by noticing the following formal identities: we have both

$$t\frac{d}{dt}\log P_1(t) = t\frac{d}{dt}\log\left(\prod_{j=1}^{2g}(1-\alpha_j t)\right) = t\sum_{j=1}^{2g}\frac{-\alpha_j}{1-\alpha_j t} = -\sum_{j=1}^{2g}\sum_{n\geq 1}(\alpha_j t)^n = -\sum_{n\geq 1} t^n\left(\sum_{j=1}^{2g}\alpha_j^n\right)$$

and

$$\begin{aligned} t\frac{d}{dt}\log\left(P_0(t)P_2(t)Z(t)\right) &= t\frac{d}{dt}\log(1-t) + t\frac{d}{dt}\log(1-qt) + t\frac{d}{dt}\left(\sum_{n\geq 1}\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q}\frac{t^n}{n}\right) \\ &= \frac{-t}{1-t} + \frac{-qt}{1-qt} + \sum_{n\geq 1}\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} t^n \\ &= -\sum_{n\geq 1} t^n - \sum_{n\geq 1} q^n t^n + \sum_{n\geq 1}\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} t^n. \end{aligned}$$

Since $P_1(t) = P_0(t)P_2(t)Z(t)$ these two expressions must be equal, and comparing the coefficients of t^n we find that (for all n) we have

$$\boxed{\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} = q^n + 1 - \sum_{j=1}^{2g}\alpha_j^n.}$$

It is now clear that (1) implies (2); for the opposite implication, to ease the notation we define $a_n := \#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} - (q^n+1)$, and notice that we have proved

$$t\frac{d}{dt}\log P_1(t) = \sum_{n\geq 1} a_n t^n.$$

Take now any $t \in \mathbb{C}$ with $|t| < q^{-1/2}$. Then (assuming (2)) we have $|a_n t^n| < 2g(|t|q^{1/2})^n$, so the series $\sum_{n\geq 1} a_n t^n$ converges absolutely. This implies that the function $t\frac{d}{dt}\log P_1(t)$ is *analytic* in the open disk of radius $q^{-1/2}$, hence in particular that $P_1(t)$ (hence $Z(t)$) has no zeroes inside this disk. The functional equation

$$Z(\frac{1}{qt}) = \pm t^{2-2g} q^{1-g} Z(t)$$

now implies that $Z(t)$ (hence $P_1(t)$) has no zeroes with $|t| > q^{-1/2}$, so it follows that all the zeroes of $P_1(t)$ have absolute value exactly equal to $q^{1/2}$. The lemma now follows from the fact that the roots of $P_1(t)$ are precisely the α_j's. □

5.1.4 Step 4: on the number of $\mathbb{F}_{q^n}$-rational points of X

In the light of Lemma 5.3 we are now reduced to showing that $|\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} - (q^n+1)| \leq 2gq^{n/2}$. Since we need to prove this for all curves X over all finite fields $\mathbb{F}_q$, it is clearly enough to establish the inequality for $n = 1$. Let therefore X be a (smooth, projective, geometrically irreducible) curve of genus g defined over $\mathbb{F}_q$: we want to estimate $\#X(\mathbb{F}_q)_{\mathbb{F}_q}$, or – which is the same given Lemma 4.3 – the cardinality of the (transverse) intersection $\Delta \cap \Gamma$, where $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, Δ is the diagonal in $S := \overline{X} \times_{\overline{\mathbb{F}_q}} \overline{X}$, and Γ is the graph of the relative Frobenius. We shall do this by exploiting the properties of the intersection product on algebraic surfaces (over algebraically closed fields). We will need the following:

Theorem 5.4 (Properties of the intersection pairing)**.** *Let S be a smooth projective surface over an algebraically closed field. There is a bilinear form $\cdot : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$ such that:*

1. *if C_1, C_2 are smooth curves intersecting transversally, then $C_1 \cdot C_2$ is the number of set-theoretic intersection points $\#(C_1 \cap C_2)$;*
2. *$\cdot$ factors via algebraic equivalence: if C_1 is algebraically equivalent to C_1' and C_2 is algebraically equivalent to C_2', then $C_1 \cdot C_2 = C_1' \cdot C_2'$.*

Remark 5.5. One can give an explicit formula for the intersection pairing:

$$\mathcal{L} \cdot \mathcal{L}' = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^{-1}) - \chi((\mathcal{L}')^{-1}) + \chi(\mathcal{L}^{-1} \otimes (\mathcal{L}')^{-1}).$$

Recall that the Euler characteristic χ of a line bundle $\mathcal{L}$ on a surface S is by definition

$$\chi(\mathcal{L}) = \dim H^0(S, \mathcal{L}) - \dim H^1(S, \mathcal{L}) + \dim H^2(S, \mathcal{L}).$$

Remark 5.6. The fact that the intersection pairing is compatible with algebraic equivalence implies that it descends to a bilinear form

$$\cdot : \mathrm{NS}(S) \times \mathrm{NS}(S) \to \mathbb{Z},$$

where $\mathrm{NS}(S)$, called the **Néron-Severi group of** S, is (by definition) the group of divisors of S modulo algebraic equivalence. It is a theorem of Severi and Néron that $\mathrm{NS}(S)$ is a finitely generated abelian group (not necessarily torsion-free). However, it is immediate to see that the intersection pairing is trivial on the torsion subgroup of $\mathrm{NS}(S)$, so that the intersection pairing induces a bilinear form on the finite-dimensional $\mathbb{Q}$-vector space

$$\mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We now recall (without proof) several important facts from algebraic geometry:

Theorem 5.7. *In the previous context we have:*

1. *(Riemann-Roch for surfaces) The following equality holds for any line bundle $\mathcal{L}$:*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_S) + \frac{1}{2}\mathcal{L} \cdot (\mathcal{L} - K_S),$$

where K_S is the canonical divisor of S.

2. *(Adjunction formula) Let D be a smooth divisor on S: then the canonical divisor K_D of D equals $(K_S + D)|_D$.*

3. *The self-intersection of the canonical divisor of a smooth curve D equals the Euler characteristic of D.*

We shall also need the following result:

Theorem 5.8 (Hodge index theorem)**.** *Let H be an ample divisor on S and let D be such that $D \cdot H = 0$. Then $D^2 \leq 0$.*

Proof. Suppose by contradiction that $D^2 > 0$ and apply the Riemann-Roch formula (part (1) of Theorem 5.7) to nD. We obtain

$$\chi(nD) = \chi(\mathcal{O}_S) + \frac{n^2}{2}D^2 - \frac{1}{2}D \cdot K_S \cdot n,$$

and the right hand side of this equality tends to infinity (as $n \to \pm\infty$) since $D^2 > 0$. Since

$$\chi(nD) = \dim H^0(S, nD) + \dim H^2(S, nD) - \dim H^1(S, nD) \leq \dim H^0(S, nD) + \dim H^2(S, nD)$$

it follows in particular that $\dim H^0(S, nD) + \dim H^2(S, nD)$ tends to infinity as $n \to \infty$. By Serre duality, $\dim H^2(S, nD) = \dim H^0(S, K_S - nD)$, so we obtain

$$\dim H^0(S, nD) + \dim H^0(S, K_S - nD) \to \infty \quad \text{ as } n \to \pm\infty.$$

Suppose first that $\dim H^0(S, nD)$ is positive for some positive n. Then (for suitable n) nD is linearly (hence also algebraically) equivalent to an effective divisor, which implies $(nD) \cdot H > 0$ (as the restriction of the ample line bundle H to the curves in nD is ample, the intersection number cannot be zero). By bilinearity of the intersection product, this implies $D \cdot H > 0$, which contradicts the assumption $D \cdot H = 0$. Similarly, if $\dim H^0(S, nD)$ is positive for some negative n, then there is some positive n such that $-nD$ is (linearly equivalent to) an effective divisor; from this it follows that $-nD \cdot H > 0$, hence $D \cdot H < 0$, contradiction. We have thus shown that $\dim H^0(S, nD)$ is identically zero for $n \neq 0$, so that $\dim H^0(S, K_S - nD)$ tends to infinity *both* as $n \to +\infty$ *and* as $n \to -\infty$. This implies that for large n both $K_S - nD$ and $K_S + nD$ are linearly equivalent to effective divisors; it follows that

$$\dim H^0(S, 2K_S) = \dim H^0(S, (K_S - nD) + (K_S + nD)) \geq \dim H^0(S, K_S + nD) \to \infty \text{ as } n \to +\infty,$$

and this is obviously a contradiction. □

We are now ready to prove part (3) of Theorem 5.1. Let $S := \overline{X} \times \overline{X}$ (this is a smooth projective surface over $\overline{\mathbb{F}_q}$) and let $v := \{\text{pt}\} \times \overline{X}$, $h := \overline{X} \times \{\text{pt}\}$ be a 'vertical' and a 'horizontal' divisor respectively (it is immediate to see that any two divisors of the form $\{\text{pt}\} \times \overline{X}$ are algebraically equivalent, and the same holds for any two divisors of the form $\overline{X} \times \{\text{pt}\}$). It is clear from the definitions that $h^2 := h \cdot h = 0$, $v^2 := v \cdot v = 0$, and $h \cdot v = 1$. Indeed, the self-intersection $h \cdot h$ can be computed by intersecting h with any divisor algebraically equivalent to it. But then it suffices to pick any other horizontal fibre different from h to get that the intersection number is zero (because the corresponding smooth curves simply don't meet set-theoretically). Similarly, $h \cdot v = 1$ follows from the fact that there is only one set-theoretic intersection point, and the intersection is transverse.

For any divisor D on $S := \overline{X} \times \overline{X}$ we can then write $D = ah + bv + D'$, where D' is orthogonal to both h and v with respect to the intersection pairing (this is clearly possible by simply taking $a = D \cdot v$ and $b = D \cdot h$). We have in particular

$$\Delta = (h + v) + \Delta' \quad \text{ and } \Gamma = h + qv + \Gamma';$$

to see this, notice that Δ intersects any horizontal or vertical divisor precisely once, while Γ (being a graph) intersects any vertical divisor once. Finally, $\Gamma \cdot h$ is the number of points in $\overline{X}$ with a given image under the relative Frobenius automorphism: since the relative Frobenius has degree q, this number is precisely q.

Lemma 5.9. *The canonical divisor K_S is algebraically equivalent to $(2g-2)(h+v)$.*

Proof. Recall that K_S is the determinant of the dual to the tangent bundle. Since $S = \overline{X} \times \overline{X}$ product, the tangent bundle T_S of S is simply $\pi_1^* T_{\overline{X}} \oplus \pi_2^* T_{\overline{X}}$: this implies immediately that K_S is $\pi_1^*(K_{\overline{X}}) + \pi_2^*(K_{\overline{X}})$. On the other hand, the canonical divisor of $\overline{X}$ is given by $2g-2$ points, so $\pi_1^*(K_{\overline{X}})$ is the sum of $2g-2$ vertical fibres. As we have already noticed, all vertical fibres are algebraically equivalent, so $\pi_1^*(K_{\overline{X}})$ is numerically equivalent to $(2g-2)v$. The same argument applies to $\pi_2^*(K_{\overline{X}})$ to show that it is algebraically equivalent to $(2g-2)h$, and this establishes the lemma. □

Lemma 5.10. *The self-intersection numbers $(\Delta')^2$ and $(\Gamma')^2$ are given by*

$$(\Delta')^2 = -2g \quad \textit{and} \quad (\Gamma')^2 = -2gq.$$

Proof. We start by noticing that both Δ and Γ are smooth curves isomorphic to $\overline{X}$: it follows from part (3) of Theorem 5.7 that

$$K_\Delta^2 = 2g-2,$$

and on the other hand (by part (2) of the same theorem) we have

$$K_\Delta^2 = (K_S + \Delta) \cdot \Delta = \Delta^2 + (2g-2)(h+v) \cdot \Delta$$

Writing $\Delta = (h+v) + \Delta'$ we then obtain

$$2g-2 = (h+v+\Delta')^2 + (2g-2)(h+v) \cdot (h+v+\Delta') = 2 + (\Delta')^2 + (2g-2) \cdot 2,$$

whence $(\Delta')^2 = -2g$. We can perform a similar computation for $(\Gamma')^2$: we have

$$\begin{aligned} 2g-2 = K_\Gamma^2 &= (K_S+\Gamma)\cdot\Gamma = (h+qv+\Gamma')^2 + (2g-2)(h+v)\cdot(h+qv+\Gamma') \\ &= 2q + (\Gamma')^2 + (2g-2)(q+1) \end{aligned}$$

whence we obtain $(\Gamma')^2 = -2gq$. □

We are now in a position to compute $\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} = \Delta \cdot \Gamma$. We have

$$\Delta \cdot \Gamma = ((h+v)+\Delta') \cdot ((h+qv)+\Gamma') = q+1+\Delta'\cdot\Gamma';$$

by Theorem 5.8 we know that the restriction of the intersection pairing to the orthogonal of the ample[4] divisor $H = h+v$ is negative semidefinite. Since Δ', Γ' are (by construction) orthogonal to H, we may apply the Cauchy-Schwarz inequality to these two divisors to conclude that

$$|\Delta' \cdot \Gamma'| \leq \sqrt{(\Delta')^2(\Gamma')^2} = \sqrt{(-2g)(-2gq)} = 2g\sqrt{q},$$

which combined with our previous computation gives

$$\left|\#X(\mathbb{F}_{q^n})_{\mathbb{F}_q} - (q+1)\right| = |\Delta\cdot\Gamma - (q+1)| = |\Delta'\cdot\Gamma'| \leq 2g\sqrt{q}$$

as claimed. This concludes the proof of Theorem 5.1.

Remark 5.11. The fact that $h+v$ is ample follows immediately from the Nakai–Moishezon criterion, see [Har77, Section V, Theorem 1.10]. Alternatively, it can also be seen in an elementary fashion: the crucial point is that we are on a product surface $X \times X$, and the intersection of $h+v$ with the horizontal/vertical fibres gives an ample divisor on each. We can then embed each of the factors X in a suitable projective space, and get an embedding of the product by composing with a suitable Segre map. We give a formal argument.

[4] see Remark 5.11 below

Fix a point $p \in X$: this clearly gives an ample divisor on X, that is, there exists some multiple mp and a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n$ such that $\mathcal{O}_X(mp) = \iota^*\mathcal{O}_{\mathbb{P}^n}(1)$. Let π_1, π_2 be the canonical projections $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$, and let s be the Segre embedding

$$s : \mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N.$$

Recall that $s^*\mathcal{O}_{\mathbb{P}^N}(1) = \pi_1^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$. Finally, ρ_1, ρ_2 be the canonical projections $X^2 \to X$, and observe that the line bundle corresponding to $h + v$ is $\rho_1^*(\mathcal{O}_X(p)) \otimes \rho_2^*(\mathcal{O}_X(p))$. Consider the map ψ given by $X \times X \xrightarrow{(\iota,\iota)} \mathbb{P}^n \times \mathbb{P}^n \xrightarrow{s} \mathbb{P}^N$. Pulling back $\mathcal{O}_{\mathbb{P}^N}(1)$ along ψ we get

$$\begin{aligned}
\psi^*\mathcal{O}_{\mathbb{P}^N}(1) &= (\iota,\iota)^* s^*\mathcal{O}_{\mathbb{P}^N}(1) \\
&= (\iota,\iota)^* \left(\pi_1^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)\right) \\
&= \rho_1^*\iota^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \rho_2^*\iota^*\mathcal{O}_{\mathbb{P}^n}(1) \\
&= \rho_1^*\mathcal{O}_X(mp) \otimes \rho_2^*\mathcal{O}_X(mp) \\
&= \mathcal{O}_{X\times X}(m(h+v)),
\end{aligned}$$

which shows that $m(h + v)$ is very ample and $h + v$ is ample as desired.

5.2 Kloosterman sums

We now give an application of Theorem 5.1 to a classical problem in analytic number theory, namely that of estimating the *Kloosterman sums*

$$S(a,b,p) := \sum_{t=1}^{p-1} e_p\left(at + b\bar{t}\right),$$

where p is a prime number, a, b are residue classes modulo p, $\bar{t}$ denotes the inverse of t modulo p, and $e_p(x) = \exp(\frac{2\pi i}{p}x)$. We shall do so by following a method due to Weil, which 'reduces' the problem to that of estimating the number of $\mathbb{F}_{p^n}$-points on a certain hyperelliptic curve over $\mathbb{F}_p$. Before doing so, however, we need an elementary preliminary about the trace map between finite fields.

Proposition 5.12. $\{x \in \mathbb{F}_{p^n} : \mathrm{tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(x) = 0\} = \{x \in \mathbb{F}_{p^n} : \exists y \in \mathbb{F}_{p^n} \ s.t. \ x = y^p - y\}$

Proof. Both sides of the equality are $\mathbb{F}_p$-vector spaces, and the one on the left (which has dimension $n-1$, being the kernel of a nontrivial linear functional) clearly contains the one on the right. The proposition then follows from the following lemma. $\square$

Lemma 5.13. *The image of $x \mapsto x^p - x : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ has order p^{n-1}.*

We provide two proof, one elementary and one using Galois cohomology.

Proof. Observe that $x \mapsto x^p - x$ is an $\mathbb{F}_p$-linear map. Thus it suffices to determine its kernel, which is clearly $\mathbb{F}_p$, and the claim follows. $\square$

Proof. Consider the following exact sequence of $\Gamma := \mathrm{Gal}\left(\overline{\mathbb{F}_{p^n}}/\mathbb{F}_{p^n}\right)$-modules:

$$0 \to \mathbb{F}_p \to \overline{\mathbb{F}_{p^n}} \xrightarrow{x^p - x} \overline{\mathbb{F}_{p^n}} \to 0.$$

Taking Galois cohomology and observing that $H^1(\Gamma, \overline{\mathbb{F}_{p^n}}) = 0$ and

$$H^1(\Gamma, \mathbb{F}_p) = \mathrm{Hom}(\Gamma, \mathbb{F}_p) \cong \mathrm{Hom}(\hat{\mathbb{Z}}, \mathbb{F}_p) \cong \mathbb{F}_p$$

we obtain

$$0 \to \mathbb{F}_p \to \mathbb{F}_{p^n} \xrightarrow{x^p - x} \mathbb{F}_{p^n} \to \mathbb{F}_p \to 0$$

as desired. $\square$

5.2.1 Estimating the sums

It is fairly natural to consider more general Kloosterman sums, namely

$$S_{m,n}(a,b,p) := \sum_{t\in\mathbb{F}_{p^n}^\times} e_p\left(m\,\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(at+b\bar{t})\right),$$

where now $\bar{t}$ denotes the inverse in $\mathbb{F}_{p^n}$. Weil's interest in estimating the number of points on curves defined over $\mathbb{F}_p$ can be traced back to his desire to estimate these sums: to see the connection, we perform the time-honoured trick of summing over *every* variable in sight, that is, we consider

$$\begin{aligned}
\sum_m S_{m,n}(a,b,p) &= \sum_{t\in\mathbb{F}_{p^n}^\times}\sum_{m=0}^{p-1} e_p(m\,\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(at+b\bar{t}))\\
&= p\#\{t\in\mathbb{F}_{p^n}^\times : \mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(at+b\bar{t})=0\}\\
&= \#\{(t,x)\in\mathbb{F}_{p^n}^\times\times\mathbb{F}_{p^n} : at+b\bar{t}=x^p-x\}\\
&= \#\{(t,x)\in\mathbb{F}_{p^n}\times\mathbb{F}_{p^n} : at^2+b=t(x^p-x)\}\\
&= \#\{(t,x)\in\mathbb{F}_{p^n}\times\mathbb{F}_{p^n} : 4a^2t^2+4ab=4at(x^p-x)\}\\
&= \#\{(t,x)\in\mathbb{F}_{p^n}\times\mathbb{F}_{p^n} : (2at-(x^p-x))^2+4ab=(x^p-x)^2\}\\
&= \#\{(y,x)\in\mathbb{F}_{p^n}\times\mathbb{F}_{p^n} : y^2+4ab=(x^p-x)^2\}
\end{aligned}$$

Thus if we denote by C the affine curve over $\mathbb{F}_p$ defined by the equation $y^2+4ab=(x^p-x)^2$ we have obtained the identity

$$\sum_m S_{m,n}(a,b,p) = \#C(\mathbb{F}_{p^n}).$$

5.2.2 Factorisation of the L-function

We have seen that, starting from Kloosterman sums, we are naturally led to consider the affine curve $C/\mathbb{F}_p$ given by

$$y^2=(x^p-x)^2-4ab;$$

we now want to go back, namely start from this curve and somehow recover the Kloosterman sums. We do so by studying the ζ (or possibly L-)function of C,

$$\zeta_C(s)=\prod_{x\in C_0}\frac{1}{1-p^{-s[\mathbb{F}_p(x):\mathbb{F}_p]}},$$

where C_0 denotes the set of closed (schematic) points of C. We observe that there is a natural fibration

$$\begin{array}{rccc}\pi: & C & \to & Q\\ & (x,y) & \mapsto & (x^p-x,y),\end{array}$$

where $Q:\{y^2=z^2-4ab\}$. It is not hard to see that $Q\cong\mathbb{P}^1\setminus\{\text{two points}\}$, so that we can establish an isomorphism

$$\begin{array}{rccc}\alpha: & Q & \to & \mathbb{P}^1\setminus\{0,\infty\}\\ & (y,z) & \mapsto & \frac{z-y}{2a};\end{array}$$

the reason for the factor $2a$ will become apparent in due time. The inverse of α is given by

$$\begin{array}{rccc}\alpha^{-1}: & Q & \to & \mathbb{P}^1\setminus\{0,\infty\}\\ & t & \mapsto & (bt^{-1}-at, at+bt^{-1}).\end{array}$$

Now let's go back to $\zeta_C(s)$. Clearly we can factor the product 'according to the fibration', that is, we can write

$$\zeta_C(s)=\prod_{(y,z)\in Q_0}\prod_{\substack{P=(x,y)\in C_0\\ \pi(P)=t}}\frac{1}{1-p^{-s[\mathbb{F}_p(P):\mathbb{F}_p]}}$$

and using our isomorphism from Q to $\mathbb{P}^1 \setminus \{0, \infty\} \cong \mathbb{A}^1 \setminus \{0\}$ we obtain

$$\zeta_C(s) = \prod_{t \in \mathbb{A}^1 \setminus \{0\}} \prod_{\substack{P=(x,y) \in C_0 \\ \alpha(\pi(P))=t}} \frac{1}{1 - p^{-s[\mathbb{F}_p(P):\mathbb{F}_p]}}$$

Now we want to express $[\mathbb{F}_p(P) : \mathbb{F}_p]$ in terms of $[\mathbb{F}_p(t) : \mathbb{F}_p]$. Two cases arise:

1. either t is of the form $x^p - x$ for some $x \in \mathbb{F}_{p^n}$: in this case $[\mathbb{F}_p(P) : \mathbb{F}_p] = [\mathbb{F}_p(t) : \mathbb{F}_p]$, and there are precisely p points in the fibre of π over t, so that the contribution to $\zeta_C(s)$ from factors corresponding to t is $(1 - p^{-s \deg(t)})^{-p}$;

2. or t is not of this form, hence the fiber of π over t consists of a single point, of degree equal to p times the degree of t, and the contribution to $\zeta_C(s)$ is $(1 - p^{-ps \deg t})^{-1}$.

In the two cases, the contribution to $\zeta_C(s)$ coming from points P with $\alpha(\pi(P)) = t$ factors as follows:

$$\underbrace{(1 - p^{-s \deg(t)})^{-1} \cdots (1 - p^{-s \deg(t)})^{-1}}_{p \text{ times}} \quad \text{or} \quad \prod_{m=0}^{p-1} (1 - \zeta_p^m p^{-s \deg(t)})^{-1}.$$

We would like to rewrite this expression in a more uniform way, namely, as

$$\prod_{m=0}^{p-1} (1 - \chi(t)^m p^{-s \deg(t)})^{-1},$$

where $\chi(t) = 1$ if the equation $x^p - x = z$ has a solution in $\mathbb{F}_{p^n}$ (where $z = z(t)$ is given by $at + bt^{-1}$ as above) and where $\chi(t)$ is a primitive p-th root of unity otherwise.

Before taking the next step, let's pause for a moment to recall that closed points of $\mathbb{A}^1_{\mathbb{F}_p}$ are in bijection with irreducible monic polynomials in $\mathbb{F}_p[u]$, and that points of $\mathbb{A}^1_{\mathbb{F}_p} \setminus \{0\}$ are in bijection with irreducible monic polynomials *different from t*.

In order to reduce the possibilities for χ (and get manageable expressions...), we might ask that it be a *multiplicative character*

$$\chi : \mathbb{F}_p[u] \setminus \{t(u) : t(0) = 0\} \to \mu_p$$

with the property that $\chi(t) = 1$ if and only if $x^p - x = at + bt^{-1}$ can be solved in $\mathbb{F}_{p^n}$, where we identify t with an element of $\mathbb{F}_{p^n}$ in the obvious way. Recall from Proposition 5.12 that the equation in question is solvable if and only if $\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(at + bt^{-1})$ is zero, which suggests that we might take

$$\chi(t) = \exp\left(\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(at + bt^{-1})\right)$$

where again we have identified an irreducible polynomial with a corresponding element of $\mathbb{F}_{p^n}$. But this is precisely one of the summands in a Kloosterman sum[5]! We must be on the right track.

Explicitly, if t is represented by $t(u) = u^n + c_{n-1}u^{n-1} + \cdots + c_1 u + c_0$, then the trace of t is $-c_{n-1}$ and the trace of t^{-1} is $-\frac{c_1}{c_0}$, so that

$$\chi(u^n + c_{n-1}u^{n-1} + \cdots + c_1 t + c_0) = e_p(-ac_{n-1}) e_p(-b\frac{c_1}{c_0}).$$

One checks that this is indeed a multiplicative function, well-defined on all monic polynomials whose constant coefficient is nonzero.

We have thus managed to represent the ζ function of C as follows:

$$\zeta_C(s) = \prod_{t \in \mathbb{A}^1 \setminus \{0\}} \prod_{m=0}^{p-1} (1 - \chi(t)^m p^{-s \deg(t)})^{-1}$$

[5] that we recover precisely this expression is due to the normalisation factor $\frac{1}{2a}$ in the definition of α

Definition 5.14. The product

$$\prod_{t\in\mathbb{A}^1\setminus\{0\}}(1-\chi(t)^m p^{-s\deg(t)})^{-1} = \prod_{\substack{t\in\mathbb{F}_p[u]\\ \text{monic irreducible}\\ t(0)\neq 0}}(1-\chi(t)^m p^{-s\deg(t)})^{-1}$$

is called the (Dirichlet) **L-function** of the character χ^m, and is denoted by $L(\chi^m, s)$.

We have thus proven:

Theorem 5.15. *The ζ function of C admits the factorisation $\zeta_C(s) = \prod_{m=0}^{p-1} L(\chi^m, s)$.*

It remains to understand the factors $L(\chi^m, s)$:

Proposition 5.16.

$$L(\chi^m, s) = \begin{cases} \dfrac{1-p^{-s}}{1-p^{1-s}}, & \text{if } m=0 \\ 1+S_{m,1}(a,b,p)p^{-s}+p^{1-2s}, & \text{if } m\neq 0 \end{cases}$$

Proof. Denote by $\mathcal{MI}$ the set of monic irreducible polynomials in $\mathbb{F}_p[u]$ whose constant term does not vanish. Then

$$\begin{aligned}\prod_{t\in\mathcal{MI}}(1-\chi(t)^m p^{-s\deg(t)})^{-1} &= \prod_{t\in\mathcal{MI}}\sum_{r=0}^{\infty}\chi(t)^{mr}p^{-rs\deg(t)}\\ &= \prod_{t\in\mathcal{MI}}\sum_{r=0}^{\infty}\chi(t^r)^m p^{-s\deg(t^r)}\\ &= \sum_{\substack{t\in\mathbb{F}_p[u]\text{ monic}\\ t(0)\neq 0}}\chi(t)^m p^{-s\deg(t)}\\ &= \sum_{n\geq 1}\sum_{\substack{t\in\mathbb{F}_p[u]\text{ monic}\\ t(0)\neq 0\\ \deg t=n}}\chi(t)^m p^{-ns}.\end{aligned}$$

To simplify the notation, set $\chi(t)=0$ for polynomials t that vanish at 0. With this convention,

$$L(\chi^m, s) = \sum_{n\geq 0}p^{-ns}\sum_{\substack{t\in\mathbb{F}_p[u]\text{ monic}\\ \deg t=n}}\chi(t)^m.$$

We now distinguish cases according to whether $m=0$ or $m\neq 0$.

1. If $m=0$, then the previous expression becomes

$$\begin{aligned}L(1,s) &= \sum_{n\geq 0}p^{-ns}\#\{t\in\mathbb{F}_p[u] : t(0)\neq 0, \deg(t)=n, t \text{ monic}\}\\ &= 1+\sum_{n\geq 1}p^{-ns}(p-1)p^{n-1}\\ &= \frac{1-p^{-s}}{1-p^{1-s}}\end{aligned}$$

Exercise 5.17. Compute the ζ function of a point. Compute the ζ function of $\mathbb{P}^1_{\mathbb{F}_p}$. Observe that $\mathbb{A}^1\setminus\{0\} = \mathbb{P}^1\setminus\{0,\infty\}$ and compare the ζ function we just obtained with the ζ function of $\mathbb{P}^1_{\mathbb{F}_p}$. What do you notice?

2. If $m > 0$ we obtain
$$L(1,s) = \sum_{n\geq 0} p^{-ns} \sum_{\substack{t\in\mathbb{F}_p[u] \\ \deg t = n}} \chi(t)^m.$$

Let us look at the inner sum for a fixed value of n.

(a) for $n = 0$ we clearly get 1;

(b) for $n = 1$ we obtain
$$\sum_{t\in\mathbb{F}_p^\times} \chi(t)^m = \sum_{t\in\mathbb{F}_p^\times} e_p(at+bt^{-1})^m = S_{m,1}(a,b,p);$$

(c) for $n = 2$ we have
$$\begin{aligned}\sum_{c_0\in\mathbb{F}_p^\times}\sum_{c_1\in\mathbb{F}_p} e_p(-ac_1)e_p(-bc_1/c_0) &= \sum_{c_1\in\mathbb{F}_p} e_p(-ac_1)\sum_{f\in\mathbb{F}_p^\times} e_p(-bfc_1) \\ &= \sum_{c_1\in\mathbb{F}_p} e_p(-ac_1)\begin{cases}0, & \text{if } c_1\neq 0\\ p, & \text{if } c_1 = 0\end{cases} \\ &= p.\end{aligned}$$

(d) finally, for $n \geq 3$ we obtain
$$\sum_{c_0\in\mathbb{F}_p^\times}\sum_{c_1\in\mathbb{F}_p}\sum_{c_{n-1}\in\mathbb{F}_p}\sum_{c_2,\dots,c_{n-2}\in\mathbb{F}_p} e_p(-ac_{n-1})e_p(-b\frac{c_1}{c_0}) = 0$$

since the sum over c_{n-1} always vanishes.

Putting together the previous computations we have
$$L(\chi^m,s) = 1 + S_{m,1}(a,b,p)p^{-s} + p\cdot p^{-2s}$$
as claimed.

□

5.2.3 Conclusion of the proof

Our efforts have led us to the identity
$$\zeta_C(s) = \frac{1-p^{-s}}{1-p^{1-s}} \times \prod_{m=1}^{p-1}\left(1+S_{m,1}(a,b,p)p^{-s}+p^{1-2s}\right).$$

On the other hand, the Weil conjectures yield[6]
$$\zeta_C(s) = \frac{(1-p^{-s})^2\prod_{i=1}^{p-1}(1-\alpha_i p^{-s})(1-\beta_i p^{-s})}{(1-p^{-s})(1-p^{1-s})}$$
or equivalently
$$\prod_{m=1}^{p-1}\left(1+S_{m,1}(a,b,p)T+pT^2\right) = \prod_{i=1}^{p-1}(1-\alpha_i T)(1-\beta_i T)$$
which (by unique factorisation and the fact that the $S_{m,1}(a,b,p)$ are real numbers, and up to renumbering) yields $S_{m,1}(a,b,p) = -\alpha_m - \beta_m$. Since the Weil conjectures imply $|\alpha_m| = |\beta_m| = \sqrt{p}$, this establishes the bound
$$|S_{m,1}(a,b,p)| \leq 2\sqrt{p}.$$

Exercise 5.18. Prove that $S_{m,n}(a,b,p) = -\alpha_m^n - \beta_m^n$ and that therefore $|S_{m,n}(a,b,p)| \leq 2p^{n/2}$.

[6] exercise! Where does the factor $(1-p^{-s})^2$ come from?

6 Kähler differentials

We wish to give a schematic analogue of the *topological* notion of a *local isomorphism* – we have already had hints that the notion of being *Zariski locally* an isomorphism is way too strong, so we wish to find a weaker notion. We start with an example: consider

$$\begin{array}{ccc} \mathbb{C}^\times & \to & \mathbb{C}^\times \\ z & \mapsto & z^2. \end{array}$$

Topologically, this is a local isomorphism; however, it is in no way a local isomorphism in the Zariski topology: a Zariski open is all of $\mathbb{C}^\times$ except for finitely many points, so any Zariski open set contains a point with two inverse images. However, we have a different way (say, in the category of differential manifolds) to say that a map is a local isomorphism: by the inverse function theorem, it is enough to require that the differential be invertible. This is the starting point for our algebraic generalization.

Given a morphism of schemes $f : X \longrightarrow Y$ we will define the sheaf of relative differential $\Omega_{X/Y}$ on X. We start by looking at the analogous situation in the differentiable category, so assume X, Y are C^∞-manifolds and f a C^∞-map. We can define the space of relative vector fields, or vertical vector fields, in the point x, $T_x(X/Y)$ as the kernel of the differential df_x:

$$0 \to T_x(X/Y) \to T_xX \to T_{f(x)}Y$$

Dually we can define the space of relative 1-forms in the point x, $T^*_x(X/Y)$ as the cokernel of its transpose

$$T^*_{f(x)}Y \to T^*_xX \to T^*_x(X/Y) \to 0$$

so that $T^*_x(X/Y)$ is the dual of $T_x(X/Y)$.

Notice that (even in the C^∞ category) it is clear how to define these objects $T_x(X/Y)$ for a given point x, but it is *not* what could be the right definition for a global object when the dimension of $T_x(X/Y)$ depends on the point! We could consider the analogous sequences at the level of sheaves

$$\mathcal{T}_X \to f^*\mathcal{T}_Y \text{ and its dual version } f^*\mathcal{E}^1_Y \to \mathcal{E}^1_X$$

and define the sheaf of relative vector fields as the kernel of the first map and the sheaf of relative one forms as the cokernel of the second. In this case the second object is not anymore the dual of the first one and the fibers of the the sheaf constructed in this way is not always the space $T_x(X/Y)$.

We can see already in a very simple example. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then $\mathcal{T}_X = \mathcal{O}_X \frac{d}{dx}$ (the tangent sheaf) and $\Omega_X = \mathcal{O}_X dx$ (cotangent sheaf). From these sheaves we obtain

$$\begin{array}{ccc} \mathcal{O} \cong \mathcal{T}_X & \to & f^*\mathcal{T}_Y \cong \mathcal{O} \\ g(x) & \mapsto & 2xg(x) \end{array}$$

As a map of *sheaves*, $0 \to \mathcal{O} \xrightarrow{df} \mathcal{O}$ is injective; however, at the level of points, in $x = 0$ we have $df = 0$, so that we have a nontrivial kernel. Thus for tangent sheaves it is not too clear how to define the relative tangent sheaf; it will be much easier to work at the level of *co*tangent sheaves, and then take the dual. In particular, for the cotangent sheaf there *is* a nontrivial cokernel, a sheaf supported at $x = 0$. This is (part of) the motivation to work with cotangent sheaves rather than tangent ones. We start by considering the affine case.

6.1 Cotangent sheaf: affine case

Let $f : X = \mathrm{Spec}(B) \to Y = \mathrm{Spec}(A)$. We wish to define a sheaf $\Omega_{X/Y}$ on X (the sheaf of derivations) and then its dual, $TX/Y = \Omega^*_{X/Y} = \mathrm{Hom}_B((\Omega, B)$. Since we know what $\Omega_{X/Y}$ should be (derivations), it is natural to define

$$\Omega_{X/Y} = \Omega_{B/A} = \frac{\bigoplus_{b\in B} db}{(d(b_1b_2) - b_1\, db_2 - b_2\, db_1, d(b_1 + b_2) - db_1 - db_2, da \mid \forall b_1, b_2 \in B, a \in A)}$$

The reason for the relations in the quotient is that we want derivations to satisfy the Leibniz rule ($d(uv) = udv + vdu$) and we want to consider functions that come from A as constants.

We also have a universal map

$$\begin{array}{rccc} d: & B & \to & \Omega_{B/A} \\ & b & \mapsto & db \end{array}$$

which is *not* a map of B-modules, but is an A-derivation:

Definition 6.1. Let M be a B-module. A map $\partial : B \to M$ is an A-derivation if it is A-linear and if $\partial(b\beta) = b\partial\beta + \beta\partial b$.

Remark 6.2. We have

$$\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_B(\Omega_{B/A}, M),$$

with the isomorphism given by

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \ni \varphi \mapsto \varphi \circ d \in \mathrm{Der}_A(B, M).$$

As a special case of this remark we have

$$\Omega^*_{B/A} = \mathrm{Hom}_B(\Omega_{B/A}, B) = \mathrm{Der}_A(B, B),$$

which is consistent with our (C^∞) view of the tangent space.

Definition 6.3. We set $TB/A := \Omega^*_{B/A}$ and call it the **relative tangent space**. The module $\Omega_{B/A}$ is called the module of **Kähler differentials**.

Remark 6.4. As we already observed in the C^∞ case, it is not always the case that $\Omega_{B/A}$ is dual to TB/A.

6.2 Properties of $\Omega_{B/A}$

Proposition 6.5. *Let $B = A[t_1, \dots, t_n]$. Then*

$$\begin{array}{ccc} \Omega_{B/A} & \to & \bigoplus Bdt \\ f & \mapsto & \sum_{i=1}^n \frac{\partial f}{\partial t_i} dt_i \end{array}$$

is an isomorphism; in particular, $\Omega_{B/A}$ is a free B-module of rank n.

Proposition 6.6 (Localization)**.** *Let $S \subset B$ be a multiplicative subset. Then*

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A},$$

with the isomorphism being given by

$$\begin{array}{ccc} \frac{\beta db}{s} & \leftarrow\!\shortmid & \frac{\beta db}{s} \\ d\left(\frac{b}{s}\right) & \mapsto & \frac{sdb - bds}{s^2} \end{array}$$

Remark 6.7. Notice that this property is already enough to glue various $\Omega_{B/A}$ corresponding to affine subschemes of a general scheme, hence the definition of $\Omega_{B/A}$ globalises. We will see later a more intrinsic definition of the global object.

Proposition 6.8 (Base change)**.** *Consider a cartesian diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' = A' \otimes_A B, \end{array}$$

or equivalently

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

Then

$$\Omega_{X'/Y'} = \Omega_{B'/A'} = A' \otimes_A \Omega_{B/A} = B' \otimes_B \Omega_{B/A}.$$

Proposition 6.9 (Fiber product)**.** *Let $R \to A, R \to B$ be maps. One may consider $\Omega_{A/R}$ (the cotangent sheaf of A), $\Omega_{B/R}$ (the cotangent sheaf of B) and $\Omega_{A\otimes_R B/R}$ (the cotangent sheaf of "the product variety"). We have*

$$\Omega_{A/R} \otimes_R B \oplus A \otimes_R \Omega_{B/R} \cong \Omega_{A\otimes_R B/R}.$$

Proposition 6.10 (Relative case)**.** *Let $X \xrightarrow{f} Y \to S$ be maps of affine schemes; write $X = \mathrm{Spec}(B), Y = \mathrm{Spec}(A), S = \mathrm{Spec}(R)$ and consider the sequence of ring maps $R \to A \to B$. We have an exact sequence*

$$B \otimes_A \Omega_{A/R} \xrightarrow{b\otimes da \mapsto b\,da} \Omega_{B/R} \xrightarrow{db \mapsto db} \Omega_{B/A} \to 0$$

or equivalently

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Proof. Clearly $db \mapsto db$ is surjective; we want to check exactness at $\Omega_{B/R}$. The composition is certainly zero: given $b \otimes da$ we first send it to $b\,da$ and then to $b\,da$, which is zero in $\Omega_{B/A}$ by construction. Now take an element $\sum b_i\, d\beta_i$ in the kernel of $\Omega_{B/R} \to \Omega_{B/A}$. By construction, such an element is zero if and only if it can be written as

$$\sum (f_{ij}(d(g_ig_j) - g_i\, dg_j - g_j\, dg_i) + \sum f_a d_a.$$

The term $\sum (f_{ij}(d(g_ig_j) - g_i\, dg_j - g_j\, dg_i)$ is already zero in $\Omega_{B/R}$, so this element is the image of $\sum f_a \otimes da$, which proves exactness at the middle spot. □

Remark 6.11. When S is a point, this recovers our original intuition of the relative cotangent space at a point as the cokernel of the map induced by f on cotangent spaces.

Corollary 6.12. *When $B = A[t_1, \ldots, t_n]$ and A is an R-algebra we have*

$$0 \to B \otimes_A \Omega_{A/R} \to \Omega_{B/R} \to \Omega_{B/A} \to 0$$

and the sequence splits. More precisely, there is a section $\sigma : \Omega_{B/R} \to B \otimes \Omega_{A/R}$; by the universal property of Kähler differentials, giving σ amounts to giving a derivation $\partial : B \to B \otimes_A \Omega_{A/R}$, and the derivation in question is

$$\partial(at^\alpha) = t^\alpha \otimes da.$$

Proposition 6.13 (Closed immersions)**.** *Consider $X \hookrightarrow Y$ a closed immersion, that is, $B = A/I$. In this case $\Omega_{B/A}$ is certainly 0, because $da = 0$ for every $a \in A$, and on the other hand every element in B is the class of an element in A. There is an exact sequence*

$$\frac{I}{I^2} \xrightarrow{d} B \otimes_A \Omega_{A/R} \to \Omega_{B/R} \to 0.$$

Proof. Notice that I/I^2 is an A/I-module, hence a B-module. The map $\overline{d}$ is defined by passing the universal derivation $d : I \to \Omega_{A/R}$ to the quotient $\frac{\Omega_{A/R}}{I\Omega_{A/R}} = \Omega_{A/R} \otimes_A B$. One then checks easily that $d : I/I^2 \to B \otimes_A \Omega_{A/R}$ is well-defined and B-linear.

The only part that requires proof is exactness at $B \otimes_A \Omega_{A/R}$. That the composition is zero is obvious: given $b \otimes df$ we send it to $b\,df$, but here f actually means the class of f in the quotient A/I, so that (since $f \in I$) this is $b\,d0 = 0$. Conversely, take $\omega = \sum b_i\,da_i$ that maps to 0. As in the proof of the previous proposition, we have

$$\sum b_i d\overline{a_i} = 0$$

where we have denoted by $\overline{a_i}$ the class of a_i in A/I. Given the relations that define the quotient, we must have

$$\sum b_i\,d\overline{a_i} = \overline{f_{ij}}(d(\overline{g_i g_j}) - \overline{g_i}\,d\overline{g_j} - \overline{g_j}\,d\overline{g_i}) + \sum \overline{f_\rho}\,d\rho.$$

Replacing ω with

$$\omega - \sum \overline{f_{ij}} \otimes (d(g_i g_j) - g_i\,dg_j - g_j\,dg_i) - \sum f_\rho \otimes d\rho$$

we may assume that $\sum b_i d\overline{a_i}$ is 0 not just in the quotient, but already in $\bigoplus_{b\in B} B\,db$, that is,

$$\forall x \in B \qquad \sum_{\overline{a_i}=x} b_i = 0,$$

or equivalently $\sum_{\overline{a_i}=x} b_i \otimes da_i \to 0$.

Hence what we need to show is that an element $\sum_{u\in I} b_u \otimes d(a+u)$ with $\sum b_u = 0$ is in the image of $d : I/I^2 \to B \otimes_A \Omega_{A/R}$ (notice that $a+u$, for u varying in I, is the set of all possible elements which have the same class $\overline{a}$ in A/I). Now

$$\begin{aligned}
\omega &= \sum_{u\in I} b_u \otimes d(u+a) \\
&= \sum b_u \otimes da + \sum b_u \otimes du \\
&= \left(\sum b_u\right) \otimes da + \sum b_u \otimes du \\
&= \sum b_u \otimes du = d\left(\sum b_u u\right).
\end{aligned}$$

This works: indeed, $d(\sum b_u u) = \sum db_u \otimes u + \sum u db_u = \omega$ since $\sum u db_u = 0$ (since $u \in I$). $\square$

6.3 Examples

Example 6.14. Take $R = \mathbb{C}$, $B = \mathbb{C}[x,y]/(y^2 - x^3)$ and $A = \mathbb{C}[x,y]$. Let $I = (y^2 - x^3)$. We wish to describe $\Omega_{B/\mathbb{C}}$. By the properties above we obtain

$$I/I^2 \to B \otimes_A \Omega_{A/R} \to \Omega_{B/R} \to 0$$

with $\Omega_{A/R} = A\,dx \oplus A\,dy$. The exact sequence becomes

$$I/I^2 \to B\,dx \oplus B\,dy \to \Omega_{B/R} \to 0$$

with the generator $y^2 - x^3$ of I/I^2 being sent to $-3x^2\,dx + 2y\,dy$. It follows that

$$\Omega_{B/R} = \frac{B\,dx \oplus B\,dy}{2y\,dy - 3x^2\,dx}$$

More generally,

Example 6.15. Let

$$\begin{array}{ccc}
\tilde{A} = \mathbb{C}[y_1, \ldots, y_n] & \longrightarrow & \tilde{B} = \mathbb{C}[x_1, \ldots, x_m] \\
\downarrow & & \downarrow \\
A = \tilde{A}/I & \longrightarrow & B = \tilde{B}/J
\end{array}$$

and suppose that the map $\tilde{A} \to \tilde{B}$ carries y_i to f_i. This corresponds to maps

$$\begin{tikzcd} \mathbb{A}^m \arrow[r] & \mathbb{A}^n \\ X \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

and we want to compute $\Omega_{X/\mathbb{C}}, \Omega_{Y/\mathbb{C}}$ and $\Omega_{X/Y}$. As before, we obtain

$$dy_i \longrightarrow \sum \frac{\partial f_i}{\partial x_j} dx_j$$

$$\begin{tikzcd} B\,dy_1 \oplus \cdots \oplus B\,dy_n \arrow[r] \arrow[d] & B\,dx_1 \oplus \cdots \oplus B\,dx_n \arrow[d] & & \\ B \otimes_A \Omega_{Y/\mathbb{C}} \arrow[r] \arrow[d] & \Omega_{X/\mathbb{C}} \arrow[r] \arrow[d] & \Omega_{X/Y} \arrow[r] & 0 \\ 0 & 0 & & \end{tikzcd}$$

so that the map induced by f at the level of the cotangent sheaves of affine space is precisely the Jacobian matrix of f.

6.4 Caveat: finiteness properties

As a consequence of the properties recalled above we obtain the following: if B is a finitely generated A-algebra, then $\Omega_{B/A}$ is a finitely generated B-module. Indeed, if $B = A[x_1, \ldots, x_n]/I$, we have $\bigoplus Bdx_i \to \Omega_{B/A} \to 0$, so that $\Omega_{B/A}$ is finitely generated, and if furthermore B is finitely presented (ie I is finitely generated) then $\Omega_{B/A}$ is also finitely presented. We will mostly be interested in this situation. We shall stick to the following situation:

Running assumption. If $f : X \to Y$ is a map of schemes, we shall usually assume that Y is locally noetherian and that the map is of finite type.

6.5 Cotangent sheaf: global construction

We describe the affine situation in a different way, which generalizes more nicely to the global case.

Consider a map of rings $A \to B$ (corresponding to a map of schemes $X \to Y$) and consider the product $X \times_Y X$ (that is, $B \otimes_A B$). The diagonal is defined by the ideal $I = (b \otimes 1 - 1 \otimes b)$: since we are in the affine case, $\Delta : X \hookrightarrow X \times_Y X$ is a closed immersion. We have that $\frac{I}{I^2}$ is a B-module, and it measures "the ways one can move away from the diagonal, up to first order". Let's make this precise:

Proposition 6.16. *The following hold:*

1. *the map $\hat{d} : B \to \frac{I}{I^2}$ given by $db = b \otimes 1 - 1 \otimes b$ is an A-linear derivation;*

2. *$(I/I^2, \hat{d}) \cong (\Omega_{B/A}, d)$.*

Proof. We start with a remark: the B-structure on I/I^2 can be defined in two equivalent ways, namely, we can either define $b \cdot (x \otimes y)$ as $bx \otimes y$ or as $x \otimes by$. The two definitions are equivalent because the former corresponds to multiplying by $(b \otimes 1)$, while the latter corresponds to multiplying by $(1 \otimes b)$; however, the difference of these two is an element of I, hence it acts trivially on I/I^2.

1. We have $\hat{d}(b\beta) = b\beta \otimes 1 - 1 \otimes b\beta$ and

$$b\,\hat{d}\beta + \beta\, db = b(\beta \otimes 1 - 1 \otimes \beta) + \beta(b \otimes 1 - 1 \otimes b) = b\beta \otimes 1 - 1 \otimes b\beta = \hat{d}(b\beta).$$

2. It suffices to show that $(I/I^2, \hat{d})$ satisfies the correct universal property. Given a B-module M, one has an isomorphism $\mathrm{Hom}_B(I/I^2, M) \to \mathrm{Der}_A(B, M)$ given by $\varphi \mapsto \varphi \circ \hat{d}$.

$\square$

Definition 6.17. Let $f : X \to Y$ be a map of schemes. Consider the diagonal $\Delta : X \to X \times_Y X$ and the corresponding exact sequence

$$\mathcal{I} \to \mathcal{O}_{X \times X} \to \Delta_* \mathcal{O}_X$$

We define $\Omega_{X/Y}$ as $\mathcal{I}/\mathcal{I}^2$; the maps $\hat{d}$ from before glue into a universal derivation.

Remark 6.18. Let $A \longrightarrow B$ be a morphism of ring and let $I \subset B \otimes_A B$ the kernel of the muplitiplication map so that $\Omega_{B/A} \simeq I/I^2$ as explained above. We want to make some remarks on the localization $\Omega_{B/A,\mathfrak{q}}$ of the module of Kähler differential at a prime $\mathfrak{q}$ of B. Let $\mathfrak{p} = \mathfrak{q}^c$ be the contraction of q to A, Q the contraction to $B \otimes_A B$ and finally Q' the contraction to $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}$. Let also $\tilde{I}$ be the kernel of the multiplication map $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} B_\mathfrak{q} \longrightarrow B_\mathfrak{q}$. To avoid possible error we notice that

- $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}$ is not local and in particular $\tilde{I}$ is not isomorphic to I_Q;
- $(B_\mathfrak{q} \otimes_{A_\mathfrak{p}} B_\mathfrak{q})_{Q'} \simeq (B \otimes_A B)_Q$ and $I_Q \simeq \tilde{I}_{Q'}$;
- $(I/I^2)_\mathfrak{q} \simeq \tilde{I}/\tilde{I}^2 \simeq (I/I^2)_Q \simeq I_Q/I_Q^2 \simeq (\tilde{I}/\tilde{I}^2)_{Q'} \simeq \tilde{I}_{Q'}/\tilde{I}^2_{Q'}$.

6.6 Fibers

Let $\mathcal{F}$ be a sheaf on X. We have a stalk $\mathcal{F}_x$ and another closely related object, which we call the **fiber** at x, defined as

$$\mathcal{F}(x) = k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x,$$

where $k(x) = \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x}$ is as usual the residue field at x. We wish to compare (the fibers of) the cotangent space as presently defined in terms of Kähler differentials with the good old Zariski cotangent space. Concretely, we wish to compare

$$\Omega_X(x) \quad \text{with } T_x^* X = \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2};$$

we do this in the special case of $f : X \to Y$ with $A = \mathrm{Spec}(k)$ and B a k-algebra. For simplicity, let us also consider the case of closed points (thus let $\mathfrak{m}$ be a maximal ideal).

Remark 6.19. One can always reduce to this case. Indeed, the properties of $\Omega_{B/A}$ with respect to localization imply that

$$\Omega_{B/k}(x) = \frac{\Omega_{B/k,x}}{\mathfrak{m}_x \Omega_{B/k,x}} = \frac{\Omega_{B_x/k}}{\mathfrak{m}_x \Omega_{B_x/k}},$$

so we are reduced to the case of a closed point.

We have

$$\Omega_X(x) = \frac{\Omega_X}{\mathfrak{m}\Omega_X} = k(m) \otimes_B \Omega_X$$

and a sequence $k \to B \to k(\mathfrak{m}) = \frac{B}{\mathfrak{m}}$, from which we obtain

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \xrightarrow{d} \Omega_{B/k}(\mathfrak{m}) \to \Omega_{k(\mathfrak{m})/k} \to 0$$

In general, d is neither injective nor surjective. However, if the composition $k \to B \to k(\mathfrak{m})$ is an isomorphism, then d induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k}(\mathfrak{m})$. To see this, notice that on the one hand the term $\Omega_{k(\mathfrak{m})/k}$ is obviously zero (so d is surjective), while to show injectivity we explicitly construct an inverse $\varphi : \Omega_{B/k} \to \frac{\mathfrak{m}}{\mathfrak{m}^2}$, given by

$$\varphi(df) = f - f(\mathfrak{m}).$$

Notice that $f(\mathfrak{m})$ means the class of f in $B/\mathfrak{m} = k(\mathfrak{m})$; in our special case, $k(\mathfrak{m}) = k$, so we may identify $f(\mathfrak{m})$ to an element of B itself. This is what allows us to take the difference $f - f(\mathfrak{m})$ in B.

7 Unramified morphisms

7.1 Kähler differentials for an extension of fields

We study $\Omega_{E/K}$ for a field extension $K \subseteq E$; in particular, we'd like to understand when this E-vector space is 0. We shall assume that E is finitely generated over K as a field. In particular, we may consider E as being obtained from K by adding one element at a time: let us add one more generator and consider $F = E(\alpha)$.

- Suppose first that α is transcendental over E: then F is the field of fractions of the ring of polynomials $E[x]$. Recall that for $B = A[t_1, \dots, t_n]$ we have an exact sequence

$$0 \to B \otimes \Omega_{A/R} \to \Omega_{B/R} \to \Omega_{B/A} \to 0,$$

  where $B = A[t_1, \dots, t_n]$, $\Omega_{B/A} = \bigoplus B dx_i$, and there is a section $\sigma : \Omega_{B/R} \to B \otimes \Omega_{A/R}$.

  We wish to understand $\Omega_{E(x)/K}$; we first notice that, due to the previous exact sequence, $\Omega_{E[x]/K} = E[x] \otimes_E \Omega_{E/K} \oplus E[x]\, dx$. By localization we obtain

$$\Omega_{E(x)/K} = E(x) \otimes_E \Omega_{E/K} \oplus E(x)\, dx,$$

  so that the dimension of $\Omega_{E(x)/K}$ over $E(x)$ is one more than the dimension of $\Omega_{E/K}$ over E.

- Suppose now that α is algebraic, so that $F = \frac{E[x]}{(f(x))}$. We have another exact sequence

$$\left(\frac{f}{f^2}\right) \xrightarrow{d} \Omega_{E[x]/K} \to \Omega_{F/K} \to 0,$$

  and as we have already described $\Omega_{E[x]/K}$ as $E[x] \otimes_E \Omega_{E/K} \oplus E[x]\, dx$ we obtain

$$\left(\frac{f}{f^2}\right) \xrightarrow{d} E[x] \otimes_E \Omega_{E/K} \oplus E[x]\, dx \to \Omega_{F/K} \to 0,$$

  which shows that the F-dimension of $\Omega_{F/K}$ is not smaller than the E-dimension of $\Omega_{E/K}$. Notice that this sequence is fairly explicit: writing $f(x) = \sum \varepsilon_i x^i$ with $\varepsilon_i \in E$, the image of f via d in $E[x] \otimes_E \Omega_{E/K} \oplus E[x]\, dx$ is

$$\left(\sum d\varepsilon_i\, x^i, f'\right).$$

  In particular, if $f' \neq 0$, then $\dim_F \Omega_{F/K} = \dim_{E/K} \Omega_{E/K}$.

As a consequence, if we have three fields $F \supset E \supset K$, with every field finitely generated over the previous one, then $\dim_F \Omega_{F/K} \geq \dim_E \Omega_{E/K}$ and

- $\dim_E \Omega_{E/K} \geq \operatorname{trdeg}_K E$

- if $E \supset K$ is a finite separable extension we have $\Omega_{E/K} = 0$.

We shall show that the second implication is in fact an equivalence.

Proposition 7.1. *Let $\Omega_{E/K} = 0$. Then E/K is finite and separable.*

Proof. Since $\dim_E \Omega_{E/K} = 0$, from our previous analysis we deduce that E/K is finite. We consider sub-extensions as follows:

$$K \subseteq L \subseteq M \subseteq E$$

where $L = E^s$ is the maximal separable subextension, M is chosen so that E/M is primitive and totally inseparable of degree q, $E = M[\alpha]$, and $\alpha^q \in M$. We have an exact sequence

$$\Omega_{E/K} \to \Omega_{E/M} \to 0,$$

so if we prove that $\Omega_{E/M}$ is nonzero also $\Omega_{E/K}$ is nonzero. But this is easy: $E = \frac{M[x]}{x^q - \beta}$, and since the derivative of $x^q - \beta$ is zero in M w ehave that $f \mapsto f'$ is a nontrivial M-derivation from E in E, so $\Omega_{E/M}$ cannot be trivial. □

Proposition 7.2. *K a field, A a finitely generated K-algebra. Then $\Omega_{A/K} = 0$ if and only if $A = \prod_{i=1}^n L_i$ with every $L_i \supset K$ a finite separable field extension of K.*

Proof. One implication is trivial, so assume that $\Omega_{A/K} = 0$. We start by showing that A is Artinian. Let $\mathfrak{p}$ be a prime of A and let $B = A/\mathfrak{p}$. We have an exact sequence

$$\frac{\mathfrak{p}}{\mathfrak{p}^2} \to B \otimes_A \Omega_{A/K} \to \Omega_{B/K} \to 0,$$

which implies $\Omega_{B/K} = 0$. Let E be the field of fractions of B: localising at E we get $E \otimes_B \Omega_{B/K} = \Omega_{E/K} = 0$, so E/K is a finite (in particular, algebraic) separable extension of K. It follows that $\dim B = \operatorname{trdeg}_K E = 0$, so every prime ideal of A is maximal, and therefore A is Artinian. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ be the maximal ideals of A. One may reduce easily to the local (Artinian) case, so consider $(A, \mathfrak{m})$ local Artinian with $\Omega_{A/K} = 0$: we claim that $\mathfrak{m} = 0$ (hence that A is a field).

The previous argument shows that $\Omega_{\frac{A}{\mathfrak{m}}/K} = 0$, so $\frac{A}{\mathfrak{m}}/K$ is finite and separable. We may therefore write $\frac{A}{\mathfrak{m}} = \frac{K[t]}{(f)}$; recall that $\mathfrak{m}^n = 0$ for some n (we are in an Artinian ring), so A is complete. Notice that the equation $f(x) = 0$ has a solution in $A/\mathfrak{m}$ (the class of t is a solution), and furthermore $f'(t)$ is invertible in $A/\mathfrak{m}$ (this is because $f(t)$ is separable), so Hensel's lemma implies that there exists $\alpha \in A$ such that $f(\alpha) = 0$. The existence of α gives a section $E := \frac{A}{\mathfrak{m}} \to A$, so that A is a E-algebra and $\mathfrak{m}$ is an E-rational point (that is, $k(\mathfrak{m}) = E$). In this situation we have proven that we have

$$\Omega_{A/K} \to \Omega_{A/E} = \frac{\mathfrak{m}}{\mathfrak{m}^2} \to 0,$$

which – since $\Omega_{A/K} = 0$ – implies $\Omega_{A/E} = 0$. Hence $\mathfrak{m} = \mathfrak{m}^2$ and – by Nakayama's Lemma – $\mathfrak{m} = 0$, which is what we wanted to show. □

Let's go back to the following situation: A a finitely generated k-algebra and $\mathfrak{m}$ one of its maximal ideals. We had a sequence

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega_{A/k}(\mathfrak{m}) \to \Omega_{k(\mathfrak{m})/k} \to 0;$$

if $k \subset k(m)$ is separable, then $\Omega_{k(\mathfrak{m})/k} = 0$, and therefore the natural map $\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega_{A/k}(\mathfrak{m})$ is surjective. As for the injectivity on the left,

Exercise 7.3. The natural map

$$d : \frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega_{A/k}(\mathfrak{m})$$

is injective if and only if $\frac{A}{\mathfrak{m}^2} \to \frac{A}{\mathfrak{m}} = k(\mathfrak{m})$ admits a section. Furthermore, the argument used in the proof of Proposition 7.2 shows that this condition holds if $k \subseteq k(\mathfrak{m})$ is finite and separable. In particular, this shows

$$\boxed{k \subseteq k(\mathfrak{m}) \text{ finite separable extension } \Rightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \cong \Omega_{A/K}(\mathfrak{m})}$$

7.2 Jacobian criterion

Recall the following: let $A = \frac{L[x_1, \cdots, x_n]}{(f_1, \ldots, f_m)}$, let $\mathfrak{m}$ be a maximal ideal, and let $Jf = \left(\frac{\partial f_j}{\partial x_i}\right)$ be the Jacobian matrix corresponding to $(f_1, \ldots, f_m)$. Let d be the dimension of $A_{\mathfrak{m}}$. We have:

1. if $\mathfrak{m}$ is k-rational, then:
   - $\dim \frac{\mathfrak{m}}{\mathfrak{m}^2} = n - \operatorname{rank} Jf(\mathfrak{m})$;
   - $\mathfrak{m}$ is regular $\Leftrightarrow \operatorname{rank} Jf(\mathfrak{m}) = n - d$.
2. In general, $\dim_{k(\mathfrak{m})} \frac{\mathfrak{m}}{\mathfrak{m}^2} \leq n - \operatorname{rank} Jf(\mathfrak{m})$
3. If $k(\mathfrak{m}) \supset k$ is separable[7], then

$$\dim_{k(\mathfrak{m})} \frac{\mathfrak{m}}{\mathfrak{m}^2} = n - \operatorname{rank} Jf(\mathfrak{m}).$$

Proof. We have already seen[8] (1) and (2). Let's prove (3). We have

$$\dim_{k(\mathfrak{m})} \frac{\mathfrak{m}}{\mathfrak{m}^2} = \dim_{k(\mathfrak{m})} \Omega_{A/k}(\mathfrak{m}),$$

and on the other hand we have a sequence

$$\frac{(f_i)}{(f_i)^2} \to A \otimes_{k[x_1,\ldots,x_n]} \Omega_{k[x_1,\ldots,x_n]/k} \to \Omega_{A/k} \to 0;$$

tensoring with $k(\mathfrak{m})$ we obtain

$$k(\mathfrak{m}) \otimes \frac{(f_i)}{(f_i)^2} \xrightarrow{d} \bigoplus k(\mathfrak{m})\, dx_i \to \Omega_{A/k}(\mathfrak{m}) \to 0,$$

which – since $\operatorname{Im} d = \langle df_1, \ldots, df_m \rangle$ – gives

$$\dim \Omega_{A/k(\mathfrak{m})} = n - \operatorname{rank} Jf(\mathfrak{m}).$$

$\square$

7.3 Unramified morphisms

Definition 7.4. Let $f : X \to Y$ be a morphism of schemes with X, Y locally Noetherian, and locally of finite type. Then we say that

1. f is **unramified at** x if $\Omega_{X/Y,x} = 0$ or equivalently $\Omega_{X/Y}(x) = 0$;
2. f is **unramified** if it is unramified at every point of x

Definition 7.5 (Standard assumptions)**.** We say that a morphism $f : X \to Y$ of schemes satsfies the standard assumptions if X, Y are locally Noetherian and f is locally of finite type.

[7] we would need *finite* separable. However, as the algebra is finitely generated and the ideal is maximal, this is automatic from the Nullstellensatz

[8] there are no notes for that lecture: sorry!

Lemma 7.6. *In the same setting, let $y = f(x)$. Then f is unramified at x if and only if $\mathfrak{m}_Y \mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(y) \subset k(x)$ is finite and separable.*

Proof. This is clearly a local statement, so we may assume that everything in sight is affine. Write $Y = \operatorname{Spec}(A), X = \operatorname{Spec}(B)$. We may assume that $(A, \mathfrak{m} = \mathfrak{m}_y)$ is local. Indeed both claims do not change if we base change to $A \longrightarrow A_{\mathfrak{m}}$.

Assume that f is unramified. By shrinking to an open neighbourhood of x we can assume that $\Omega_{B/A} = 0$. Then by the base change $A \longrightarrow k(y)$ we obtain $\Omega_{(B/\mathfrak{m}B)/k(y)} = 0$. Hence, being $B \supset A$ finitely generated, $B/\mathfrak{m}B$ is a product of finite separable extension of $k(y)$. By localizing in x we obtain $B_x/\mathfrak{m}B_x$ is a field and it is a finite separable extension of $k(y)$. In particular $\mathfrak{m}B_x$ is maximal, hence equal to $\mathfrak{m}_x$.

Assume now that $\mathfrak{m}B_x = \mathfrak{m}_x$ and $k(y) \subset k(x)$ is separable. We have

$$\Omega_{B/A}(x) = \frac{\Omega_{B_x/A}}{\mathfrak{m}_x \Omega_{B_x/A}} = \frac{\Omega_{B_x/A}}{\mathfrak{m}\Omega_{B_x/A}} = k(\mathfrak{m}) \otimes_A \Omega B_x/A = \Omega_{(B_x/\mathfrak{m}B_x)/k(y)} = \Omega_{k(x)/k(y)} = 0.$$

Finally by Nakayama's Lemma we deduce $\Omega_{B/A,x} = 0$. □

Remark 7.7.
1. It is easy to check that unramified morphisms are stable under base change, fibre product, and composition.
2. Open immersions are unramified: $\Omega_{S^{-1}A/A} = S^{-1}\Omega_{A/A} = 0$.
3. Closed immersions are unramified: $\Omega_{\frac{A}{I}/A} = 0$.
4. Under the standard assumptions, let $f : X \to Y$ be an unramified morphism and let $y \in Y$. The fibre $X_y = \operatorname{Spec} k(y) \times_Y X$ consists of a finite number of points, and for every $x \in X_y$ the extension $k(y) \subseteq k(x)$ is finite and separable. In other words, these are not (necessarily) finite morphisms, but fibre by fibre the inverse image of a point consists of a finite number of points.

Lemma 7.8. *Let $f : X \to Y$ be unramified at x. Then $\Delta : X \to X \times_Y X$ is an open immersion in a neighbourhood of x.*

Proof. We are only considering a neighbourhood of x, so this is a local problem. Let $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$, $f : A \to B$ and

$$B \otimes_A B \xrightarrow{\Delta} B.$$

Let $I = \ker \Delta$: then $I/I^2 \cong \Omega_{B/A}$, and in particular

$$\left(\frac{I}{I^2}\right)_{\mathfrak{p}} = 0,$$

where $\mathfrak{p}$ is the prime corresponding to x. Now as I/I^2 is finitely generated and 0 at $\mathfrak{p}$, it is zero in a neighbourhood of $\mathfrak{p}$, so up to shrinking B and passing to an open set we may assume $I/I^2 = 0$, so that in particular

$$\left(\frac{I}{I^2}\right)_{\Delta(x)} = 0 \quad \forall x.$$

As I is contained in the maximal ideal $\mathfrak{m}_{\Delta(x)}$ (because $\Delta(x)$ is a point of the diagonal), the equality

$$I_{\Delta(x)} = I^2_{\Delta(x)}$$

implies

$$I_{\Delta(x)} = \mathfrak{m}_{\Delta(x)} I_{\Delta(x)},$$

which by Nakayama means $I_{\Delta(x)} = 0 \forall x$. Under our standard assumptions, this implies that there exists a neighbourhood U of $\Delta(X)$ such that $I|_U = 0$: this implies that Δ is a local isomorphism. □

7.4 Étale morphisms

Definition 7.9. Let $f : X \to Y$ be a morphism of schemes that satisfies the standard assumptions, let $x \in X$, and let $y = f(x)$. We say that f is **étale at** x if it is both unramified and flat[9] at x. We say that f is **étale** if it is étale at every point.

Closed immersions are almost never étale:

Exercise 7.10. Let A be noetherian. A ring map $A \to A/I$ is flat if and only if A can be written as $B \times C$ with $I = B \times 0$.

Lemma 7.11. *Étale morphisms are stable under base change, fibre products and composition.*

Proof. We have already noticed this for unramified morphisms, and flatness has the same properties. For example, if $A \to B$ is flat and we consider a base change

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' = A' \otimes_A B, \end{tikzcd}$$

then we need to check that $B' \otimes'_A -$ is exact. However, $B' \otimes_{A'} - = (A' \otimes_A B) \otimes_{A'} - = B \otimes_A -$, and this is exact by assumption; the case of compositions is even easier, and the case of fibre products follows. □

Exercise 7.12. Let $B = \frac{A[t]}{(f(t))}$. Then $A \to B$ is étale if and only if $(f, f') = A$, if and only if $\Omega_{B/A} = 0$.

Remark 7.13. In general, checking $\Omega_{B/A} = 0$ is easy and checking that $A \to B$ is flat is hard. This is an example where the second condition is automatic: we shall soon discuss more cases where the same happens (that is, flatness is automatic).

The next Propositions shows that étale morphisms induces isomorphism at the level of differentials.

Proposition 7.14. *Consider morphisms that satisfy the standard assumptions. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

If f is étale in x the $(f^*\Omega_{Y/S})_x \xrightarrow{\sim} \Omega_{X/S,x}$. In particular if f is étale then $f^*\Omega_{Y/S} \xrightarrow{\sim} \Omega_{X/S}$ is an isomorphism.

Remark 7.15. This is an algebraic avatar of the fact that a local isomorphism induces an isomorphism on tangent spaces.

Proof. This is a local statement, so we may assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $S = \operatorname{Spec} R$, A and B are R-algebras and f is induced by a map $\varphi : A \to B$, of R-algebras. Let $\mathfrak{q}$ be the prime corresponding to x, let $\mathfrak{p}$ its contraction to A and ℓ its contraction to R. Let also Q and Q' the contraction of $\mathfrak{q}$ to $B_\mathfrak{q} \otimes_{R_\ell} B_\mathfrak{q}$ and to $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}$.

Let $\tilde{I}_B$ be the kernel of the muplication map $B_\mathfrak{q} \otimes_{R_\ell} B_\mathfrak{q} \longrightarrow B_\mathfrak{q}$, $\tilde{I}_A$ be the kernel of the muplication map $A_\mathfrak{p} \otimes_{R_\ell} A_\mathfrak{p} \longrightarrow A_\mathfrak{p}$, and let $\tilde{J}$ be the kernel of the map $B_\mathfrak{q} \otimes_{R_\ell} B_\mathfrak{q} \longrightarrow B_\mathfrak{q} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}$.

[9] Recall that flat at x means that the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat

We have the following commutative diagram

$$\begin{tikzcd}
A_{\mathfrak{p}} \otimes_{R_\ell} A_{\mathfrak{p}} \arrow[r, two heads] \arrow[d] & A_{\mathfrak{p}} \arrow[d] \arrow[dr, "\varphi"] & \\
B_{\mathfrak{q}} \otimes_{R_\ell} B_{\mathfrak{q}} \arrow[r, two heads] \arrow[d] & B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} \arrow[r, two heads] \arrow[d] & B_{\mathfrak{q}} \arrow[d, equal] \\
(B_{\mathfrak{q}} \otimes_{R_\ell} B_{\mathfrak{q}})_Q \arrow[r, two heads] & (B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}})_{Q'} \arrow[r, two heads, "\simeq"] & B_{\mathfrak{q}}
\end{tikzcd}$$

We notice that

i) The square on the top left corner is cartesian since in general for $U \longrightarrow V \longrightarrow W$ a sequence of unitary rings we have

$$(W \otimes_U W) \otimes_{V \otimes_U V} V \simeq W \otimes_V W$$

through the maps $w_1 \otimes w_2 \otimes v \mapsto vw_1 \otimes w_2$ and $w_1 \otimes w_2 \otimes 1 \leftarrow\!\shortmid w_1 \otimes w_2$.

ii) The square on the bottom left corner is cartesian since the morphism of the last row is obtained by the one in the middle row by localizing at Q as $B_{\mathfrak{q}} \otimes_{R_\ell} B_{\mathfrak{q}}$-modules, hence by tensoring by $(B_{\mathfrak{q}} \otimes_{R_\ell} B_{\mathfrak{q}})_Q \otimes_{B_{\mathfrak{q}} \otimes_{R_\ell} B_{\mathfrak{q}}} \cdot$. Similarly for the square on the bottom right corner.

iii) The second morphism in the last row is an isomorphism since by 7.8 the map f is an open immersion in a neighborhood of x.

iv) Finally the vertical maps on the left are flat. The first one is flat becouse $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$ is flat by assumption and if $V \longrightarrow W$ is a flat morphism of U-algebras then it is flat also $V \otimes_U V \longrightarrow W \otimes_U W$. The second one is flat becouse it is a localization.

Hence by *iii*) we get $\tilde{J} = \tilde{I}_B$, and by the flatness of vertical morphisms on the left column and by i) and ii) we get

$$(B_{\mathfrak{q}} \otimes_{R_\ell} B_{\mathfrak{q}})_Q \otimes_{A_{\mathfrak{p}} \otimes_{R_\ell} A_{\mathfrak{p}}} \tilde{I}_A = \tilde{J}$$

By flatness and by iv) we get also

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \frac{\tilde{I}_A}{I_A^2} = (B_{\mathfrak{q}} \otimes_{R_\ell} B_{\mathfrak{q}})_Q \otimes_{A_{\mathfrak{p}} \otimes_{R_\ell} A_{\mathfrak{p}}} \frac{I_A^2}{I_A^2} = \frac{\tilde{J}}{\tilde{J}^2} = \frac{\tilde{I}_X}{\tilde{I}_X^2}.$$

Finally we recall from remark 6.18 that

$$\frac{\tilde{I}_A}{\tilde{I}_A^2} \simeq \Omega_{A/R,\mathfrak{p}} \qquad \text{and} \qquad \frac{\tilde{I}_B}{\tilde{I}_B^2} \simeq \Omega_{B/R,\mathfrak{q}}$$

proving the claim. □

7.5 Exercises

Exercise 7.16. Let A be a finitely generated R-algebra and let I be an ideal of A. Prove that the sequence

$$0 \longrightarrow \frac{I}{I^2} \stackrel{d}{\longrightarrow} \frac{A}{I} \otimes_A \Omega_{A/R} \longrightarrow \Omega_{\frac{A}{I}/R} \longrightarrow 0$$

is exact if and only if there exists a natural section of the projection map $A/I^2 \longrightarrow A/I$. Moreover, when the sequence is exact then it is also split.

Exercise 7.17. Let $f : A \longrightarrow B$ be a ring homomorphism. Prove that $\Omega_{B/A} = 0$ if and only if for all commutative diagrams of rings

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[r, "\pi"] & C/I \\ B \arrow[urr, "h"'] & & \end{tikzcd}$$

such that I is an ideal of C with $I^2 = 0$, there exists at most one ring homomorphism $h' : B \longrightarrow C$ such that the two triangles commutes.

Exercise 7.18. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms in our standard hypotheses. If the composition gf is étale and g is unramified then f is étale. [Describe f as the composition of the graph $\Gamma : X \longrightarrow X \times_Z Y$ and the projection $X \times_Z Y \longrightarrow Y$ and prove that these maps are étale since they are obtained by base change from étale maps.]

8 Flatness criteria

8.1 Recap

We had defined étale morphisms: a map $f : X \to Y$ is étale (at $x \in X$) if X, Y are locally noetherian, f is locally finitely presented, and

- f is flat at x;
- $\Omega_{X/Y,x} = 0$ or (equivalently by Nakayama) $\Omega_{X/Y}(x) = 0$.

We had shown a number of consequences of étaleness, among which:

Proposition 8.1. *If f is étale (in fact, even just unramified) the diagonal $\Delta : X \to X \times_Y X$ is an open immersion. If f is étale in x, the diagonal embedding is open in a neighbourhood of x.*

Proposition 8.2. *Let $f : X \to Y$ be étale at $x \in X$ and consider the diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S. & \end{tikzcd}$$

Then $f^*\Omega_{Y/S,x} \cong \Omega_{X/S,x}$; if f is étale everywhere, $f^*\Omega_{Y/S} \cong \Omega_{X/S}$.

The following was left as an exercise:

Exercise 8.3. Let $f : X \to Y$, $x \in X$. If f is étale at x, setting $y = f(x)$ we have

1. $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y}$
2. $k(x) \supset k(y)$ is finite and separable
3. $df : k(x) \otimes_{k(y)} T_y^*Y \to T_x^*X$ is an isomorphism.

Solution 8.4. The statement is plainly local, so we may assume $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$. We can also assume that A is local with maximal ideal equal to y.

By considering the base change $A \longrightarrow k(y)$ and by localizing to x we see that $k(y) \longrightarrow B_x/yB_x$ is étale and as we have already seen this implies that B_x/yB_x has dimension 0 and that $k(y) \subset k(x)$ is finite and separable. So we proved (2).

By flatness we have also

$$\dim B_x = \dim A + \dim \left(\frac{B_x}{yB_x} \right);$$

and since the second summand of the right is zero we obtain (1).

We now prove (3). Consider the composition $A \to B \to B_x$. This may fail some finite presentation conditions, but it's still unramified and flat. We may therefore assume that B is also local. For this computation we denote the maximal ideals x and y with $\mathfrak{m}_x$ and $\mathfrak{m}_y$. We have a local map $(A, \mathfrak{m}_y) \to (B, \mathfrak{m}_x)$ with $\mathfrak{m}_y B = \mathfrak{m}_x$ (this is equivalent to the map being unramified). Our claim is

$$k(x) \otimes_{k(y)} \frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \cong \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}.$$

Notice that $\mathfrak{m}_x^2 = (\mathfrak{m}_y B)^2 = \mathfrak{m}_y^2 B$. We also have an exact sequence

$$0 \to \mathfrak{m}_y^2 \to \mathfrak{m}_y \to \frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \to 0$$

which we may tensor with B to obtain a sequence (exact also on the left since $A \to B$ is flat!)

$$0 \to \mathfrak{m}_y^2 \otimes_A B \to \mathfrak{m}_y \otimes B \to \frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \otimes_A B \to 0. \tag{7}$$

If we now consider $0 \to \mathfrak{m}_y \to A$ and tensor it by B, we get an injection $\mathfrak{m}_y \otimes_A B \hookrightarrow B$ with image $\mathfrak{y}B$, which is therefore isomorphic to $\mathfrak{m}_y \otimes_A B$. We can therefore rewrite sequence (7) as

$$0 \to \mathfrak{m}_x^2 \to \mathfrak{m}_x \to \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \to 0;$$

it now suffices to note that

$$\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \cong \frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \otimes_A B \cong \frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \otimes_{A/\mathfrak{m}_y} \frac{B}{\mathfrak{m}_y B} = \frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \otimes_{k(x)} k(y).$$

Corollary 8.5. *x is regular if and only if y is regular.*

We want now to give some conditions for proving flatness. Usually proving that $\Omega_{B/A}$ vanishes is easy, while the hard part is to show that a map is flat. We would therefore like to have some criteria for flatness.

8.2 Flatness and criteria for flatness

We shall use the following facts, that we recall without proof:

1. an A-module M is flat over A if and only if $M_\mathfrak{m}$ is flat over $A_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A.

2. Given a map $A \to B$ and a B-module M, M is flat over A (and not over B!) if and only if for every maximal ideal $\mathfrak{m}$ of B the module $M_\mathfrak{m}$ is flat over $A_{\mathfrak{m}^c}$.

8.2.1 Artin-Rees

A a ring, M an A-module, I an ideal of A. Consider a filtration $\mathcal{M}$ of M,

$$M = M_0 \supset M_1 \supset \cdots \supset M_n \supset ...$$

This is called an I-filtration if for every j we have $IM_j = M_{j+1}$, and it's called I-stable if $IM_j = M_{j+1}$ for j sufficiently large.

Lemma 8.6. *Let A be noetherian and M be finitely generated. Let N be a submodule. Let $\mathcal{M}$ be an I-stable filtration of M. Then setting $N_i := M_i \cap N$ we get an I-stable filtration of N.*

Lemma 8.7. *In the same situation, let $N = \bigcap_n I^n M$. Then $IN = N$. In particular, if A is local and I is contained in the maximal ideal, $N = 0$ (Nakayama).*

8.2.2 Main criterion for flatness

We wish to generalise the following well-known statement:

Proposition 8.8. *$(A, \mathfrak{m})$ noetherian local ring, M a finitely generated A-module. Then M is flat if and only if it is free if and only if* $\operatorname{Tor}(M, A/\mathfrak{m}) = 0$.

Proof. We have a sequence

$$0 \to K \to A^n \to M \to 0$$

obtained by lifting generators of $M/\mathfrak{m}M$ to $k(\mathfrak{m})^n$. Tensoring with $A/\mathfrak{m} = k(\mathfrak{m})$ we obtain that

$$0 \to K \otimes k(\mathfrak{m}) \to k(\mathfrak{m})^n \to M/\mathfrak{m}M \to 0$$

is exact. But the surjection is an isomorphism by construction, so $K \otimes k(\mathfrak{m}) = 0$, which by Nakayama implies $K = 0$, so that $A^n \to M$ is an isomorphism. $\square$

The general theorem we will show is the following:

Theorem 8.9. $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ *a local*[10] *map of rings. Suppose A, B are noetherian and let M be a finitely generated B-module. Then M is flat over A if and only if*

$$\mathrm{Tor}(A/\mathfrak{m}, M) = 0.$$

Proof. One implication is trivial, so let us assume that $\mathrm{Tor}(A/\mathfrak{m}, M) = 0$.

Notice that a simple A-module is necessarily isomorphic to $A/\mathfrak{m}$. In particular: **First remark.** Suppose that the length of a module N is finite. Then $\mathrm{Tor}_A(N, M) = 0$. Indeed, we can proceed by induction on the length of a minimal Jordan-Hölder sequence, and since – as observed – the simple Jordan-Hölder quotients are all isomorphic to $A/\mathfrak{m}$ the claim follows immediately.

To prove flatness of M we choose to use the following criterion: M is flat if and only if for every ideal I of A the map $I \otimes_A M \to M$ is injective. Let $\tau \in I \otimes_A M$ be an element in the kernel. Notice that B acts on $I \otimes M$ by $b(x \otimes m) = x \otimes bm$. Applying Artin-Rees (lemma 8.7) it suffices to show that τ is in $\mathfrak{n}^i(I \otimes M)$ for every i, and in fact it's enough to so show that τ is in $\mathfrak{m}^i(I \otimes M)$ for every i for every i (since $f(\mathfrak{m}) \subseteq \mathfrak{n}$). Thus it's enough to prove that $\tau \in \mathfrak{m}^i I \otimes M$ (notice that the advantage of working with A is that we can act on the ideal I instead of acting on the module M).

We have two $\mathfrak{m}$-stable filtrations of I: one is given by $\mathfrak{m}^i I$ (obvious), the other by $\mathfrak{m}^h \cap I$ (Artin-Rees again, lemma 8.6). Since $\mathfrak{m}^i I$ is the largest $\mathfrak{m}$-stable filtration, for every i there exists an h such that $\mathfrak{m}^h \cap I \subset \mathfrak{m}^i I$. We are thus reduced to showing that $\tau \in (\mathfrak{m}^h \cap I) \otimes M$ for all h.

Consider the exact sequence

$$0 \to \mathfrak{m}^h \cap I \to I \to \frac{I}{\mathfrak{m}^h \cap I} \to 0$$

and tensor it with M to obtain the exact sequence

$$(\mathfrak{m}^h \cap I) \otimes M \to I \otimes M \xrightarrow{\pi_h} \frac{I}{\mathfrak{m}^h \cap I} \otimes M \to 0.$$

To prove our claim $\tau \in (\mathfrak{m}^h \cap I) \otimes M$ it is therefore enough to show that $\pi_h(\tau) = 0$ for every h. Consider

$$\begin{tikzcd}
\tau \in I \otimes M \arrow[r, "\pi_h"] \arrow[d] & \frac{I}{\mathfrak{m}^h \cap I} \otimes M \arrow[d, "\varphi"] \\
M = A \otimes M \arrow[r] & \frac{A}{\mathfrak{m}^h} \otimes M
\end{tikzcd}$$

By diagram chase, since τ maps to 0 in M by definition, it suffices to show that the vertical arrow on the right, φ, is injective. Now φ is obtained by tensoring with M the sequence

$$0 \to \frac{I}{\mathfrak{m}^h \cap I} \to \frac{A}{I} \to \frac{A}{\mathfrak{m}^h + I} \to 0,$$

and therefore it suffices to show that $\mathrm{Tor}(M, A/(\mathfrak{m}^h + I)) = 0$. But this follows from our first remark, because $A/(\mathfrak{m}^h + I)$ certainly has finite length. $\square$

8.2.3 Some further flatness criteria

Theorem 8.10. $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ *a local map of rings. Suppose A, B are noetherian and let M be a finitely generated B-module and I be an ideal of A contained in $\mathfrak{m}$. Then M is flat over A if and only if the following two conditions hold:*

- *M/IM is flat over A/I*
- $\mathrm{Tor}(M, A/I) = 0$.

[10] that is, $f(\mathfrak{m}) \subseteq \mathfrak{n}$

Proof.

($\Rightarrow$) flatness is stable by base change.

($\Leftarrow$) We want to show that $\mathrm{Tor}(M, A/\mathfrak{m})$ vanishes. Notice that we know both $\mathrm{Tor}_A(M, A/I) = 0$ (by assumption) and $\mathrm{Tor}_{A/I}\left(\frac{M}{IM}, \frac{A}{\mathfrak{m}}\right) = 0$ (since M/IM is flat over A/I).

We want to show that (under the first assumption) we have

$$\mathrm{Tor}_A(M, A/\mathfrak{m}) = \mathrm{Tor}_{A/I}(M/IM, A/\mathfrak{m}).$$

This will in fact follow from the following more general claim: if $\mathrm{Tor}_A(X, A/I) = 0$, then

$$\mathrm{Tor}_{A/I}\left(\frac{X}{IX}, Y\right) = \mathrm{Tor}_A(X, Y).$$

Here X is an A-module and Y is an A/I-module. Consider the beginning of a free resolution of X:

$$0 \to G \to F \to X \to 0$$

with F a free A-module. Tensoring by A/I we get

$$0 \to G/IG \to F/IF \to X/IX \to,$$

which is exact on the left because the 0 is $\mathrm{Tor}_A(X, A/I)$. Denote by $\overline{F}, \overline{X}$ the A/I-modules $F/IF, X/IX$ and notice that F/IF is A/I-free. We get the sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_A(X,Y) \arrow[r] & G \otimes_A Y \arrow[r] \arrow[d, "\cong"] & F \otimes_A Y \arrow[r] \arrow[d, "\cong"] & X \otimes_A Y \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Tor}_{A/I}(\overline{X},Y) \arrow[r] & \overline{G} \otimes_{A/I} Y \arrow[r] & \overline{F} \otimes_{A/I} Y \arrow[r] & \overline{X} \otimes_A Y \arrow[r] & 0
\end{tikzcd}$$

which implies $\mathrm{Tor}_A(X, Y) = \mathrm{Tor}_{A/I}(\overline{X}, Y)$ as desired. ☐

Exercise 8.11. Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local map of noetherian local rings. Let M be a finitely generated B-module, $a \in A$ not a zerodivisor in M. Then M is flat over A if and only if M/aM is flat over $A/(a)$.

We list some other criteria for flatness which can be deduced from the previous one.

Theorem 8.12. *Let $(R, \ell) \to (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be local maps of noetherian local rings. Let M be a finitely generated B-module which is flat over R. Then M is flat over A if and only if $\frac{M}{\ell M}$ is flat over $\frac{A}{\ell A}$.*

Remark 8.13. This corresponds to the geometric situation

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

in which we know that $X \to S$ are flat and we want to know something about the flatness of $X \to Y$. The statement is then: provided that the family $X \to S$ is flat, in order to decide flatness of $X \to Y$ it's enough to check what happens on the fiber.

Proof. ($\Rightarrow$) Flatness is stable by base change.

($\Leftarrow$) Let $I = \ell A$. We need to check two conditions: that M/IM is flat over A/I (true by assumption) and that $\mathrm{Tor_A}(M, A/I) = 0$. We prove directly that the multiplication map, $\beta : I \otimes M \to M$ is injective and this immediately implies that the required Tor vanishes. Consider the following composition

$$\ell \otimes_R M \xrightarrow{\alpha} I \otimes_A M \xrightarrow{\beta} M.$$

The composition $\beta \circ \alpha$ injective, since M is flat over R and the map α is surjective, by the definition of I. Hence β is injective as required.

□

Theorem 8.14. *$R \to A \to B$ noetherian, M a finitely generated B-module that is flat over R. Then M is flat over A if and only if for every prime ℓ of R the module $\frac{M}{\ell M}$ is flat over $\frac{A}{\ell A}$.*

In fact, we have a slightly more precise statement:

Theorem 8.15. *$R \to A \to B$ noetherian, M a finitely generated B-module that is flat over R. Let $\mathfrak{q}$ be a prime of B, let $\mathfrak{p}$ the contraction of $\mathfrak{q}$ to A and let ℓ be the contraction of $\mathfrak{q}$ to R. Assume that $M_\mathfrak{q}$ is flat over R_ℓ. Then $M_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ if and only if $\frac{M_\mathfrak{q}}{\ell M_\mathfrak{q}}$ is flat over $\frac{A_\mathfrak{p}}{\ell A_\mathfrak{p}}$.*

In particular, it is enough to check the condition in theorem 8.14 not for every prime, but only for those that are contractions of ideals in B.

8.3 Proving that some maps are étale

In the next lectures we will use the flatness criteria proved above to prove some criteria for étaleness. The criteria will have the following pattern: under appropriate conditions, in order to prove that a map is étale it suffices to check some condition on Kähler differentials.

To explain what we mean we begin by proving a more elementary result, for which the criteria proved above are not necessary, and that was left as an exercise (exercise 7.12). First we recall the statement:

Exercise 8.16. The natural map

$$A \to \frac{A[t]}{(f(t))}$$

is étale if and only if $(f, f') = (1)$. This is equivalent to $\Omega_{B/A} = 0$.

Proof. The equivalence of $(f, f') = 1$ and $\Omega_{B/A} = 0$ follows from the exact sequence

$$\frac{f}{f^2} \to Bdt \to \Omega_{B/A} \to 0,$$

where the map $f/f^2 \to B\,dt$ sends the class of $f = \sum a_j t^j$ to $\sum j a_j t^{j-1}$. Now $\Omega_{B/A}$ is zero iff this map is surjective, iff $(f, f') = 1$.

Let's now prove that $(f, f') = 1$ implies that $A \to B$ is étale. We start by showing that $(f, f') = 1$ implies that f is not a zerodivisor. Recall the following standard lemma:

Lemma 8.17. *Let $f, g \in A[t]$ be elements in a polynomial ring. If $gf = 0$ with $g \neq 0$, then there exists $a \neq 0$ such that $af = 0$.*

If f were a zerodivisor, there would be a scalar a such that $af = 0$ (hence also $af' = (af)' = 0$). But on the other hand we have $1 = \alpha f + \beta f'$, and multiplying by a we get $a = 0$, contradiction. We now want to show that $A \to B$ is flat, that is, $I \otimes_A B \to B$ is injective for every I ideal of A, or equivalently that $\mathrm{Tor}(B, A/I) = 0$.

We start resolving B with the obvious sequence

$$0 \to A[t] \xrightarrow{f} A[t] \to B \to 0;$$

tensoring by A/I we get

$$0 \to A/I[t] \xrightarrow{\overline{f}} A/I[t] \to B/IB \to 0.$$

Notice that $(f, f') = 1$ implies that multiplication by f is injective, and $(\overline{f}, \overline{f}') = 1$ implies that multiplication by $\overline{f}$ is injective in the quotient. By comparing the two sequences we get that $\mathrm{Tor}(B, A/I) = 0$ as claimed. $\square$

We now begin pursuing the following objective. Consider a diagram

$$\begin{tikzcd} x \in X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & s \in S & \end{tikzcd}$$

with $x \mapsto s$: we would like to deduce informations on the flatness of $X \to Y$ from the flatness of

$$\begin{tikzcd} X_s \arrow[rr] \arrow[dr] & & Y_s \arrow[dl] \\ & k(s) & \end{tikzcd}$$

Furthermore, we'd like to replace $k(s)$ with its algebraic closure.

Definition 8.18 (Faithfull flatness)**.** $f : A \to B$ is **faithfully flat** if for every pair of sequences

$$0 \to X \to Y \to Z \to 0 \tag{8}$$

$$0 \to X \otimes_A B \to Y \otimes_A B \to Z \otimes_A B \to 0 \tag{9}$$

we have that (8) is exact if and only if (9) is.

Remark 8.19. In particular, an A-module M is zero iff $M \otimes_A B$ is zero: just look at $0 \to M \to 0$.

Proposition 8.20. *The following are equivalent:*

1. $f : A \to B$ *is faithfully flat*
2. $f : A \to B$ *is flat and* $M = 0 \Leftrightarrow M \otimes B = 0$
3. $f : A \to B$ *is flat and* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective*
4. $f : A \to B$ *is flat and* $\operatorname{Im}(\operatorname{Spec} B \to \operatorname{Spec} A)$ *contains the maximal ideals of* A*.*

Definition 8.21 (Faithful flatness for schemes)**.** A map of schemes $f : X \to Y$ is **faithfully flat** if it is flat and surjective.

Lemma 8.22. *Let $S' \to S$ be faithfully flat (and suppose for simplicity S, S' affine). Consider a diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

and the corresponding diagram after base-change,

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr, "g'"'] & & Y' \arrow[dl, "h'"] \\ & S' & \end{tikzcd}$$

Then f is étale if and only if f' is.

A similar statement holds for being étale at x and x', where $\alpha(x') = x$ and α is the canonical map $X' \to X$.

Proof. As usual, the statement is local in all the data, so we can assume X, Y, S affine. Let $X = \operatorname{Spec} B, Y = \operatorname{Spec} A, S = \operatorname{Spec} R$ and A', B', R' be the corresponding rings after base change. We need to show:

1. $\Omega_{B/A} = 0$ iff $\Omega_{B'/A'} = 0$.
2. $A \to B$ flat iff $A' \to B'$ flat.

For (1), recall that $\Omega_{B'/A'} = R' \otimes_R \Omega_{B/A}$, so since R' is faithfully flat over R the claim follows.

For (2), one implication is clear (basechange of a flat map is flat). For the other implication, assume that $A' \to B'$ is flat. Take an exact sequence of A-modules

$$0 \to X \to Y \to Z \to 0.$$

Tensoring this sequence by R' over R and then by B' over A' we get another exact sequence. Since $B' \otimes_{A'} R' \otimes_R M \simeq R' \otimes_R B \otimes_A M$ for every A module M, this new exact sequence can be written as

$$0 \to R' \otimes_R B \otimes_A X \to R' \otimes_R B \otimes_A Y \to R' \otimes_R B \otimes_A Z \to 0.$$

Since R' is faithfully flat over R we deduce that

$$0 \to B \otimes_A X \to B \otimes_A Y \to B \otimes_A Z \to 0,$$

is exact, proving that $A \to B$ is flat. □

9 Étale morphisms from a 'differential' point of view

Yesterday we've proven that in order to check that f is étale in the following situation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

it suffices to do so after a faitfhully flat base change $S' \to S$. We now prove something stronger:

Lemma 9.1. *Consider a diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

with g flat. Then the following are equivalent:

1. *f is étale;*
2. *$\forall s \in S$, the induced map $f_s : X_s \to Y_s$ is étale;*
3. *for every algebraically closed field k and every $\varphi : \operatorname{Spec} k \to S$ the induced map $f_\varphi : X_\varphi \to Y_\varphi$ is étale.*

Furthermore, checking étaleness at the single point x is equivalent to checking (2) at $s = f(x)$ or (3) at $\operatorname{Spec} \overline{k(s)}$.

Proof. (2) and (3) are equivalent because a field extension is faithfully flat (so we can apply lemma 8.22). That (1) implies (2) is obvious, and so it suffices to prove that (2) implies (1). Thus we have to check that

- f is unramified at x. As usual, the question is local, so we study

$$\begin{tikzcd} X = \operatorname{Spec} B \arrow[rr, "f"] \arrow[dr, "g"'] & & Y = \operatorname{Spec}(A) \arrow[dl, "h"] \\ & S = \operatorname{Spec}(R) & \end{tikzcd}$$

  Let x correspond to a prime $\mathfrak{p}$ in B and let $\ell = \mathfrak{p} \cap R$. We need to prove $\Omega_{B/A,\mathfrak{p}} = 0$ (or $\Omega_{B/A}\cdot(\mathfrak{p}) = 0$). What we do know is that

$$\Omega_{(B/\ell)_\ell/(A/\ell)_\ell,\mathfrak{p}} = 0.$$

  But on the other hand

$$\Omega_{(B/\ell)_\ell/(A/\ell)_\ell,\mathfrak{p}} = k(\ell) \otimes_R \Omega_{B/A,\mathfrak{p}} = \left(\frac{\Omega_{B/A,\mathfrak{p}}}{\ell\Omega_{B/A,\mathfrak{p}}}\right)_\ell,$$

  while

$$\Omega_{B/A}(\mathfrak{p}) = \left(\frac{\Omega_{B/A}}{\mathfrak{p}\Omega_{B/A}}\right)_\mathfrak{p},$$

  where with respect to $\left(\frac{\Omega_{B/A,\mathfrak{p}}}{\ell\Omega_{B/A,\mathfrak{p}}}\right)_\ell$ we're quotient out by more elements and inverting more elements, so if $\left(\frac{\Omega_{B/A,\mathfrak{p}}}{\ell\Omega_{B/A,\mathfrak{p}}}\right)_\ell$ is zero then $\Omega_{B/A}(\mathfrak{p}) = \left(\frac{\Omega_{B/A}}{\mathfrak{p}\Omega_{B/A}}\right)_\mathfrak{p}$ is a fortiori zero.

- f is flat at x: this follows at once from theorem 8.15.

□

We now come to a completely differential characterisation of étale morphisms:

Theorem 9.2. *Let $f : X \to Y$ be a morphism of schemes, let $x \in X, y \in Y$ be regular points with $y = f(x)$, and assume that $k(x) \subset k(y)$ is separable. If the differential $df : k(x) \otimes_{k(y)} T_y^* Y \to T_x^* X$ is an isomorphism, then f is étale at x.*

Remark 9.3. We have already seen (exercise 8.3) that if f is étale then df is an isomorphism; this is a (partial) converse of that fact.

Proof. We need to show that f is unramified and flat.

- It suffices to prove that $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_{X,x}$. The second assumption yields

$$k(x) \otimes_{k(y)} \frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \cong \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2};$$

  if some elements y_i are in $\mathfrak{m}_y$ and there images $\overline{y_i}$ are a basis of $\mathfrak{m}_y/\mathfrak{m}_y^2$, then the isomorphism above guarantees that they are also a basis in $\mathfrak{m}_x/\mathfrak{m}_x^2$, and by Nakayama this suffices to show that the y_i generate $\mathfrak{m}_x$ over $\mathcal{O}_{X,x}$.

- Flatness. By induction on $\dim \mathcal{O}_{X,x} = d$. Notice that this dimension (by the assumption of regularity combined with the fact that df is an isomorphism) is the same as the dimension of $\mathcal{O}_{Y,y}$.

  If $d = 0$, then $\mathfrak{m}_x = \mathfrak{m}_y = 0$, $k(x) = \mathcal{O}_{X,x}$ and $k(y) = \mathcal{O}_{Y,y}$, which (being a separable field extension) is étale.

  For $d > 0$, let $y_1, \ldots, y_d \in \mathfrak{m}_y$ be elements whose classes generate $\mathfrak{m}_y/\mathfrak{m}_y^2$. Since a regular local ring is integral, y_d is not a zerodivisor in $\mathcal{O}_{Y,y}$, so $f^\#(y_d)$ is not a zerodivisor in $\mathcal{O}_{X,x}$. We are then in the following situation:

$$\mathcal{O}_{X,x} = B \longleftarrow A = \mathcal{O}_{Y,y},$$

  with an element $(y_d =)a \in A$ that is not a zerodivisor in B or in A. An exercise from yesterday (exercise 8.11) shows that B is flat over A if and only if B/aB is flat over A/aA; now we have

  - $\dim(B/aB) = \dim(A/aA) = d - 1$ (by Krull's hauptidealsatz the dimension goes down at most by one; and since A, B are domains, the dimension does go down)
  - the maximal ideal in B/aB and in A/aA is generated by $y_1, \ldots, y_{d-1}$, so the two rings are regular
  - the map induced by df is again an isomorphism (we're quotienting out by corresponding 1-dimensional subspaces in both $k(x) \otimes_{k(y)} \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ and $\mathfrak{m}_y/\mathfrak{m}_y^2$).

  We now conclude by induction.

□

Remark 9.4. If we only assume that $df(x) : k(x) \otimes_{k(y)} T_y Y^* \to T_x^* X$ is injective, then (with the same proof) we can conclude that

1. f is flat at x;
2. X_y is regular at x.

This is the algebraic version of the implicit function theorem (the assumption that x, y are regular is automatic in the usual setting of the implicit function theorem).

Now we want to reformulate the previous criterion in terms of Kähler differentials.

Lemma 9.5. *Let R be a ring, let $f : X = \mathbb{A}^n_R \to \mathbb{A}^n_R = Y$, let $p \in X$ and $q = f(p)$. Assume that the differential*

$$df : f^*\Omega_{Y/R}(p) \to \Omega_{X/R}(p)$$

is an isomorphism. Then f is étale at p, and in fact in a whole neighbourhood of p.

Proof. We start by checking flatness. We have $\Omega_{Y/R} = \bigoplus S\, dy_i$, where $S = R[y_1, \dots, y_n]$, while its pullback is $f^*\Omega_{Y/R} = \bigoplus T\, dx_i$ (with $T = R[x_1, \dots, x_n]$), and finally $\Omega_{X/R} = \bigoplus T\, dx_i$. Let $g_i \in T$ denote the image of y_i. We are then studying the map

$$\begin{array}{ccc} \bigoplus T\, dy_i & \to & \bigoplus T\, dx_i \\ y_i & \mapsto & dg_i = \sum \frac{\partial g_i}{\partial x_j} dx_j \end{array}$$

The fact that this is an isomorphism at p means that $\Delta := \det\left(\frac{\partial g_i}{\partial x_j}\right)$ is nonzero in $k(p)$, hence it is nonzero in the local ring at P, and therefore also in a neighbourhood of P. In particular, there exists an affine neighbourhood U of P such that Δ is invertible on U. Consider the diagram

$$\begin{tikzcd} U \arrow[rr, "f|_U"] \arrow[dr, "h"'] & & \mathbb{A}^n_R = Y \arrow[dl] \\ & \operatorname{Spec} R & \end{tikzcd}$$

where the structure map h is flat (since it's a localisation of the flat map $R \to R[x_1, \dots, x_n]$; notice that this is flat because the ring of polynomials is R-free). By lemma 9.1 we can therefore work fibre by fibre, and in fact we only need to look at geometric fibres. In other words, we can assume R to be an algebraically closed field. We now want want to check the flatness of $U \to \mathbb{A}^n_R$: it suffices to do so on the *closed* points of U (since it's enough to check flatness at the *maximal* ideals of the ring). We can therefore assume that p corresponds to a maximal ideal of $k[x_1, \dots, x_n]$, and under these assumptions $\Omega(p) \cong p/p^2$. We can then apply theorem 9.2 to conclude that $f|_U$ is flat.

It remains to check that f is unramified. But this is easy: there is an exact sequence

$$f^*\Omega_{Y/R} \to \Omega_{X/R} \to \Omega_{X/Y} \to 0,$$

and the first map is an isomorphism not just at p, but over all of U (since the determinant of the Jacobian matrix is invertible there). Hence $\Omega_{X/Y}$ vanishes over all of U, and we are done. □

Remark 9.6. This Lemma generalises to a more general setting, that we will be defined in the lecture, of a morphism $f : X \longrightarrow Y$ over a base scheme S that in ur case is $\operatorname{Spec} R$. We notice now that the proof depends only the following facts:

1. to define Δ we need that $\Omega_{X/S}, \Omega_{Y/S}$ are locally free of the same rank.

2. the geometric fibres of X and Y over S consist of regular points.

We will see that these properties are not unrelated and they we will guide us to the definition of a smooth point/morphism; before doing so, however, let's finish discussing étale morphisms.

9.1 Further properties of étale morphisms

Theorem 9.7. *Let*

$$\begin{tikzcd} X \arrow[rr, hook, "f"] \arrow[dr, "\alpha"'] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a diagram of morphisms of schemes in our standard hypotheses with α étale and $f : X \to Y$ a closed immersion. Let I be the ideal sheaf that defines X and consider the surjection $\frac{I}{I^2} \to f^*\Omega_{Y/S}$. This surjection is in fact an isomorphism.

Proof. As usual everything is local, so $S = \operatorname{Spec} R, X = \operatorname{Spec} B, Y = \operatorname{Spec} A$. In our setting, $B = A/I$. We proceed in stages.

1. Suppose $Y = \mathbb{A}^n_R$, $I = (t_1, \dots, t_n)$. Then the claim is obvious.

2. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, hook, "f'"] \arrow[d, "g"'] & Y' \arrow[d, "g"] \\ X \arrow[r, hook, "f"'] & Y \end{tikzcd}$$

with g étale. Suppose that the claim holds for $X \to Y$: then it also holds for $X' \to Y'$. So: we know $f^*\left(\Omega_{Y/S}\right) \cong f^*\left(\frac{I}{I^2}\right)$ and we want to prove

$$f^*\left(\Omega_{Y'/S}\right) \cong f^*\left(\frac{J}{J^2}\right).$$

Pulling back the hypothesis by g' we obtain

$$(g')^* f^*\left(\Omega_{Y/S}\right) \cong (g')^* f^*\left(\frac{I}{I^2}\right)$$

or equivalently (by commutativity of the diagram)

$$(f')^* g^*\left(\Omega_{Y/S}\right) \cong (f')^* g^*\left(\frac{I}{I^2}\right)$$

As g is étale, $g^*\Omega_{Y/S} = \Omega_{Y'/S}$ and $g^*(I/I^2) = g^*I/g^*I^2 = J/J^2$ (since the diagram is Cartesian). The claim follows.

3. Suppose $X \hookrightarrow Y = \mathbb{A}^n_R$ with $X \to \operatorname{Spec} R$ étale. In this case, $I/I^2 = B \otimes_A \Omega_{A/R}$ and we have a surjection

$$\begin{array}{ccc} I/I^2 & \twoheadrightarrow & \bigoplus B\, dt_i \\ g_i & \mapsto & dt_i \end{array}$$

The Jacobian matrix is congruent to the identity modulo the ideal I, so $\Delta := \det\left(\frac{\partial g_i}{\partial x_j}\right)$ is invertible on an open U that contains X. Morally, we'd like to show that the g_i define X: this is **not true**, but close enough to the truth to let us finish the proof. Consider the sequence

$$\begin{tikzcd} \frac{R[t_1,\dots,t_n]}{I} & \frac{R[t_1,\dots,t_n]}{(g_i)} \arrow[l] & R[t_1, \dots, t_n] \arrow[l] \\ & R \arrow[u] & R[t_1, \dots, t_n] \arrow[l] \arrow[u] \end{tikzcd}$$

The map $R[t_1, \dots, t_n] \to R$ sends t_i to 0. The vertical map $R[t_1, \dots, t_n] \to R[t_1, \dots, t_n]$ sends t_i to g_i. The square on the right is Cartesian. At the level of varieties, recalling that U is the open subset of $\mathbb{A}^n_R$ where Δ is invertible we get the following diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\alpha"'] & U \times_{\mathbb{A}^n_R} \mathbb{A}^0 \arrow[r, hook, "h"] \arrow[d, "g'"] & U \arrow[d, "g"] \\ & \mathbb{A}^0_R \arrow[r] & \mathbb{A}^n_R \end{tikzcd}$$

with the square on the right Cartesian, and the compitions on the top equal to our map f. Notice that U is precisely the open where Δ is invertible, so we may apply lemma 9.5 to conclude étaleness). By base change also g' is étale and by Exercise 7.18 also $X \hookrightarrow U \times_{\mathbb{A}^n_R} \mathbb{A}^0$ is étale. Being also a closed immersion we deduce, by Exercise 7.10 that it is an isomorphism between X and a union of connected components of $U \times_{\mathbb{A}^n_R} \mathbb{A}^0$. Now: the claim is true for the bottom map, so by the previous step is true for h (the map on top). This gives in particular

$$h^* \left(\Omega_{U/S}\right) = \frac{(g_1, \dots, g_n)}{(g_1, \dots, g_n)^2};$$

upon restriction to the connected component that is X, this is precisely what we needed to prove.

4. General case. Since Y is of finite over S we can consider a diagram of the following form

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[dr, "\alpha"'] & Y \arrow[r, hook, "g"] \arrow[d] & \mathbb{A}^n_R \arrow[dl] \\ & S & \end{tikzcd}$$

and we set $h = g \circ f$. Let $B = A/I = C/H$, $A = C/J$ where $C = R[t_1, \dots, t_n]$ and $H = I/J$. Consider the exact sequence

$$\frac{H}{H^2} \longrightarrow g^*\Omega_{\mathbb{A}^n/S} \longrightarrow \Omega_{Y/S} \longrightarrow 0$$

We can pull back this sequence using f and complete it to the following diagrams with exact rows:

$$\begin{tikzcd} & f^*\frac{H}{H^2} \arrow[r] & h^*\Omega_{\mathbb{A}^n/S} \arrow[r] & f^*\Omega_{Y/S} \arrow[r] & 0 \\ 0 \arrow[r] & \frac{J^2+H}{J^2} \arrow[r] & \frac{J}{J^2} \arrow[r] \arrow[u] & \frac{I}{I^2} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

By step 3, the vertical map $J/J^2 \cong h^*\Omega_{\mathbb{A}^n/S}$ is an isomorphism. Now observe that

$$B \otimes_A \frac{H}{H^2} = \frac{C}{J} \otimes_{C/H} \frac{H}{H^2} = \frac{C}{J} \otimes_C \frac{H}{H^2} = \frac{H/H^2}{J(H/H^2)}$$

and notice that the map $\frac{J^2+H}{J^2} \to \frac{J}{J^2}$ has image equal to the map $H \to \frac{J}{J^2}$ induced by the inclusion of H in J. Thus (by losing injectivity on the left) we get a commutative diagram

$$\begin{tikzcd} H \arrow[r] & h^*\Omega_{\mathbb{A}^n/S} \arrow[r] & f^*\Omega_{Y/S} \arrow[r] & 0 \\ H \arrow[r] \arrow[u, "\cong"] & \frac{J}{J^2} \arrow[r] \arrow[u, "\cong"] & \frac{I}{I^2} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

so the map $I/I^2 \to f^*\Omega_{Y/S}$ is an isomorphism because both objects are isomorphic to the cokernel of the same map.

□

Theorem 9.8 (Characterisation of étale morphisms by infinitesimal thickenings). *A map $A \to B$ is étale if and only if for every diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & D \arrow[r, "p"] & D/I \\ B \arrow[ur, dashed, "g"] \arrow[urr] & & \end{tikzcd}$$

with $I^2 = 0$ there exists a unique h that makes the diagram commute.

Proof. The uniqueness of h is equivalent to the fact that the map $A \longrightarrow B$ is unramified, and we leave it as an exercise.

We first show the existence of h assuming that the map $A \longrightarrow D$ is étale. Consider an exact sequence of A-modules

$$0 \to I \to C \xrightarrow{\varphi} B \to 0$$

where B, C are also A-algebras. Since $I^2 = 0$, the action of C on I descends to an action of B on I. In particular, I is a B-module. We claim that $C \cong I \oplus B$, not just as an A-module, but also as a ring, where the product on $I \oplus B$ is $(i,b)(j,\beta) = (\beta i + bj, b\beta)$.

To see this, notice that (by the previous theorem) there is an isomorphism $\delta : B \otimes \Omega_{C/A} \to I/I^2 = I$ that allows us to construct a morphism $\begin{array}{ccc} C & \to & I \\ c & \mapsto & \delta(1 \otimes dc) \end{array}$ which is a section of the inclusion $I \subset C$. Everything else is just straightforward verifications.

Now consider

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & D \arrow[r, "p"] & D/I \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "g"] & 0
\end{tikzcd}$$

where C is the set-theoretic fibre product

$$C = \{(b,d) : g(b) = p(d)\} \subset B \times D.$$

Notice that C is in fact a ring. We are in the setting of the previous remark, so $C = B \oplus I$ and we have constructed a map $B \to C \to D$ as desired (up to checking that the diagram really commutes, but it's true).

For the converse, assume that every diagram as in the statement can be completed and we prove that f is étale. As already recalled the fact that f is unramified follows from the unicity of h.

Now we show that f enjoys the following property (which we know that étale maps do possess!): given a diagram

$$\begin{tikzcd}
X = \operatorname{Spec} B \arrow[rr, hook] \arrow[dr] & & \operatorname{Spec} C \arrow[dl] \\
& \operatorname{Spec} A &
\end{tikzcd}$$

with $B = C/I$ (so the top arrow is a closed immersion) then the map induced by the differential $d : I/I^2 \longrightarrow B \otimes_C \Omega_{C/A}$ which already know to be surjective since the map is unramified. By exercise 7.16 to prove injectivity is enough to prove that the projection $p : C/I^2 \longrightarrow C/I$ has a section. The existence of this section follows from the property of map $f : A \longrightarrow B$ applied to the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & C/I^2 \arrow[r, "p"] & B = C/I \arrow[dll, equal] \\
B. & &
\end{tikzcd}$$

Since, by assumption, we are working with locally finitely presented schemes/morphisms, we may assume that $B = \frac{A[x_1,\ldots,x_n]}{I}$. So, by the property we have just proved, we have that $d : I/I^2 \longrightarrow \bigoplus B\, dx_i$ is an isomorphism. Let $g_i \in I$ be such that $dg_i = dx_i$. Now, as in the last step of the proof of Theorem 9.7, by Lemma 9.5, we have that $C = \frac{A[x_1,\ldots,x_n]}{(g_i)}$ is étale over A. Finally, we prove that $(g_1 \ldots, g_n) = I$. Indeed, consider $J = I/(g_1, \ldots, g_n)$ so that $B = C/J$. Since C is étale over A, by the property proved above we deduce $J/J^2 = 0$ which implies that $J_{\mathfrak{p}} = 0$ for all containing J. Hence $J = 0$ in a neighbourhood of $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} C$ is the union of some connected components, hence is étale over A. $\square$

Exercise 9.9. Let A be a notherian ring and let B be a finitely generated A algebra. Then B is étale over A if and only if there is an isomorphism

$$B \simeq \frac{A[t_1, \ldots, t_n]}{(f_1, \ldots, f_n)}$$

and the determinant of the Jacobian matrix $\left(\partial f_j / \partial t_i\right)_{i,j=1,\ldots,n}$ is invertible in B.

10 Smooth varieties and morphisms

We start by motivating the definition of a smooth scheme/morphism. Recall that yesterday we proved the following:

Theorem 10.1 (Cf. lemma 9.5). *Let $f : \mathbb{A}^n_R \to \mathbb{A}^n_R$ be a morphism. Suppose that $f^*\Omega \cong \Omega$: then f is étale.*

As already remarked, the only ingredients necessary for the proof of this result are the following:

- Ω is locally free (and of the same rank);
- the geometric points of the fibres of X, Y over $\operatorname{Spec} R$ are regular.

These two properties are not unrelated, and are captured by the following definition:

Definition 10.2 (Smooth variety). Let X be a variety (in the context of this definition, this simply means *scheme of finite type*) over a field k. Let $\overline{X} = X \times_k \overline{k}$. We say that $x \in X$ is **smooth** if there exists a **regular** point $\overline{x} \in \overline{X}$ that projects to x under the canonical map $\overline{X} \to X$.

Remark 10.3. Part of the motivation for this definition is the fact that over an algebraically closed field we have a very simple version of the Jacobian criterion to check for regularity. We now discuss more precisely the case of closed and non-closed points.

10.1 Smoothness of closed points

The property of x of being smooth is clearly local, so we may write $X = \operatorname{Spec} \frac{k[X_1,\dots,x_n]}{(f_1,\dots,f_m)}$. Let $d = \dim \mathcal{O}_{X,x}$, then[11] we have

$$\operatorname{rank} Jf(x) \leq n - d$$

and if equality holds then x is a regular point, moreover if x is a k rational then also the converse holds: if x is regular then $\operatorname{rank} Jf(x) = n - d$. Let now $\overline{x} \in \overline{X}$ be a point whic maps to x. Since $\overline{X} \longrightarrow X$ is an integral extension we have that $\overline{x}$ is a closed point, in particular is $\overline{k}$ rational, and $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{\overline{X},\overline{x}}$. Finally as already remarked $\operatorname{rank} Jf(x) = \operatorname{rank} Jf(\overline{x})$ and by base change $\dim_{\overline{k(\overline{x})}} \Omega_{\overline{X},\overline{x}} = \dim_{k(x)} \Omega_{X,x} = n - \operatorname{rank} Jf(x)$. Hence the condition of x to be smooth translate into:

$$x \text{ is smooth} \Leftrightarrow \operatorname{rank}_{k(x)} Jf(x) = n - d \Leftrightarrow \dim_{k(x)} \Omega_{X/k}(x) = d :$$

Notice in particular that in this case if x is smooth then it is also regular.

This also proves that, for closed points, the condition "there exists $\overline{x} \in \overline{X}$ with image x..." is equivalent to "for every $\overline{x} \in \overline{X}$ with image x...": indeed, the rank of the Jacobian does not depend on the point $\overline{x}$ chosen.

Furthermore, since $n - d$ is the maximal possible value for the rank of $Jf(x)$, the smoothness condition is open, or more precisely

$$\{x \in X : x \text{ closed and smooth}\} = \{x \in X : x \text{ closed}\} \cap \text{open set}.$$

One final equivalence: X is smooh at x if and only if $\Omega_{X/k,x}$ is free of rank d. One implication is obvious (if $\Omega_{X/k,x}$ is free of rank d, then $\dim_{k(x)} \Omega_{X/k}(x)$ has dimension d. The opposite implication is proven in the following lemma (where we take $M = \Omega$ in an open in which all closed points are smooth).

Lemma 10.4. *Let A be a noetherian domain and M a finitely generated A-module. Suppose that for every maximal ideal $\mathfrak{m}$ of A we have $\dim_{k(\mathfrak{m})} \frac{M}{\mathfrak{m}M} = d$. Then M is locally free of rank d.*

[11] as we recalled in a lecture which is not part of these notes

Proof. Fix a point $\mathfrak{m}$ and generators of $M/\mathfrak{m}M$. By Nakayama, these lift to generators $x_1, \ldots, x_d$ of $M_\mathfrak{m}$. It follows that $x_1, \ldots, x_d$ generate M in an open neighbourhood: there exists f such that M_f is generated by $x_1, \ldots, x_d$. We now prove that $x_1, \ldots, x_d$ are a basis of M_f as an A_f-module. Without loss of generality we can assume that f is 1.

Suppose that we have a nontrivial relation $\sum a_i x_i = 0$: then for every maximal ideal $\mathfrak{n}$ we have $\sum a_i(\mathfrak{n})x_i(\mathfrak{n}) = 0$; but the x_i are generators of the vector space $M/\mathfrak{n}M$, and are the correct number, so they are a basis (and in particular linearly independent). It follows that $a_i = 0$ in $A_\mathfrak{n}$ for every $\mathfrak{n}$, so $a_i = 0$. $\square$

10.2 Smoothness of not necessarily closed points

The situation is more complicated; our analysis will rely on the following nontrivial results (that we will not prove):

Theorem 10.5. *Let A be noetherian and $\mathfrak{p} \supset \mathfrak{q}$. If $A_\mathfrak{p}$ is regular, then $A_\mathfrak{q}$ is regular.*

Definition 10.6. A field extension $k \subset K$ is **separably generated** if there exists a transcendence basis $x_1, \ldots, x_n$ of K/k such that $k(x_1, \ldots, x_n) \subseteq K$ is finite separable.

Theorem 10.7. *If k is a perfect field and $k \subseteq K$ is a finitely generated (but not necessarily finite) field extension. Then K is* ***separably generated*** *over k. In particular by 7.1* $\dim_K \Omega_{K/k} = \operatorname{trdeg}_k K$.

Remark 10.8. Notice that the theorem is nontrivial: consider for example the extension $k \subseteq k(t) \subseteq k(\sqrt[q]{t})$, where $k = \overline{\mathbb{F}_q}$. Then $k(\sqrt[q]{t})/k(t)$ is certainly inseparable. But the reason is that we chose the 'wrong' transcendence basis: we should just have taken $\sqrt[q]{t}$ itself as a basis!

Theorem 10.9. *Let A be a finitely generated k-algebra and $\mathfrak{p}$ a prime of A. We have a sequence*

$$\left(\frac{\mathfrak{p}}{\mathfrak{p}^2}\right)_\mathfrak{p} \to \Omega_{A/k}(\mathfrak{p}) \to \Omega_{k(\mathfrak{p})/k} \to 0$$

If $k \subseteq k(\mathfrak{p})$ is separably generated, then the first arrow is injective.

Definition 10.10. In the following discussion the following definitions will be usefull. Let X a scheme of finite type over a field $\Bbbk$. Let $x \in X$ and let $X_1, \ldots, X_n$ be the irreducible component of X containing x. We define the dimension of X at x, and we denote it with $\dim_x X$ as the maximum of the dimensions of $X_1, \ldots, X_n$.

Moreover we say that X has pure dimension if all its irreducible component have the same dimension.

Combining these results, we obtain yet another version of the Jacobian criterion:

Theorem 10.11 (Jacobian Criterion 3)**.** *Let $S = k[x_1, \ldots, x_n] \supset I = (f_1, \ldots, f_m)$ and let $A = S/I$. Let $\mathfrak{p}$ be a prime of A and suppose that* $\operatorname{Spec} A$ *has dimension d at the prime p. The following hold:*

1. $\operatorname{rank}_{k(\mathfrak{p})} Jf(p) = n - \operatorname{rank} \Omega_{A/k}(\mathfrak{p})$
2. $\dim \Omega_{A/k}(\mathfrak{p}) \geq d$
3. *if $k(\mathfrak{p}) \supset k$ is separably generated, then*

$$\mathfrak{p} \textit{ is regular} \Leftrightarrow \dim_{k(\mathfrak{p})} \Omega_{A/k}(\mathfrak{p}) = d$$

4. *$\mathfrak{p}$ is smooth if and only if* $\dim_{k(\mathfrak{p})} \Omega_{A/k}(\mathfrak{p}) = d$.
5. *$\mathfrak{p}$ is smooth if and only if every $\overline{\mathfrak{p}}$ that maps to $\mathfrak{p}$ is regular.*

Remark 10.12. Part 3 depends on theorem 10.9 and 10.7 and part 4 depends on theorem 10.7.

Theorem 10.13. *Let X be a variety over k. Then:*

1. *the set of smooth points is open;*
2. *if $\mathfrak{p}$ is smooth, then it is regular;*
3. *x is a smooth point if and only if $\Omega_{X,x}$ is free of rank $\dim_x X$.*

Proof. We first prove (1) and (2). We study first the case k algebraically closed. In this case the second statement is trivial. In particular smooth points are contained in the set of points belonging only to one irreducible component of X. Since this set is open, this reduces to proving (1) when X is irreducible of dimension d. Under this assumption smooth points are caracterized by the fact that the rank of Jf is the maximal possible, hence they form an open subset.

We now prove part (2) in general. Let $x \in X$ be a smooth point and let $\overline{x} \longrightarrow x$. Since the set of smooth point is open in $\overline{X}$ there exists a closed smooth point $\overline{y} \in \overline{X}$ in the closure of x. Let y be the image of this point in X. Then y is a smooth closed point, in particular it is regular by the discussion we have done for closed points, and y is in the closure of x. Hence x is regular by Serre's regularity theorem (theorem 10.5). We can now complete the proof of part (1) as in the algebraically closed case.

We now prove part (3). Let x be smooth. Then by (2) there exists an integral neighbourhood of x. In particular we can assume X irreducible of dimension d, and the claim follows from the discussion we have given for closed point.

Assume now that $\Omega_{X,x}$ is free of rank equal to the dimension $d = \dim_x X$. Then $\Omega_{X,x}$ has dimension d, and the conclusion follows from part (4) of the Jacobian Criterion. □

10.3 Application to étale morphisms

By what noticed in Remark 9.6 and the discussion above we can now generalize lemma 9.5 as follows:

Proposition 10.14. *Let $f : X \to Y$ be a morphism of smooth k-varieties. Then f is étale at p if and only if $f^*\Omega_{Y/k}(p) \cong \Omega_{X/k}(p)$. Moreover if f is étale in p it is étale in a neighbourhood of p.*

Recall that the proof of that lemma 9.5 relied on an algebraic version of the inverse function theorem (theorem 9.2). Here is the version for smooth varieties which follows from Remark 9.4:

Proposition 10.15. *Let $f : X \to Y$ with $f(x) = y$. Suppose that x, y are smooth points. If the differential*

$$df : T_y^*Y \otimes_{k(y)} k(x) \hookrightarrow T_x^*X$$

is injective, then X_y is smooth at x.

Here is another connection between étale and smooth morphisms:

Theorem 10.16. *Let X be a k-variety and let x be a smooth point. There exists an open set U containing x and an étale morphism $U \to \mathbb{A}_k^n$.*

Proof. Being a regular local ring at x, the ring $\mathcal{O}_{X,x}$ is a domain. It's easy to prove that there is a neighbourhood U which is affine, irreducible, reduced, and smooth, and we can assume $X = U$. Let $X = \operatorname{Spec} A$ where $A = \frac{k[x_1,\dots,x_n]}{I}$ and $I = (f_1, \dots, f_m)$ and let d denote its dimension. By smoothness of X the module, $\Omega_{X/k}$ is projective. Hence the image M of the map

$$\frac{I}{I^2} \longrightarrow \bigoplus_{i=1}^{n} A\, dx_i \to \Omega_X \to 0,$$

is projective. By further shrinking U and by reordering the indeces we can assume that M is free of rank $c = n - d$, that $df_1, \dots, df_c$ is a basis of M and that $\det \left(\frac{\partial f_j}{\partial x_i}\right)_{i,j=1,\dots,c}$ is invertible.

We claim that the map

$$\begin{array}{ccc} A & \leftarrow & k[x_{c+1}, \dots, c_n] \\ X & \rightarrow & \operatorname{Spec} k[x_{c+1}, \dots, c_n] \end{array}$$

is étale. But for this we can use proposition 10.14. □

10.4 Smooth morphisms

As always with schemes, we like to work with relative versions of all our notions:

Definition 10.17. A morphism of schemes $f : X \to Y$ is smooth at x if and only if x is a smooth point of X_y and f is flat at x. It is smooth if it is smooth at every point.

Remark 10.18. Consider a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

Then f is smooth iff $f_s : X_s \to Y_s$ is smooth for every s, iff this holds for the geometric fibres.

Lemma 10.19 (Implicit function theorem)**.** *Let $f : X \to Y$ be a morphism of schemes. Then f is smooth at x if and only if there exists a neighbourhood U of x and a morphism $g : U \to \mathbb{A}^n_Y$ étale at x such that*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

Proof. If the factorisation exists, then f is smooth: indeed f is flat at x (composition of flat morphisms), and we need to check that X_y is smooth at x, or equivalently that Ω_{X_y} is (locally) smooth of rank n. But as g is étale at x, then it is étale at x also its restriction to the fibers over y, and by Proposition 7.14 we have that $\Omega_{X_y,x}$ is isomorphic to $g^*(\Omega_{\mathbb{A}^n_y/y})_x$, which is free of rank n.

We now assume f to be smooth at x. By theorem 10.16 we get (locally) an étale map to $\mathbb{A}^n_k(y)$. We can assume that X is affine, $X = \operatorname{Spec} A$. Localising if necessary, we get an étale map $X_y \to A^n_{k(y)}$, and we can assume that $Y = \operatorname{Spec} B$ is also affine. This corresponds to a map of rings

$$A \otimes k(y) \leftarrow k(y)[t_1, \dots, t_n].$$

By further localising A, we can assume that the images of the t_i are in A. Denote by $g_i \in A$ elements such that t_i is sent to $1 \otimes g_i$. We then have a map

$$h : B[t_1, \dots, t_n] \to A$$

which sends t_i to g_i. Upon tensorisation with $k(y)$, this recovers the map described above. If we show that g is étale we are done. We are in the situation of the following diagram:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[dr, "\text{flat at } x"'] & \mathbb{A}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

In this context, in order to prove étaleness at x it suffices to do so on the fibre over $y = f(x)$, but on the fibre this is the function $h_y : X_y \to \mathbb{A}^n_{k(y)}$ we started with, and this was étale by construction. □

This result often allows us to deduce results about smooth maps from corresponding results for étale maps. For example we can use it to prove another characterization of smooth maps which is often used as definition.

Proposition 10.20. *A morphism $f : X \to Y$ is smooth at x if and only if it is flat and $\Omega_{X/Y,x}$ is free of rank $\dim_x X_{f(x)}$.*

Proof. ($\Leftarrow$) f is flat by assumption; we need to check smoothness of the point. which is implied by $\Omega_{X_y/k(y)}(x)$ has the correct rank (point 4 of the Jacobian Criterion 10.11)

($\Rightarrow$) Locally,

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

and therefore, by Proposition 7.14 $\Omega_{X/Y,x} \cong (g^*\Omega_{\mathbb{A}^n_Y/Y})_x$, which is free of the correct dimension. Flatness is true by assumption. □

We can now give this "global version" of the Implicit function theorem 10.19. Firt we prove the following simple lemma.

Lemma 10.21. *Let M be a finitely presented A-module and let $\mathfrak{p}$ be a prime of A. Assume that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module, then there exists $a \in A$ such that M_a is a free A_a-module*

Proof. Notice first that if N is finitely generated and $\varphi : A^n \longrightarrow N$ is a morphism of A module such that $\varphi_\mathfrak{p} : A^n_\mathfrak{p} \longrightarrow N_\mathfrak{p}$ is surjective then there exists $a \in A$ such that $\varphi_a : A^n_a \longrightarrow N_a$ is surjective.

Let $\varphi : A^n \longrightarrow M$ be such that its localization in $\mathfrak{p}$ is an isomorphism, K the kernel of φ and let $A^a \longrightarrow A^b \longrightarrow M$ be a finite presentation of M. We have the following commutative diagram:

$$\begin{tikzcd} & A^a \arrow[r] \arrow[d, "\alpha"] & A^b \arrow[r] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0K \arrow[r] & A^n \arrow[r, "\varphi"] & M \arrow[r] & 0 & \end{tikzcd}$$

When we localize at $\mathfrak{p}$ the map φ is an isomorpshim, so $\beta_\mathfrak{p}$ is surjective. So localizing by an element a we can assume that β is surjective. It follows that also α is surjective. In particular K is finitely generated and $K_\mathfrak{p} = 0$. Hence there exist an element a such that $K_a = 0$ and the claim of the lemma follows. □

We can now prove the global version of the implicit function theorem.

Lemma 10.22 (Implicit function theorem, second version). *Let $f : X \to Y$ be a morphism of schemes. f is smooth at x if and only if for every x there exists a neighborhood U of x and an étale morphism $g : U \to \mathbb{A}^n_Y$ such that*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

Moreover if f is smooth at x there exist a neighborhood of x such that $\dim_y X_{f(y)}$ i constant.

Proof. If the factorisation exists, then f by Lemma 10.19 we deduce that f is smooth at every x, hence it is smooth. Moreove the dimension of fibers of f is equal to the dimension of the projection $\mathbb{A}^n_Y \longrightarrow Y$ which is n.

We now assume f to be smooth. We know by Proposition 7.14 and Lemma 10.21 that in a neighborhood V of X the sheaf $\Omega_{X/Y}$ is free, let's say of rank n. We know also that n the rank of this sheaf is equal to the dimension of the fibers V_y.

Now by 10.19 we know there esists a neighborhood $W \subset V$ and a diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[dr, "f"'] & \mathbb{A}_Y^m \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

with h étale in x. Now we have that $\Omega_{X/Y}$ is free of rank n, $f^*(\Omega_{\mathbb{A}_Y^m/Y})$ is free of rank m and the two are isomorphic in x. So $m = n$ and in a neighborhood $U \subset W$ of x we have that the two sheaves are isomorphic. Hence f is unramified. Moreover since f is flat and the projection is flat to check the flatness it is enough to check it on geometric fibers. Hence we can assume that Y is the spectrum of an algebraically closed field $\Bbbk$ the neighborhoof U is a smooth $\Bbbk$-variety of dimension n and $g : U \longrightarrow \mathbb{A}_n^k$ is a map with which is an isomorphism at the level of tangent bundles. Then g is flat and étale by Proposition 10.14. $\square$

10.5 Openness of the étale locus; standard étale morphisms

Let $f : X \to Y$ be a morphism. In some special cases (for example when X, Y are smooth, in which case the Jacobian criterion applies) we have seen that $\{x \in X : f \text{ is étale at } x\}$ is open. In fact, what is true is that the locus where f is *flat* is open; this is hard (certainly harder than the étale case), but we still record it as a fact (see stack project 36.15.1 and 10.128.4):

Theorem 10.23. *Let $f : X \to Y$ be a morphism of schemes.The locus where f is flat is open; the same holds with 'flat' replaced by 'étale'.*

In the étale case this can be deduced from the description of étale morphisms as smooth morphisms, which is simpler than this theorem, but still not easy (stack project 10.141.16).

Definition 10.24. Let A be a ring and $f(t) \in A[t]$ be a monic polynomial such that $(f, f') = 1$. Then $A \to \left(\frac{A[t]}{(f(t))}\right)_g$ is an étale map; maps of this form are called **standard étale**.

Theorem 10.25. *Let $f : A \to B$ be a morphism of ring in our standard hypotheses. Then f is étale at a prime $\mathfrak{q}$ of B if and only if there exists $g \in B \setminus \mathfrak{q}$ such that B_g is standard étale over A.*

Finally we want to briefly recall a last characterisation of étale morphisms.

Definition 10.26. Let A be a noetherian ring, $\mathfrak{p}$ an ideal of A. We can define three 'progressively more local' objects,

1. the localisation $A_\mathfrak{p}$;
2. the cone $C_\mathfrak{p}A = \bigoplus \frac{\mathfrak{p}^n}{\mathfrak{p}^{n+1}}$
3. the completion $\hat{A}_\mathfrak{p} = \varprojlim_n \frac{A}{\mathfrak{p}^n}$.

The cone gives us yet another characterisation of étale morphisms, which captures in some sense the notion of étale maps as being local isomorphisms (in terms not of tangent spaces, but of tangent cones):

Theorem 10.27. *Let $f : A \to B$ be a morphism satisfying our standard hypotheses. Let $\mathfrak{q}$ be a prime of B and let $\mathfrak{p}$ be the contraction. Suppose that $k(\mathfrak{p}) = k(\mathfrak{q})$. Then f is étale at $\mathfrak{p}$ if and only if $C_\mathfrak{p}(A) \cong C_\mathfrak{q}B$, if and only if $\hat{A}_\mathfrak{p} \cong \hat{B}_\mathfrak{q}$.*

11 Sites, sheaves and topoi

Recall that our purpose is that of *refining* the Zariski topology on a scheme so as to be able to compute a meaningful cohomology. A key insight of Grothendieck was the realisation that since one only computes cohomology *of sheaves*, it is not necessary to refine the topology *per se* (that is, one doesn't necessarily need extra open sets), but rather we need to specify the value taken by sheaves on these extra open sets. We now try to make all of this precise, starting with the categorical interpretation of (traditional) sheaves on a topological space.

11.1 From topological spaces to sites

Let X be a topological space. With X we can canonically associate a category (temporarily denoted by X_{top}) in the following way:

1. the *objects* $|X_{top}|$ of X_{top} are the open sets in X;
2. given two objects U, V of X_{top}, the set of morphisms $\mathrm{Hom}(U, V)$
   - consists of a single arrow $\iota_U : U \to V$ if $U \subseteq V$; topologically, ι_U should simply be considered as the inclusion of U into V.
   - is empty, if $U \not\subseteq V$.

Let us now recall the traditional definition of a presheaf (of sets):

Definition 11.1 (Presheaves, topological version)**.** Let X be a topological space. A **presheaf** $\mathcal{F}$ (of sets) on X is the data of the following:

1. for every open set U of X, a set $\mathcal{F}(U)$;
2. for every open subset V of an open set U, a restriction morphism $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$.

This data is supposed to satisfy the following compatibility condition: if $W \subseteq V \subseteq U$ are three open sets, then $r_{UW} = r_{VW} \circ r_{UV}$.

Remark 11.2. Given $V \subseteq U$ and $s \in \mathcal{F}(U)$, the element $r_{UV}(s) \in \mathcal{F}(V)$ is often denoted by $s|_V$.

The category attached to X allows us to recast this definition in a much more compact form:

Definition 11.3 (Presheaves, categorical version)**.** A **presheaf of sets** on X is a contravariant functor $\mathcal{F} : X_{top} \to \mathbf{Sets}$, or – equivalently – a functor $X_{top}^{\mathrm{op}} \to \mathbf{Sets}$. More generally, a presheaf with values in a category $\mathcal{C}$ is a contravariant functor $X_{top} \to \mathcal{C}$.

Indeed, the action of $\mathcal{F}$ on the *objects* of X_{top} recovers the data of the sets $\mathcal{F}(U)$, and the action on the (inclusion) *arrows* $V \hookrightarrow U$ recovers the restriction morphisms.

Even more generally, we may define presheaves on a category $\mathcal{C}$:

Definition 11.4 (Presheaves on a category)**.** Let $\mathcal{C}, \mathcal{D}$ be categories. A presheaf on $\mathcal{C}$ with values in $\mathcal{D}$ is simply a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. For fixed $\mathcal{C}, \mathcal{D}$, the $\mathcal{D}$-valued presheaves on $\mathcal{C}$ form a category; for our purposes, the category $\mathbf{PAb}(\mathcal{C})$ of presheaves of abelian groups on $\mathcal{C}$ will be especially relevant.

The definition of a presheaf in these terms suggests that – in order to "refine" the topology on X (where now X is a scheme) – what we really need to do is enrich the category X_{top}. Before doing so, however, we wish to also understand in categorical terms the more subtle notion of a sheaf. Recall the traditional definition:

Definition 11.5 (Sheaves, topological version)**.** Let X be a topological space. A **sheaf of sets** on X is a presheaf of sets $\mathcal{F}$ that further satisfies:

- (locality) let U be an open set in X and let $(U_i)_{i\in I}$ be an open cover of U. Then if $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for every $i \in I$ we have $s = t$.
- (gluing) if (U_i) is an open covering of an open set U, and if for each i an elment $s_i \in \mathcal{F}(U_i)$ is given such that for each pair U_i, U_j of covering sets the restrictions of s_i and s_j agree on the overlaps (that is, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$), then there is an element $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.

Remark 11.6. Due to the geometric origin of sheaves, elements of $\mathcal{F}(U)$ are often called **sections** of $\mathcal{F}$ over U. This is due to the fact that if $p : E \to X$ is (for example) a vector bundle, then the association

$$U \mapsto \mathcal{F}(U) = \{f : U \to E \mid f \text{ continuous}, p \circ f = \mathrm{id}_U\}$$

with the obvious restriction morphism

$$\begin{array}{rccc} r_{UV}: & \mathcal{F}(U) & \to & \mathcal{F}(V) \\ & f & \mapsto & f|_V \end{array}$$

is indeed a sheaf.

Let us try to rephrase this definition in more categorical terms: on the one hand, the locality axiom states that (for every open cover $U_i \to U$) the joint restriction morphism

$$\mathcal{F}(U) \xrightarrow{\prod_{i\in I} r_{UU_i}} \prod_i \mathcal{F}(U_i)$$

is injective; this combines with the gluing axiom to say that

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \tag{10}$$

is an **exact sequence of sets**. This requires some explanation. Let $\pi_i : \prod_i \mathcal{F}(U_i) \to \mathcal{F}(U_i)$ be the canonical projection on the factor indexed by i.

The two arrows $\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i\cap U_j)$ are defined as follows: to give a map to $\prod_{i,j} \mathcal{F}(U_i\cap U_j)$ is to give a map to each $\mathcal{F}(U_i \cap U_j)$. This can be done in two ways: either we take the map to $\mathcal{F}(U_i \cap U_j)$ to be $r_{U_i,U_i\cap U_j} \circ \pi_i$, or we take it to be $r_{U_j,U_i\cap U_j} \circ \pi_j$. In particular, given a collection $(s_i) \in \prod_i \mathcal{F}(U_i)$, the two images of (s_i) in $\prod_{i,j} \mathcal{F}(U_i \cap U_j)$ are the same if and only if for every i, j we have $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, that is, if and only if the collection satisfies the gluing condition. By the gluing axiom, this implies that there is some $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. The locality axiom implies that this s is unique.

In categorical terms, this says that the first arrow in sequence (10) is an **equaliser** in the category of sets.

Definition 11.7 (Equaliser)**.** Let $\mathcal{C}$ be a category, let X, Y be objects of $\mathcal{C}$ and let f, g be morphisms from X to Y. The **equaliser** of f and g consists of an object E and a morphism $eq : E \to X$ satisfying

$$f \circ eq = g \circ eq$$

and such that, given any object O and morphism $m : O \to X$, if $f \circ m = g \circ m$, then there exists a unique morphism $u : O \to E$ such that $eq \circ u = m$. In other words, the equaliser (if it exists) is the limit of the diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y.$$

We now wish to further explore the categorical nature of sheaves. In order to do so, it is important to realise that (topological) intersections correspond to category-theoretic products:

Lemma 11.8. *Let X be a topological space and let U_1, U_2 be open sets in X. Then the categorical product $U_1 \times U_2$ in X_{top} exists and is given by the intersection $U_1 \cap U_2$ together with its two obvious morphisms to U_1, U_2.*

Proof. Let $V \in |X_{top}|$ be an open set in X and let $\varphi_1 : V \to U_1, \varphi_2 : V \to U_2$ be two morphisms. The very existence of these morphisms implies by definition that (topologically) we have $V \subseteq U_1$ and $V \subseteq U_2$, which clearly gives $V \subseteq U_1 \cap U_2$. By the definition of the morphisms in X_{top} this means that there is a unique arrow $V \to U_1 \cap U_2$. Furthermore, the composition $V \to U_1 \cap U_2 \to U_i$ is certainly equal to φ_i, because φ_i is the *unique* morphism $V \to U_i$. $\square$

With this observation at hand, sequence (10) becomes

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times U_j),$$

so that the sheaf axioms are all defined categorically (they simply boil down to the fact that this sequence is an equaliser) except for the notion of an open cover $(U_i)_{i \in I}$ of U. The idea behind Grothendieck (pre)toplogies is then to also axiomatize in categorical terms the notion of an open cover:

Definition 11.9 (Grothendieck pretopology)**.** Let $\mathcal{C}$ be a small category with fibred products. A **Grothendieck pretopology** on $\mathcal{C}$ is the data, for each object $U \in \mathcal{C}$, of a collection of **covering families** (or simply **coverings**). Each covering family is a set of morphisms $U_i \to U$, indexed by an arbitrary set of indices I. The set of all covering families of U is denoted by $\mathrm{Cov}(U)$. These covering families should satisfy the following axioms:

(PT1) If $f : V \to U$ is an isomorphism, then $\{f\}$ is a covering family of U.

(PT2) If $(U_i \to U)_{i \in I}$ is a covering of U, and if $g : V \to U$ is any morphism, then $(V \times_U U_i \to V)_{i \in I}$ is a covering of V.

(PT3) If $(U_i \to U)_{i \in I}$ is a cover of U and, for every $i \in I$, $(U_{ij} \to U_i)_{j \in J_i}$ is a covering of U_i, then $(U_{ij} \to U_i \to U)_{i,j}$ is a covering of U.

Remark 11.10. These three axioms are modelled over the topological notion of a covering. Axiom (PT1) corresponds to the idea that an open set covers itself. Axiom (PT2) encodes the idea that if some open sets U_i cover an open set U, then for every subset V of U the sets $U_i \cap V$ cover V: notice that in the categories obtained from topological spaces the only morphisms are inclusions and fibre products are simply intersections, so *in these categories* (PT2) gives back precisely the topological notion. Finally, (PT3) corresponds to the topological idea that if we have a family U_i of open sets covering a topological space, and we then cover each U_i with smaller open sets U_{ij}, then the collection of all the U_{ij} covers U.

Of these three axioms, in a sense the least intuitive is (PT1), because we are not just allowing the *identity* as a covering, but any isomorphism. This axiom is sometimes replaced by a weaker one, namely that $\{\mathrm{id}_U\}$ is a covering of U for every object U (axiom (PT1')). When coverings are used to generate Grothendieck topologies, the difference between (PT1) and (PT1') disappears.

Continuing with our analogy with the topological world, a Grothendieck pretopology can be considered as somewhat similar to the choice of a basis of a topology: we are declaring *some specific* collections of open sets to be coverings. Just like the case of traditional topology, there is a (more intrinsic) notion of Grothendieck **topology**, and very different choices of a basis (ie, a pretopology) can lead to the same topology, and there is no easy criterion (in the language of pretopologies) to decide whether two different pretopologies lead to the same topology.

We will not need Grothendieck topologies, and we will formulate all our statements in the language of coverings – which has the advantage of being closer to our geometric intuition of how sheaves and coverings should behave.

We are now ready for the fully categorical definition of sheaves; before giving it, we introduce the notion of site:

Definition 11.11 (Site)**.** A **site** is a small[12] category $\mathcal{C}$ equipped with a Grothendieck pretopology.

Example 11.12. Let X be a topological space and let X_{top} be the associated category. We may turn X_{top} into a site by declaring that a collection $(U_i \xrightarrow{f_i} U)_{i\in I}$ is a covering of U if and only if $\bigcup_{i\in I} f_i(U_i) = U$, where this equality is to be read at the level of sets (that is: we identify the categorical morphism $f_i : U_i \to U$ with the set-theoretic inclusion $U_i \hookrightarrow U$).

Definition 11.13 (Sheaves, categorical version)**.** Let $\mathcal{C}$ be a site. A **sheaf of sets** on $\mathcal{C}$ is a presheaf $\mathcal{F} : \mathcal{C}^{\text{op}} \to \mathbf{Sets}$ such that, for every $U \in \mathcal{C}$ and for every covering $(U_i \xrightarrow{f_i} U)_{i\in I}$ of U, the first arrow in the diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times U_j),$$

is an equaliser. More generally, a **separated presheaf** on $\mathcal{C}$ is a presheaf on $\mathcal{C}$ such that the first arrow in the above sequence is injective (a monomorphism in the category of sets).

We may also define sheaves of abelian groups, rings, R-modules, etc, by simply replacing **Sets** with $\mathbf{AbGrp}, \mathbf{Rings}, \mathbf{R-Mod}$, etc, in the previous definition.

Remark 11.14. Presheaves on a site $\mathcal{C}$ form a category, denoted by $\mathbf{PSh}(\mathcal{C})$. Sheaves on $\mathcal{C}$ are then a full subcategory of $\mathbf{PSh}(\mathcal{C})$: this category is denoted by $\mathbf{Sh}(\mathcal{C})$ and called the **topos** of $\mathcal{C}$. We shall also be interested in the category $\mathbf{Ab}(\mathcal{C})$ of abelian sheaves on $\mathcal{C}$.

Sheaves on a site $\mathcal{C}$ form a category, called the **topos** corresponding to $\mathcal{C}$. The notion of a morphism of (pre)sheaves becomes that of a natural transformation of functors:

Definition 11.15 (Morphism of presheaves)**.** Let $\mathcal{F}, \mathcal{G}$ be presheaves on a site $\mathcal{C}$. A **morphism of presheaves** $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation $\varphi : \mathcal{F} \Rightarrow \mathcal{G}$, that is, a collection of maps $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for any morphism $f : U \to V$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_V"'] & \mathcal{F}(U) \arrow[d, "\varphi_U"] \\ \mathcal{G}(V) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(U) \end{tikzcd}$$

commutes (recall that $\mathcal{F}$ is a contravariant functor!). A morphism of sheaves is simply a morphism of the corresponding presheaves.

From the fact that the category of abelian groups is abelian, one obtains easily that the category $\mathbf{PAb}(\mathcal{C})$ is also abelian. Given a map of presheaves $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$, the kernel of φ is the abelian presheaf $U \mapsto \ker(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$ and the cokernel of φ is the presheaf $U \mapsto \operatorname{Coker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$.

11.2 Sheafification; the category $\mathbf{Ab}(\mathcal{C})$ is abelian

Let $\mathcal{C}$ be a site. In this section we construct a sheafification function $\mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C})$ and use it to show that the category of abelian sheaves on $\mathcal{C}$ is an abelian category.

We start by recalling (without proof) that limits exist in the category of sheaves:

Lemma 11.16. *Limits in the category of sheaves exist and coincide with the corresponding limits in the category of presheaves.*

[12]that is, objects and morphisms in the category form a set (as opposed to a proper class)

11.2.1 Morphisms of coverings

Let $\mathcal{C}$ be a site, let U, V be objects in $\mathcal{C}$, and let $\mathcal{U} = (U_i \xrightarrow{f_i} U)_{i \in I}$ and $\mathcal{V} = (V_j \xrightarrow{g_j} V)_{j \in J}$ be coverings of U, V respectively. A **morphism of coverings** is the data of

1. a morphism $h : U \to V$;
2. a function $\alpha : I \to J$;
3. morphisms $U_i \xrightarrow{h_i} V_{\alpha(i)}$ such that all the diagrams

$$\begin{tikzcd} U_i \arrow[r, "h_i"] \arrow[d, "f_i"'] & V_{\alpha(i)} \arrow[d, "g_{\alpha(i)}"] \\ U \arrow[r, "h"'] & V \end{tikzcd}$$

   commute.

When $U = V$ and $h = \mathrm{id}$, we say that the morphism is a **refinement** and that $\mathcal{U}$ **refines** $\mathcal{V}$.

11.2.2 The $\mathcal{F}^+$ construction

In order to construct the sheafification functor, we begin by introducing the first Čech cohomology group of a presheaf *with respect to a covering*:

Definition 11.17. Let $\mathcal{U} = (U_i \to U)_{i \in I}$ be a covering of U. For any presheaf $\mathcal{F}$, we set

$$H^0(\mathcal{U}, \mathcal{F}) = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}\}.$$

As in the construction of the usual Čech cohomology we now wish to define $H^0(U, \mathcal{F})$ by passing to the limit over all coverings of U. Before doing so, we make sure that the limit we're taking is well-behaved:

Lemma 11.18. *The following hold:*

1. *Any two coverings $\mathcal{U} = (U_i \to U)_{i \in I}, \mathcal{V} = (V_j \to U)_{j \in J}$ of U admit a common refinement.*
2. *Let $\mathcal{U} = (U_i \to U)_{i \in I}$ and $\mathcal{V} = (V_j \to V)_{j \in J}$ be coverings and let $h : \mathcal{U} \to \mathcal{V}$ be a morphism. Then h induces $h^* : H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$.*
3. *Let $\mathcal{U}, \mathcal{V}$ be as above. Let $f, g : \mathcal{U} \to \mathcal{V}$ be two morphisms that induce the same map $U \to V$: then the induced morphisms f^*, g^* agree.*

Proof. 1. Guided by our topological intuition, we would like to take as common refinement of $\mathcal{U}, \mathcal{V}$ the set of all pairwise intersections $U_i \cap V_j$. Given that, as we have already seen, *intersections* are replaced by *fibre products* in the categorical language, we try to take $\mathcal{W} = \{U_i \times_U V_j \to U\}_{(i,j) \in I \times J}$. In order to show that $\mathcal{W}$ is a refinement of $\mathcal{U}$ (the argument is the same for $\mathcal{V}$), we need to describe:

   (a) a map $\alpha : I \times J \to I$, which we take to be the canonical projection;

   (b) maps $U_i \times_U V_j \to U_{\alpha(i,j)} = U_i$, which again we take to be the canonical projections.

   It is clear that the relevant diagrams commute; we only need to show that $\mathcal{W}$ is in fact a covering. By axiom (PT2), for fixed i the morphisms $(U_i \times_U V_j \to U_i \times_U U = U_i)_j$ form a covering of U_i, and by axiom (PT3) since $(U_i \to U)_{i \in I}$ is a covering we obtain that $\mathcal{W} = \{V_j \times_U U_i \to U\}_{i,j}$ is indeed a covering.

2. Given $s = (s_j) \in H^0(\mathcal{V}, \mathcal{F})$ the only possible definition is to set h^*s to be the collection whose i-th component is
$$(h^*s)_i := \mathcal{F}(h_i)(s_{\alpha(i)}).$$
In order to check that this is well-defined, we need to prove that h^*s satisfies the compatibility conditions in the definition of $H^0(\mathcal{U}, \mathcal{F})$. These amount to saying that
$$(h^*s)_i|_{U_i \times_U U_{i'}} = (h^*s)_{i'}|_{U_i \times_U U_j}$$
holds for every i, i', that is,
$$\mathcal{F}(h_i)(s_{\alpha(i)})|_{U_i \times_U U_{i'}} = \mathcal{F}(h_{i'})(s_{\alpha(i')})|_{U_i \times_U U_{i'}}.$$
Recalling that 'restricting from U_i to $U_i \times_U U_{i'}$' really means applying $\mathcal{F}(\pi_i)$, where π_i is the canonical projection $\pi_i : U_i \times_U U_{i'} \to U_i$, we see that we need to check
$$\mathcal{F}(\pi_i)\mathcal{F}(h_i)(s_{\alpha(i)}) = \mathcal{F}(\pi_{i'})\mathcal{F}(h_{i'})(s_{\alpha(i')}),$$
or equivalently
$$\mathcal{F}(h_i \circ \pi_i)(s_{\alpha(i)}) = \mathcal{F}(h_{i'} \circ \pi_{i'})(s_{\alpha(i')}).$$
On the other hand, it is easy to check that we have a commutative diagram
$$\begin{tikzcd}
& U_i \arrow[rr, "h_i"] & & V_{\alpha(i)} \\
U_i \times_U U_{i'} \arrow[ur, "\pi_i"] \arrow[rr, "\varphi"'] \arrow[dr, "\pi_{i'}"'] & & V_{\alpha(i)} \times_V V_{\alpha(i')} \arrow[ur, "\tilde{\pi}_{\alpha(i)}"'] \arrow[dr, "\tilde{\pi}_{\alpha(i')}"] & \\
& U_{i'} \arrow[rr, "h_{i'}"'] & & V_{\alpha(i')}
\end{tikzcd},$$
so that the equality we need to check becomes
$$\mathcal{F}(\tilde{\pi}_{\alpha(i)} \circ \varphi)(s_{\alpha(i)}) = \mathcal{F}(\tilde{\pi}_{\alpha(i')} \circ \varphi)(s_{\alpha(i')}) \Leftrightarrow \mathcal{F}(\varphi)\mathcal{F}(\tilde{\pi}_{\alpha(i)})(s_{\alpha(i)}) = \mathcal{F}(\varphi)\mathcal{F}(\tilde{\pi}_{\alpha(i')})(s_{\alpha(i')}),$$
which holds since
$$\mathcal{F}(\tilde{\pi}_{\alpha(i)})(s_{\alpha(i)}) = s_{\alpha(i)}|_{V_{\alpha(i)} \times_V V_{\alpha(i')}} = s_{\alpha(i')}|_{V_{\alpha(i)} \times_V V_{\alpha(i')}} = \mathcal{F}(\tilde{\pi}_{\alpha(i')})(s_{\alpha(i')}).$$

3. By definition, for every $i \in I$ we have a commutative diagram
$$\begin{tikzcd}
& V_{\alpha(i)} \arrow[dr] & \\
U_i \arrow[ur, "f_i"] \arrow[dr, "g_i"'] & & V \\
& V_{\beta(i)} \arrow[ur] &
\end{tikzcd}$$
which, by the universal property of the fibre product, induces
$$\begin{tikzcd}
& V_{\alpha(i)} \\
U_i \arrow[ur, "f_i"] \arrow[r, "\varphi"] \arrow[dr, "g_i"'] & V_{\alpha(i)} \times_V V_{\beta(i)} \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
& V_{\beta(i)}
\end{tikzcd}$$

Now let $s = (s_j) \in H^0(\mathcal{V}, \mathcal{F})$. We get

$$(f^* s)_i = f_i^* s_{\alpha(i)} = \varphi^* \pi_1^* s_{\alpha(i)} = \varphi^* \pi_2^* s_{\beta(i)} = g_i^* s_{\beta(i)},$$

where we have used the notation p^* as a shorthand for $\mathcal{F}(p)$. Notice that the middle equality holds since $s \in H^0(\mathcal{V}, \mathcal{F})$, so that $\pi_1^* s_{\alpha(i)} = s_{\alpha(i)}|_{V_{\alpha(i)} \times_V V_{\beta(i)}} = s_{\beta(i)}|_{V_{\alpha(i)} \times_V V_{\beta(i)}} = \pi_2^* s_{\beta(i)}$. □

We are now ready to define the presheaf $\mathcal{F}^+$:

Definition 11.19. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We define

$$\mathcal{F}^+(U) = \varinjlim_{\mathcal{U} \in \mathrm{Cov}(U)} H^0(\mathcal{U}, \mathcal{F}),$$

where the limit is taken over the directed set of coverings of $\mathcal{U}$, with transition morphisms given by refinement.

Remark 11.20. By lemma 11.18 this is a **filtered colimit**. This is guaranteed by the fact that

1. given any two coverings $\mathcal{U}_1, \mathcal{U}_2$, there is a third covering $\mathcal{V}$ that refines both;
2. given any two morphisms $f, g : \mathcal{U}_1, \mathcal{U}_2 \to \mathcal{V}$, the induced maps $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}_1, \mathcal{F})$ and $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}_2, \mathcal{F})$ agree.

The filtered structure of the colimit allows us to give a fairly concrete description of $\mathcal{F}^+(U)$: for every element $s \in \mathcal{F}^+(U)$ there is a covering $\mathcal{U} = (U_i \to U)_{i \in I}$ of U and an element $(s_i)_i \in H^0(\mathcal{U}, \mathcal{F})$ such that the image of (s_i) in $\mathcal{F}^+(U)$ is s. Furthermore, two elements $(s_i) \in H^0(\mathcal{U}, \mathcal{F})$ and $(s'_j) \in H^0(\mathcal{U}', \mathcal{F})$ define the same element in $\mathcal{F}^+(U)$ if and only if there exists a common refinement $\mathcal{V}$ of $\mathcal{U}, \mathcal{U}'$ such that the images of s, s' in $H^0(\mathcal{V}, \mathcal{F})$ (images taken along any refinement morphism) coincide.

Lemma 11.21. *$\mathcal{F}^+$ is a presheaf.*

Proof. We need to describe the action of $\mathcal{F}^+$ on morphisms: given a morphism $f : U \to V$ we want to construct a map $\mathcal{F}^+(V) \to \mathcal{F}^+(U)$. The universal property of colimits (applied to $\mathcal{F}^+(V)$) implies that giving such a map is equivalent to giving a map $H^0(\mathcal{V}, \mathcal{F}) \to \varinjlim_{\mathcal{U}} H^0(\mathcal{U}, \mathcal{F})$ for every covering $\mathcal{V}$ of V. Moreover, the universal property of colimits, applied to $\mathcal{F}^+(U, \mathcal{F})$, yields – for every cover $\mathcal{U}$ of U – a universal map $H^0(\mathcal{U}, \mathcal{F}) \to \mathcal{F}^+(U)$. Thus it suffices to construct, for every $\mathcal{V} \in \mathrm{Cov}(V)$, a map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ for a suitable covering $\mathcal{U}$ of U.

It is now enough to notice that we can define a pulback cover $f^*\mathcal{V}$ in such a way that there is a canonical pullback map $f^* : H^0(\mathcal{V}, \mathcal{F}) \to H^0(f^*\mathcal{V}, \mathcal{F})$. Indeed, if $\mathcal{V} = (V_j \xrightarrow{g_j} V)_{j \in J}$, we can set $f^*\mathcal{F} = (V_j \times_V U \to U)_{j \in J}$, which by (PT2) is a covering of U. The universal property of fibre products yields maps $U \times_V V_j \to V_j$ that (taken together with $f : U \to V$ and with the identity $J \to J$) in turn give a morphism of coverings $f^*\mathcal{V} \to \mathcal{V}$. Finally, this morphism induces the desired pullback $H^0(\mathcal{V}, \mathcal{F}) \to H^0(f^*\mathcal{V}, \mathcal{F})$.

We leave it to the reader to check that if $f : U \to V$ and $g : V \to W$ are two morphisms, then the map $(g \circ f)^* : \mathcal{F}^+(W) \to \mathcal{F}^+(U)$ coincides with f^*g^* (this follows immediately from the analogous statements fors pullbacks of coverings, which in turn is clear because of the canonical isomorphism $(W_k \times_W V) \times_V U = W_k \times_W U$). □

Definition 11.22 (Canonical map $\mathcal{F} \to \mathcal{F}^+$)**.** For every $U \in \mathcal{C}$ we have a canonical map

$$\theta : \mathcal{F}(U) \to \mathcal{F}^+(U).$$

This follows from the fact that $(\mathrm{id}_U : U \to U)$ is a covering of U, hence by the universal property of colimits we get

$$\mathcal{F}(U) = H^0(\{\mathrm{id}_U\}, \mathcal{F}) \to \varinjlim_{\mathcal{U}} H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}^+(U).$$

Since θ is clearly compatible with pullbacks, it induces a morphism of sheaves $\mathcal{F} \to \mathcal{F}^+$.

We now show that sections of $\mathcal{F}^+(U)$, when restricted to a sufficiently fine covering, come from sections of $\mathcal{F}$:

Lemma 11.23. *For every object $U \in \mathcal{C}$ and every $s \in \mathcal{F}^+(U)$ there is a covering $\mathcal{U} = (U_i \to U)_{i \in I}$ of U such that, for every $i \in I$, the restriction of s to U_i is in the image of $\theta : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.*

Proof. This is almost tautological. We have already remarked that for every $s \in \mathcal{F}^+(U)$ there is a covering $\mathcal{U} = (U_i \to U)_{i \in I}$ and a collection $(s_i)_{i \in I}$ such that s is the image of (s_i) in the colimit $\varinjlim_{\mathcal{U}} H^0(\mathcal{U}, \mathcal{F})$. We take this $\mathcal{U}$ as our covering. In order to compute the restriction of s to U_i, we may equally well compute the pullback of (s_i) from $H^0(\mathcal{U}, \mathcal{F})$ to $H^0(\{\mathrm{id}_{U_i}\}, \mathcal{F})$. According to lemma 11.18, we can choose whichever morphism of coverings we prefer, hence in particular we may choose the obvious morphism $(U_i \to U_i) \to (U_j \to U)_{j \in I}$. Using this morphism we find that the pullback of s to $H^0(\{\mathrm{id}_{U_i}\}, \mathcal{F})$ is the collection consisting of a single element, given by the pullback along $\mathrm{id}_{U_i} : U_i \to U_i$ of s_i. Finally, in order to get the pullback of s to $\mathcal{F}^+(U_i)$ we need to consider the image of this element in $\mathcal{F}^+(U_i)$: by definition of θ, this is precisely $\theta(s_i)$. $\square$

Lemma 11.24. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves. There is a commutative diagram of presheaves*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \mathcal{F}^+ \arrow[d, "\varphi^+"] \\ \mathcal{G} \arrow[r, "\theta"'] & \mathcal{G}^+ \end{tikzcd}$$

Proof. This is an easy verification; we only describe how to construct the morphism $\mathcal{F}^+ \to \mathcal{G}^+$. As in the proof of lemma 11.21, it suffices to construct a map $H^0(\mathcal{U}, \mathcal{F}) \to \mathcal{G}^+(U)$ for every covering $\mathcal{U}$ of U; one checks without difficulty that φ induces a map $H^0(\mathcal{U}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{G})$ for every covering $\mathcal{U}$ of U (by sending $(s_i)_{i \in I}$ to $(\varphi(s_i))_{i \in I}$), and it then suffices to compose with the universal map $H^0(\mathcal{U}, \mathcal{G}) \to \mathcal{G}^+(U)$. $\square$

We finally come to the main theorem of this section:

Theorem 11.25. *Let $\mathcal{F}$ be a presheaf on a site $\mathcal{C}$.*

1. *The presheaf $\mathcal{F}^+$ is separated.*
2. *If $\mathcal{F}$ is separated, then $\mathcal{F} \to \mathcal{F}^+$ is injective, and for every refinement $\mathcal{U} \to \mathcal{V}$ of coverings of an object U the map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ is injective.*
3. *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf.*
4. *The presheaf $\mathcal{F}^{++}$ is always a sheaf, and there is a canonical map $\theta^2 : \mathcal{F} \to \mathcal{F}^{++}$.*
5. *$\mathcal{F}$ is a sheaf if and only if $\theta : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*

Proof. 1. Let U be an object in $\mathcal{C}$, let $\mathcal{U}$ be a covering of U, and let $s, s' \in \mathcal{F}^+(U)$ be two sections with the same restriction to $\mathcal{F}^+(\mathcal{U})$. We want to show that $s = s'$ in $\mathcal{F}^+(U)$.

Lemma 11.23 yields the existence of covers $\mathcal{V} = (V_j \to U)_{j \in J}, \mathcal{V}' = (V'_k \to U)_{k \in K}$ of U such that the restriction of s (respectively s') to every V_j (respectively every V'_k) is in the image of θ. Fix a common refinement $\mathcal{W} = (W_h \to U)_{h \in H}$ of $\mathcal{U}, \mathcal{V}$, and $\mathcal{V}'$. Then for every h we have that:

(a) the restriction of s to W_h lies in the image of θ: indeed, given a refinement map $\mathcal{W} \to \mathcal{V}$ (with index map $\alpha : H \to J$), we can obtain $s|_{W_h}$ by pulling back $s|_{\alpha(h)}$, which is in the image of θ. Since θ is a map of presheaves, $s|_{W_h}$ is also in the image of θ. The same holds for the restriction of s' to W_h, and we write $s|_{W_h} = \theta(s_h), s'|_{W_h} = \theta(s'_h)$ for some $s_h, s'_h \in \mathcal{F}(W_h)$.

(b) Moreover, $\theta(s_h) = \theta(s'_h)$: indeed, by assumption we have a covering $\mathcal{U}$ such that $s|_{U_i} = s'|_{U_i}$ for every $i \in I$. Fixing again a refinement $W_h \xrightarrow{g_h} U_{\beta(h)}$ with index map $\beta : H \to I$, we obtain

$$s|_{W_h} = g_h^*(s|_{U_{\beta(h)}}) = g_k^*(s'|_{U_{\beta(h)}}) = s'|_{W_h}$$

and in particular $\theta(s_h) = \theta(s'_h)$.

As we already discussed, given the structure of a filtered colimit this last equality implies that for every h there is some covering of W_h, call it $(W_{hl} \to W_h)$, such that the restriction of s_h and of s'_h to each $W_{h,l}$ is the same (for every h and l). This implies that s, s' give the same element in $H^0((W_{hl} \to U), \mathcal{F})$, hence the same element in $\mathcal{F}^+(U)$ as desired.

2. The second statement is true by the assumption that $\mathcal{F}$ is separated. This implies that for every object U the set $H^0(\{\mathrm{id}_U\}, \mathcal{F}) = \mathcal{F}(U)$ injects into the colimit defining $\mathcal{F}^+(U)$, that is, the first statement.

3. We need to show that the gluing axiom holds, namely, that given a covering $\mathcal{U} = (U_i \to U)_{i\in I}$ of U and sections $s_i \in \mathcal{F}^+(U_i)$ that satisfy the gluing condition

$$s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$$

we may find a section $s \in \mathcal{F}^+(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$. Again the only difficulty is that sections of $\mathcal{F}^+$ are not necessarily sections of $\mathcal{F}$, but we may get around this problem by using lemma 11.23. For every i, choose a covering $(U_{ij} \to U_i)$ such that the restriction of s_i to U_{ij} is $\theta(s_{ij})$ for a (unique, by the separatedness of $\mathcal{F}$) section $s_{ij} \in \mathcal{F}(U_{ij})$. We now have a covering $\mathcal{V} = (U_{ij} \to U)$ and a collection of sections (s_{ij}): if these satisfy the gluing condition, then they give rise to an element in $H^0(\mathcal{V}, \mathcal{F})$, hence to a section $s \in \mathcal{F}^+(U)$; it remains to check that the gluing condition holds, and that s does the job (ie, that $s|_{U_i} = s_i$).

- gluing condition: take s_{ij} and $s_{i'j'}$ and restrict them to $U_{ij} \times_U U_{i'j'}$. Since θ is injective, we may instead compare $\theta(s_{ij})|_{U_{ij} \times_U U_{i'j'}}$ and $\theta(s_{i'j'})|_{U_{ij} \times_U U_{i'j'}}$. But these sections are pullbacks of $s_i|_{U_i \times_U U_{i'}}$ and $s_{i'}|_{U_i \times_U U_{i'}}$ respectively, and these sections agree by assumption, so that indeed $s_{ij}|_{U_{ij} \times_U U_{i'j'}} = s_{i'j'}|_{U_{ij} \times_U U_{i'j'}}$.
- Checking that $s|_{U_i} = s_i$ amounts, by separatedness of $\mathcal{F}^+$ and the fact that $(U_{ij} \to U_i)$ is a covering of U_i, to showing that $s|_{U_{ij}} = s_i|_{U_{ij}}$ for every j. But by definition $s_i|_{U_{ij}} = \theta(s_{ij}) = s|_{U_{ij}}$, and we are done.

4. Follows immediately from (1) and (3); the canonical map is simply the composition $\mathcal{F} \to \mathcal{F}^+ \to \mathcal{F}^{++}$.

5. If $\mathcal{F}$ is a sheaf, then by definition we have $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U)$ for every covering of U, so that $\mathcal{F}^+(U)$, which is the colimit of the various $H^0(\mathcal{U}, \mathcal{F})$, is simply $\mathcal{F}(U)$ again (notice that if we identify $H^0(\mathcal{U}, \mathcal{F})$ with $\mathcal{F}(U)$ the transition maps are all given by the identity). In particular, θ is an isomorphism in this case. Conversely, suppose that $\mathcal{F}$ and $\mathcal{F}^+$ are isomorphic: then $\mathcal{F}$ is separated (because $\mathcal{F}^+$ is), and therefore $\mathcal{F}^+$ is a sheaf, so that $\mathcal{F} \cong \mathcal{F}^+$ is also a sheaf.

□

Definition 11.26. We define the **sheafification functor** $\mathcal{F} \mapsto \mathcal{F}^\#$ as $\mathcal{F} \mapsto \mathcal{F}^{++}$. We denote the canonical map $\mathcal{F} \to \mathcal{F}^{++}$ by θ^2.

Lemma 11.27. *The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: for any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. By lemma 11.24 we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^+ \arrow[r] \arrow[d] & \mathcal{F}^{++} \arrow[d] \\
\mathcal{G} \arrow[r] & \mathcal{G}^+ \arrow[r] & \mathcal{G}^{++}
\end{tikzcd}$$

and by theorem 11.25 the lower horizontal maps are isomorphisms. This gives the existence of the map. Uniqueness follows from the fact that sections of $\mathcal{F}^{++}$ over U are determined by their restriction to an arbitrarily fine covering of U, and by lemma 11.23, for every section $s \in \mathcal{F}^{++}(U)$ we can find a covering $\mathcal{U} = (U_i \to U)$ of U such that $s|_{U_i}$ lies in the image of $\theta^2 : \mathcal{F}(U_i) \to \mathcal{F}^{++}(U_i)$. Since the action of the map $\mathcal{F}^{++} \to \mathcal{G}$ on elements in the image of θ^2 is given, this proves uniqueness. $\square$

Lemma 11.28. *Let $\iota : \mathbf{Sh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{C})$ be the inclusion functor. Then:*

1. *${}^{\#} : \mathbf{PSh}(\mathcal{C}) \rightleftarrows \mathbf{Sh}(\mathcal{C}) : \iota$ are an adjoint pair of functors.*
2. *${}^{\#}$ is left exact, hence exact.*

Proof.
1. As we have already seen, there is an injective map $\mathrm{Hom}_{\mathbf{PSh}}(\mathcal{F}, \iota\mathcal{G}) \to \mathrm{Hom}_{\mathbf{Sh}}(\mathcal{F}^{\#}, \mathcal{G})$; a map in the opposite direction is simply given by precomposing a morphism of sheaves $\mathcal{F}^{\#} \to \mathcal{G}$ with the canonical map $\mathcal{F} \to \mathcal{F}^{\#}$. That these two constructions are inverse to each other is obvious.
2. The sheafification functor is left adjoint, hence right exact, so it suffices to prove that it is left exact. For this, one combines two facts:
   (a) limits in the category of sheaves are limits in the category of presheaves, lemma 11.16;
   (b) $\mathcal{F} \mapsto \mathcal{F}^+$ commutes with finite limits as $\mathcal{F}^+$ is obtained by taking the (co)limit over a directed set.

$\square$

It is now easy to prove that $\mathbf{Ab}(\mathcal{C})$ is an abelian category. In fact, this will follow formally from the following easy lemma in category theory:

Lemma 11.29. *Let $b : \mathcal{B} \rightleftarrows \mathcal{A} : a$ be an adjoint pair of additive functors between additive categories. Suppose that:*

1. *$\mathcal{B}$ is abelian;*
2. *b is left exact;*
3. *$ba = \mathrm{Id}_{\mathcal{A}}$.*

Then:

1. *$\mathcal{A}$ is an abelian category;*
2. *for every morphism ψ in $\mathcal{A}$, $\ker \psi = b(\ker a\psi)$ and $\mathrm{Coker}\, \psi = b(\mathrm{Coker}\, a\psi)$.*

Proof. This is completely formal, so we only give some indications. Notice that b is left exact by assumption and is right exact since it is a left adjoint, so it is exact; similarly, a is left exact. Everything now follows easily. $\square$

Theorem 11.30. *Let $\mathcal{C}$ be a site.*

1. *The category $\mathbf{Ab}(\mathcal{C})$ is an abelian category.*

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on $\mathcal{C}$.

2. *The kernel* $\ker \varphi$ *of* φ *is the same as the kernel of* φ *as a morphism of presheaves.*

3. *The morphism* φ *is injective (in the sense of category theory) if and only if for every* $U \in |\mathcal{C}|$ *the map* $\mathcal{F}(U) \xrightarrow{\varphi} \mathcal{G}(U)$ *is injective.*

4. *The cokernel* $\mathrm{Coker}(\varphi)$ *of* φ *is the sheafification of the cokernel of* φ *as a morphism of presheaves.*

5. *The morphism* φ *is surjective if and only if* φ *is surjective as a map of sheaves, that is, if and only if for every object* U *of* $\mathcal{C}$ *and for every section* $s \in \mathcal{G}(U)$ *there exists a covering* $\mathcal{U} = (U_i \to U)_{i \in I}$ *of* U *such that* $s|_{U_i}$ *is in the image of* $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

6. *More generally, a complex of abelian sheaves*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ *if and only if for all* $U \in |\mathcal{C}|$ *and all* $s \in \mathcal{G}(U)$ *mapping to zero in* $\mathcal{H}(U)$ *there exists a covering* $\{U_i \to U\}_{i \in I}$ *such that each* $s|_{U_i}$ *is in the image of* $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Proof.

1. Apply lemma 11.29 to the adjoint pair $^\# : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}) : \iota$. The hypotheses of this lemma are satisfied thanks to lemma 11.28 and our previous discussion.

2. Kernels are limits, so this follows from lemma 11.16.

3. In an abelian category, a morphism $\mathcal{F} \to \mathcal{G}$ is injective if and only if its kernel is the zero map $0 \to \mathcal{F}$. This is clearly equivalent to $\mathcal{F}(U) \to \mathcal{G}(U)$ being injective for every U.

4. Follows immediately from lemma 11.29.

5. Surjectivity of $\mathcal{F} \to \mathcal{G}$ is equivalent to the fact that the cokernel of this map is the zero map $\mathcal{G} \to 0$. Hence $\mathcal{F} \to \mathcal{G}$ is surjective if and only if the sheafification of the presheaf $\mathcal{H} : U \mapsto (\mathrm{coker}(\mathcal{F}(U) \to \mathcal{G}(U)))$ is the trivial sheaf. This means that for every U the colimit $\varinjlim_{\mathcal{U}} H^0(\mathcal{U}, \mathcal{H})$ is zero; equivalently, for every $(s_i) \in H^0(\{V_i \to U\}, \mathcal{H})$ we can find a refinement $\mathcal{U}$ of $\mathcal{V}$ such that $s|_{U_i}$ is zero. But this is precisely the requirement that $s|_{U_i}$ belongs to the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

6. This is proven as the previous part, by noticing that exactness at $\mathcal{G}$ is equivalent to

$$\ker(\mathcal{F} \to \mathcal{G}) = \mathrm{Coker}(\mathcal{G} \to \mathcal{H}).$$

□

Notice that parts (3), (5) and (6) recover exactly the usual results for sheaves on topological spaces!

11.3 $\mathbf{Ab}(\mathcal{C})$ has enough injectives

We are now close to (finally!) defining the cohomology of abelian sheaves on sites, of which étale cohomology is a special case. We intend to define the cohomology of an abelian sheaf as the (collection of) derived functors of the global sections functor $U \mapsto H^0(U, \mathcal{F})$: in order to do this, however, we need to know that the category $\mathbf{Ab}(\mathcal{C})$ has enough injectives. Proving this is technically quite demanding, so we only give an outline of the main steps involved:

1. consider the category $\mathbf{Ob}(\mathcal{C})$ whose objects are $|\mathcal{C}|$ and whose only morphisms are the identities of the various objects. Then there are adjoint functors

$$v : \mathbf{PAb}(\mathcal{C}) \to \mathbf{PAb}(\mathbf{Ob}(\mathcal{C}))$$

and

$$u : \mathbf{PAb}(\mathcal{C}) \to \mathbf{PAb}(\mathbf{Ob}(\mathcal{C})).$$

Here v is the natural forgetful functor, while u takes the presheaf $\mathcal{F}$ to the presheaf

$$u\mathcal{F} : U \mapsto \prod_{U' \to U} \mathcal{F}(U'),$$

where the product is over all morphisms in $\mathcal{C}$ with target U. Functors u and v are exact and adjoint; more precisely,

$$\mathrm{Hom}_{\mathbf{PAb}(\mathcal{C})}(\mathcal{F}, u\mathcal{G}) = \mathrm{Hom}_{\mathbf{PAb}(\mathbf{Ob}(\mathcal{C}))}(v\mathcal{F}, \mathcal{G}).$$

2. every presheaf – and in particular every sheaf – admits a (functorial) embedding into an injective presheaf. This is built out of the functorial embedding of an abelian group into an injective object given by $M \hookrightarrow J(M) := F(M^\vee)^\vee$ (here F is the functor that sends an abelian group A to the free abelian group with generators indexed by A, while $A^\vee := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$). More precisely, one checks that for every presheaf $\mathcal{F}$ on $\mathbf{Ob}(\mathcal{C})$ the sheaf

$$U \mapsto J(\mathcal{F}(U))$$

is injective, and since u, v are exact and adjoint we get that for any presheaf $\mathcal{F} \in \mathbf{PAb}(\mathcal{C})$ the presheaf $U \mapsto vJ(u\mathcal{F})$ is injective. We denote this presheaf by $J(\mathcal{F})$.

3. now we turn to sheaves. Define $J_1(\mathcal{F})$ as the sheafification of $J(\mathcal{F})$ and, by transfinite induction,

$$J_{\alpha+1}(\mathcal{F}) = J_1(J_\alpha(\mathcal{F})), \quad J_\beta(\mathcal{F}) = \varinjlim_{\alpha<\beta} J_\alpha(\mathcal{F}) \text{ if } \beta \text{ is a limit ordinal.}$$

Since the colimit is directed by the very definition of ordinals, for ordinals $\alpha < \beta$ we always have an injection $J_\alpha(\mathcal{F}) \hookrightarrow J_\beta(\mathcal{F})$. Then one proves that *the collection* $J_\alpha(\mathcal{F})$ is injective, in the sense made precise by the following lemma:

Lemma 11.31. *Let $\mathcal{G}_1 \to \mathcal{G}_2$ be an injective morphism of sheaves. Let α be an ordinal and let $\mathcal{G}_1 \to J_\alpha(\mathcal{F})$ be a morphism of sheaves. Then there exists a commutative diagram as follows:*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[d] \\ J_\alpha(\mathcal{F}) \arrow[r] & J_{\alpha+1}(\mathcal{F}) \end{tikzcd}$$

Proof. Denote by ι the forgetful functor from $\mathbf{Ab}(\mathcal{C})$ to $\mathbf{PAb}(\mathcal{C})$. Then $\iota\mathcal{G}_1 \to \iota\mathcal{G}_2$ is injective, so we get a map $\iota\mathcal{G}_2 \to J(\iota J_\alpha(\mathcal{F}))$. Applying the sheafification functor we get the desired map $\mathcal{G}_2 \to J(\iota J_\alpha(\mathcal{F}))^\# = J_{\alpha+1}(\mathcal{F})$. □

4. we would now like to say that $\varinjlim_\alpha J_\alpha(\mathcal{F})$ is an injective object into which $\mathcal{F}$ embeds; however, this does not make sense, because ordinals form a proper class and not a set, so we cannot take the colimit. The technical tool around this problem is given by the following lemma:

Lemma 11.32. *Let $\mathcal{G}_i$ be a collection of sheaves (indexed by a set I). There is an ordinal β such that, for every map $\mathcal{G}_i \to J_\beta(\mathcal{F})$, there is an ordinal $\alpha < \beta$ such that the given map factors via $J_\alpha(\mathcal{F})$.*

In other words, *provided that we only have to deal with a collection of sheaves that is a set*, we can "stop the transfinite induction at some point". Notice that this is quite subtle, and depends on the existence of ordinals of arbitrarily large cofinality[13]. Furthermore, it seems to be essential in the proof that $|\mathcal{C}|$ is a *set*, that is, that $\mathcal{C}$ is small.

[13] I don't pretend to fully understand how this works

5. we still need to reduce to a collection of sheaves which is a set (the class of *all* sheaves on a site can easily fail to be a set!). This reduction is made possible by lemma 11.33 below. To state this lemma, for every object X of $\mathcal{C}$ we define a sheaf $\mathbb{Z}_X^{\#}$ as the sheafification of the presheaf
$$\mathbb{Z}_X : U \mapsto \oplus_{U \to X} \mathbb{Z}.$$
One can check that $\mathbb{Z}_X$ satisfies the following universal property:
$$\mathrm{Mor}_{\mathbf{Ab}(\mathcal{C})}(\mathbb{Z}_X^{\#}, \mathcal{G}) = \mathcal{G}(X),$$
the isomorphism being given by sending φ on the left hand side to $\varphi(1 \cdot \mathrm{id}_X)$.

Lemma 11.33. *Suppose $\mathcal{J}$ is a sheaf of abelian groups with the following property: for any $X \in |\mathcal{C}|$, for any abelian subsheaf $\mathcal{S} \subseteq \mathbb{Z}_X^{\#}$ and any morphism $\varphi : \mathcal{S} \to \mathcal{J}$, there exists a morphism $\mathbb{Z}_X^{\#} \to \mathcal{J}$ extending φ. Then $\mathcal{J}$ is an injective sheaf of abelian groups.*

This is similar to what happens for abelian groups: starting from a diagram
$$\begin{tikzcd} I & \\ \mathcal{F}_1 \arrow[u] \arrow[r, hook] & \mathcal{F}_2, \end{tikzcd}$$
we want to complete it with an arrow $\mathcal{F}_2 \to I$. Consider all pairs $(\mathcal{F}, i)$ with $\mathcal{F}_1 \subseteq \mathcal{F} \subseteq \mathcal{F}_2$ and $i : \mathcal{F} \to I$ that extends $\mathcal{F}_1 \to I$. There is an obvious ordering on such pairs, and Zorn's lemma applies to show that there is a maximal element. Let $(\mathcal{F}, i)$ be this maximal element and suppose that $\mathcal{F} \neq \mathcal{F}_2$: to finish the proof it is enough to find an extension of i to any sheaf strictly larger than $\mathcal{F}$. Fix an object X for which $\mathcal{F}(X) \neq \mathcal{F}_2(X)$ and an $s \in \mathcal{F}_2(X) \setminus \mathcal{F}(X)$. Then we get a morphism $\psi : \mathbb{Z}_X^{\sharp} \to \mathcal{F}_2(X)$ by sending 1 to s; let S be $\psi^{-1}(\mathcal{F})$. Then the map $S \to I$ (which is given) extends to $\mathbb{Z}_X^{\sharp} \to I$ by assumption, and this yields the existence of a map $i' : \mathcal{F} + \mathrm{Im}(\psi) \to I$ extending i (the crucial remark here is that i and the map $\mathbb{Z}_X^{\sharp} \to I$ agree on $S = \mathcal{F} \cap \mathbb{Z}_X^{\sharp}$, hence they can be glued together). This contradicts the maximality of $\mathcal{F}$ and finishes the proof.

Putting everything together, one deduces:

Theorem 11.34. *Let $\mathcal{C}$ be a site. The category $\mathbf{Ab}(\mathcal{C})$ has enough injectives.*

We shall see later (see theorem 16.7) a more direct proof in the special case of the étale site.

11.4 Topologies on categories of schemes

We are finally ready to define the étale site of a scheme.

Definition 11.35. Let S be a scheme. The **small étale site** of S, denoted by $S_{ét}$, is the category of schemes X/S such that the structure map $X \to S$ is étale, equipped with the following Grothendieck pretopology: a family of S-morphisms $(X_i \xrightarrow{f_i} X)_{i \in I}$ is a covering of $X \in |S_{ét}|$ if and only if $\bigcup_{i \in I} f_i(|X_i|) = |X|$ (that is, if and only if the f_i are jointly surjective at the level of the underlying topological spaces). Any covering as above will be called an **étale covering** of X.

Despite its name, the small étale site is *not* a small category. It is, however, what one calls an *essentially small* category – that is, it is equivalent to a small one. This is sufficient for our applications, and we will not insist on these foundational difficulties.

Since this is our first nontrivial site, let's check that the definition makes sense, that is, that the coverings described satisfy axioms (PT1) through (PT3). Before doing so, let's study a more general situation:

Lemma 11.36. *Let $\mathcal{C}$ be a subcategory of* $\mathbf{Sch}/S$ *with fibre products. Let (P) be a property of morphisms of $\mathcal{C}$ satisfying:*

1. *(P) is true for isomorphisms of $\mathcal{C}$.*
2. *(P) is stable by base-change.*
3. *(P) is stable by composition.*

Define a family $(f_i : Ti \to T)_{i\in I}$ in $\mathcal{C}$ to be a covering family if for every $i \in I$ the arrow $f_i : T_i \to T$ satisfies (P), and $|T| = \bigcup_{i\in I} f_i(|Ti|)$. This defines a pretopology on $\mathcal{C}$.

Proof. (PT1) and (PT3) are true by assumption. As for (PT2), we need to check that, given a morphism $T' \to T$ and a covering $f_i : T_i \to T$, the family $f_i' : T' \times_T T_i \to T'$ has property (P) and is jointly surjective. The first statement is true by assumption; as for the second, notice that by the definition of a covering family we have that $\coprod_i T_i \xrightarrow{f_i'} T$ is surjective; taking the product with T' we get a map

$$\left(\coprod_i T_i \times_T T'\right) \to T'$$

which is surjective because the underlying set of a fibre product of schemes surjects on the fibre product of the underlying sets. This clearly implies that $f_i' : T_i' \to T'$ are jointly surjective on the underlying topological spaces. $\square$

In the case of étale coverings, it's clear that isomorphisms are étale maps, and étale maps are stable under base-change and composition (lemma 7.11), so the small étale site is indeed a site! And since $\mathbf{Ab}(\mathcal{C})$ is an abelian category with enough injectives by theorem 11.34, we can finally define étale cohomology:

Definition 11.37 (Étale cohomology of sheaves)**.** Let S be a scheme and $S_{ét}$ be its associated small étale site. Given an abelian sheaf $\mathcal{F} \in \mathbf{Ab}(S_{ét})$, we define the i-th étale cohomology group of $\mathcal{F}$ over S as

$$H^i_{ét}(S, \mathcal{F}) := R^i\Gamma(S, \mathcal{F}),$$

the i-th right derived functor of the global sections functor $\Gamma : \mathcal{F} \mapsto H^0(S, \mathcal{F})$.

Our main interest lies with étale cohomology, so we will mostly be interested in the site $S_{ét}$, but lemma 11.36 implies that one can construct many different sites, corresponding to different properties of morphisms. We describe at least one of them: the (small) Zariski site.

Definition 11.38. Let S be a scheme. The **small Zariski site** S_{Zar} is the subcategory of $\mathbf{Sch}/S$ whose objects are schemes X/S such that the structure map $X \to S$ is an open immersion. We equip S_{Zar} with the pretopology described by lemma 11.36 when the property (P) is 'being an open immersion'.

We also introduce the *big* variants of the étale and Zariski sites.

Definition 11.39. The big étale and Zariski sites of S are both given by the category of schemes over S; covering families are defined in the same way as for the small sites. They are denoted by $(\mathbf{Sch}/S)_{\mathrm{Zar}}$ and $(\mathbf{Sch}/S)_{\text{ét}}$.

Remark 11.40. Notice that the big étale site is immensely bigger than the small one: for example, when $S = \operatorname{Spec} k$, the objects in $S_{ét}$ are schemes with an étale map to $\operatorname{Spec} k$, which are of the form $\coprod \operatorname{Spec} k_i$ with k_i/k a finite separable extension. The big étale $(\mathbf{Sch}/S)_{\text{ét}}$ site contains (for example) *all* k-varieties!

The problem of reducing to small categories is even more serious for the big sites, but we will not discuss it.

Let $(P), (P')$ be properties of morphisms of schems that satisfy the conditions of lemma 11.36. If (P) implies (P'), then every (P)-covering is in particular a (P')-covering, so that the pretopology generated by (P') is **finer** than the topology generated by (P), in the sense that the pretopology corresponding to (P') has more open covers than that corresponding to (P). The following properties lead to sites that have found use in arithmetic geometry:

Definition 11.41. Let S be a scheme. For each of the following classes of morphisms τ, the construction of lemma 11.36 equips the category $\mathbf{Sch}/S$ with the structure of a site:

- open immersions, leading to the (big) Zariski site $(\mathbf{Sch}/S)_{\mathrm{Zar}}$;
- étale morphisms, leading to the (big) étale site $(\mathbf{Sch}/S)_{\text{ét}}$;
- smooth morphisms, leading to the smooth site $(\mathbf{Sch}/S)_{\mathrm{smooth}}$;
- faithfully flat morphisms of finite presentation, leading to the fppf[14] site $(\mathbf{Sch}/S)_{\mathrm{fppf}}$.

We shall (greatly) need yet one more site, namely the fpqc[15] site $(\mathbf{Sch}/S)_{\mathrm{fpqc}}$, for which the covering families are defined as follows:

Definition 11.42. Let $T \to S$ be a scheme over S. An **fpqc covering** of T is a family $(T_i \xrightarrow{f_i} T)_{i\in I}$ such that:

1. every $f_i : T_i \to T$ is a flat morphism and $|T| = \bigcup_{i\in I} f_i(|T_i|)$.
2. for every affine open subset $U \subseteq T$ there exists a **finite** subset of indices $J \subseteq I$ and affine open subsets $U_j \subseteq T_j$ such that $U = \bigcup_{j\in J} f_j(U_j)$.

11.5 Exercises

Exercise 11.43. Let S be a scheme, C be an abelian group, and let $\mathcal{F}$ be the presheaf on $S_{\text{ét}}$ which takes the value C on every object of $S_{\text{ét}}$ (with restriction maps given by the identity). Describe the sheafification of $\mathcal{F}$.

Exercise 11.44 (Étale sheaves on $\operatorname{Spec} K$)**.** Let L/K be a finite Galois (in particular, separable) extension of fields, let $S = \operatorname{Spec} K$, and let $\mathcal{F}$ be a sheaf on the small étale site of S.

1. Show that there is a natural action of $\mathrm{Gal}(L/K)$ on $\mathcal{F}(\operatorname{Spec} L)$.
2. Show that there is a natural identification $\mathcal{F}(\operatorname{Spec} L)^{\mathrm{Gal}(L/K)} = \mathcal{F}(K)$.
3. Deduce that the category of étale sheaves on the small étale site of $\operatorname{Spec} K$ is equivalent to the category of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-sets, that is, discrete sets equipped with a continuous action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, where K^{sep} denotes a separable closure of K.

[14] *fidèlement plat de présentation finie*
[15] *fidèlement plat quasi-compact*

12 More on the étale site

Last time we defined the étale cohomology of an étale abelian sheaf. We are now in the slightly akward position of knowing how to define étale cohomology, but not having a single interesting étale sheaf to compute the cohomology *of*! Our purpose for today is thus to introduce several natural classes of étale sheaves. As it turns out, essentially all the étale sheaves we will be interested in are also sheaves for much finer topologies – in particular, for the fpqc topology that we introduced last time.

12.1 A criterion for a Zariski sheaf to be an fpqc sheaf

The following (simple) lemma will be fundamental to construct sheaves for our various topologies:

Lemma 12.1. *A presheaf $\mathcal{F}$ on* $\mathbf{Sch}/S$ *is an* fpqc-*sheaf*[16] *if and only if:*

1. *it is a sheaf for the Zariski pretopology;*
2. *for any $(V \to U) \in \mathrm{Cov}_{\mathrm{fpqc}}(U)$, with U and V affine, the sequence*

$$\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is exact.

Proof of lemma 12.1. That the two conditions stated are necessary is clear. For the other implication, notice that if $\mathcal{F}$ is a Zariski sheaf then $\mathcal{F}\left(\coprod_{i\in I} U_i\right) = \prod \mathcal{F}(U_i)$. Hence the sheaf condition for the covering $(U_i \to U)_{i\in I}$ is equivalent to the sheaf condition for the covering $\coprod_i U_i \to U$, because

$$\left(\coprod_i U_i\right) \times_U \left(\coprod_i U_i\right) = \coprod_{i,j} U_i \times_U U_j.$$

This implies in particular that the fpqc-sheaf condition is satisfied for coverings $(U_i \to U)_{i=1,\ldots,n}$ which involve only a finite number of U_i, each of which affine, for in this case the disjoint union $\coprod_{i=1}^n U_i$ is affine.

Step 1. $\mathcal{F}$ is separated. Let $(U_i \to U)_{i\in I}$ be a fpqc-covering. As before, we may reduce to a single morphism $f : U' \to U$. Choose an open affine cover $(V_j)_{j\in J}$ of U and for each j write V_j as $\bigcup_{k\in K_j} f(U'_{jk})$ for open affines U'_{jk} and for a **finite** set of indices K_j. This is possible by definition of the fpqc topology. We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(U) & \longrightarrow & \mathcal{F}(U') \\
\downarrow & & \downarrow \\
\prod_j \mathcal{F}(V_j) & \longrightarrow & \prod_j \prod_k \mathcal{F}(U'_{jk})
\end{array}$$

where the arrow in the first column is injective (since $\mathcal{F}$ is a Zariski sheaf and $(V_j \to U)$ is a Zariski covering) and the arrow in the second row is also exact (since for every j the covering $(U'_{jk} \to V_j)_{j\in K_j}$ is a finite fpqc-covering by affines, for which the sheaf property holds by the above). It follows immediately that $\mathcal{F}(U) \to \mathcal{F}(U')$ is injective, that is, that $\mathcal{F}$ is separated.

Step 2. Sections are defined over a maximal open subset. Consider an arbitrary fpqc-covering $(U_i \to U)_{i\in I}$ and sections $s_i \in \mathcal{F}(U_i)$ such that the restrictions of s_i, s_j to $U_i \times_U U_j$ agree.

[16] one should rather say *satisfies the sheaf condition with respect to fpqc coverings*, because (due to significant set-theoretic difficulties) there is no *category of fpqc sheaves on a scheme*. We will not make such a fine distinction.

Also consider two open subschemes V, V' in U such that there exist sections $s_V \in \mathcal{F}(V), s_{V'} \in \mathcal{F}(V')$ with

$$s_V|_{U_i \times_U V} = s_i|_{U_i \times_U V} \text{ and } s_{V'}|_{U_i \times_U V'} = s_i|_{U_i \times_U V'}.$$

Let $W = V \cup V'$. We claim that there is a section $s_W \in \mathcal{F}(W)$ such that $s_W|_V = s_V$ and $s_W|_{V'} = s_{V'}$. As $\mathcal{F}$ is by assumption a Zariski sheaf and $V \cup V'$ covers W, it suffices to check that $s_V|_{V \times_U V'} = s_{V'}|_{V \times_U V'}$. As $\mathcal{F}$ is separated, it suffices to do this after restricting to a fpqc-covering of $V \times_U V'$, which we take to be $V \times_U V' \times_U U_i$: but upon restriction to $V \times_U V' \times_U U_i$ both s_V and $s_{V'}$ become equal to $s_i|_{V \times_U V' \times_U U_i}$, hence s_V and $s_{V'}$ agree on $V \times_U V'$ as desired. Notice that the argument generalises from two open sets V, V' to an arbitrary (not necessarily finite) family of opens. By taking the union of all V's as above, it follows that there is a unique maximal open subscheme W in U for which there exists a (necessarily unique, by step 1) section $s_W \in \mathcal{F}(W)$ with $s_W|_{W \times_U U_i} = s_i|_{W \times_U U_i}$.

Step 3. Sections glue over open affines. Let again $(U_i \to U)_{i \in I}$ be an arbitrary fpqc-covering and let $(s_i) \in \mathcal{F}(U_i)$ be sections that agree on $U_i \times_U U_j$ for every i, j. Let V be an open **affine** subscheme of U. We claim that there exists a section $s_V \in \mathcal{F}(V)$ such that $s_V|_{U_i \times_U V} = s_i|_{U_i \times_U V}$. Indeed, by definition of an fpqc covering we may find a finite set of indices J and affine opens $T_i \subseteq U_i$ for $i \in J$ such that $(T_i \times_U V \to U_i \times_U V \to V)_{i \in J}$ is a fpqc-covering. But as J is finite and the T_i are affine, our preliminary argument shows that $\mathcal{F}$ satisfies the sheaf condition for this covering, so the sections s_i (which agree on $U_i \times_U U_j$ for every i, j, hence a fortiori agree on $(T_i \times_U V) \times_U (T_j \times_U V)$) glue to give the desired section s_V.

To conclude this step, we need to check that the equality $s_V|_{U_i} = s_i$ holds for every index i. To do this, fix i and notice that by Step 1 we can check equality on any fpqc covering of U_i, for example $\{U_i \times_U T_j \times_U V\}_{j \in J}$. As $U_i \times_U T_j \times_U V$ maps to $T_j \times_U V$, we know that $s_V|_{U_i \times_U T_j \times_U V} = s_j|_{U_i \times_U T_j \times_U V}$, and since $U_i \times_U T_j$ refines $U_i \times_U U_j$, by compatibility of the sections $s_i \in \mathcal{F}(U_i)$ and $s_j \in \mathcal{F}(U_j)$ when restricted to $U_i \times_U U_j$ we obtain

$$s_V|_{U_i \times_U T_j \times_U V} = s_j|_{U_i \times_U T_j \times_U V} = s_i|_{U_i \times_U T_j \times_U V}.$$

Since this holds for every j, by separatedness of $\mathcal{F}$ we obtain $s_V|_{U_i} = s_i|_{U_i}$ as desired.

Step 4. $\mathcal{F}$ is a sheaf. As $\mathcal{F}$ is separated we only need to check that compatible collections of sections $(s_i) \in \prod \mathcal{F}(U_i)$ glue. And indeed, the maximal open $W \subseteq U$ constructed in step 2 must necessarily be all of U, because every point of U is contained in some open affine. □

As we have already remarked, in the following list:

Zariski, étale, smooth, fppf, fpqc

every topology is *finer* than the previous one – that is, for every object X the covering families of X for a given topology are a superset of the covering families for the previous topology . Since the sheaf condition is expressed in terms of covering, one immediately obtains the following important fact:

Proposition 12.2. *Let $\mathcal{F} : (\mathbf{Sch}/S) \to \mathbf{Ab}$ be a presheaf. Then if $\mathcal{F}$ is a sheaf for the fpqc topology it is also a fppf sheaf; if $\mathcal{F}$ is a fppf sheaf it is also a smooth sheaf; if it is a smooth sheaf it is also an étale sheaf; and finally, if it is an étale sheaf then it is a Zariski sheaf.*

An important consequence of this proposition is that in order to construct étale sheaves it is enough to construct presheaves that satisfy the sheaf condition for the fpqc topology. This is precisely what we are going to do in the next section.

12.2 Étale sheaves from fpqc descent

Our objective in this section is to show that many natural presheaves are in fact sheaves for the étale topology. The precise statements we are aiming for are the following:

Theorem 12.3 (fpqc sheaf associated with a quasi-coherent sheaf)**.** *Let S be a scheme and let $\mathcal{G}$ be a quasi-coherent sheaf on S. The presheaf*

$$\begin{array}{rccc} \mathcal{F}: & \mathbf{Sch}/S & \to & \mathbf{Ab} \\ & f: T \to S & \mapsto & \Gamma(T, f^*\mathcal{G}) \end{array}$$

is an fpqc sheaf, hence in particular an étale sheaf.

Theorem 12.4 (fpqc sheaf associated with a scheme)**.** *Let $X \in \mathbf{Sch}/S$. Then h_X is a sheaf on $(\mathbf{Sch}/S)_{\mathrm{fqpc}}$, hence in particular it is an étale sheaf.*

In order to prove these results we need to first discuss what is usually called *fpqc descent.*

12.2.1 fpqc descent

fpqc descent is the idea that in order to give an object on a scheme X, it is enough to give an object on a faithfully flat cover of X, together with 'gluing' or 'descent' data. This partially explains in what sense Grothendieck (pre)topologies are refinements of the Zariski topology: in order to describe a morphism/sheaf/scheme/... over a scheme S, it is enough to do so locally *for a Grothendieck (pre)topology*, which is precisely our intuitive notion of what a finer topology should behave like. In order to make all this a little more concrete, consider for example the following theorem:

Theorem 12.5. *Let $(U_i \to X)_{i \in I}$ be an fpqc covering of X, and let Z be a scheme. Suppose we have morphisms $\varphi_i : U_i \to Z$ such that for all $i, j \in I$*

$$\varphi_i|_{U_i \times_X U_j} = \varphi_j|_{U_i \times_X U_j}.$$

Then there exists a unique morphism $\varphi : X \to Z$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Remark 12.6. Unwinding the definitions one sees that this statement is equivalent to theorem 12.4, see the proof of theorem 12.4 below.

The proof of this (and similar) theorems relies on the following lemma:

Lemma 12.7 (Main lemma of fpqc descent, basic version)**.** *Let $\phi : A \to B$ be a faithfully flat map of rings. Then*

$$0 \to A \to^{\phi} B \xrightarrow{d_0} B \otimes_A B \tag{11}$$

is an exact sequence of A-modules, where $d_0(b) := b \otimes 1 - 1 \otimes b$.

Proof. Since ϕ is injective, we consider A as a subset of B. We proceed in three stages:

1. Suppose that there exists a section $g : B \to A$ such that $g|_A = \mathrm{Id}_A$. Consider the map $h := g \times id : B \otimes_A B \to B$. Then $d_0(b) = 0$ implies $0 = h \circ d_0(b) = b - g(b)$, which in turn gives
$$b = g(b) \in A.$$

2. For any ring morphism $A \to C$ we have
$$(B \otimes_A B) \otimes_A C \cong (B \otimes_A C) \otimes_C (B \otimes_A C).$$
Now suppose that $A \to C$ is faithfully flat: then if we tensor (11) with C over A we obtain
$$0 \to C \xrightarrow{\phi} B \otimes_A C \xrightarrow{d_0} (B \otimes_A C) \otimes_C (B \otimes_A C).$$
Since C is faithfully flat, it suffices to prove that this latter sequence is exact. Thus, we can replace the pair (A, B) with $(C, B \otimes_A C)$.

3. Finally, consider an arbitrary faithfully flat map $A \to B$. Applying the previous reduction with $C = B$ we get the ring map

$$B \hookrightarrow B \otimes_A B, \ b \mapsto b \otimes 1,$$

for which we can construct a section by setting $g(b \otimes b') = bb'$. By case (1) this completes the proof.

□

Lemma 12.8 (Main lemma of fpqc descent, general version). *Let $\phi : A \to B$ be a faithfully flat map of rings, and let M be any A-module. Then*

$$0 \to M \to M \otimes_A B \xrightarrow{d_0} M \otimes_A B^{\otimes 2} \to \cdots \xrightarrow{d_{r-2}} M \otimes_A B^{\otimes r}$$

is an exact sequence of A-modules, where

$$d_{r-1}(b) = \sum_i (-1)^i e_i(b)$$

and

$$e_i \ (b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}$$

Proof. That the composition of any two consecutive maps is 0 is straightforward. To check exactness, we reduce as before to the case where there exists a section $g : B \to A$. Then consider the morphism $k_r : B^{\otimes r+2} \to B^{\otimes r+1}$ defined as

$$k_r \ (b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0) \cdot b_1 \otimes b_2 \otimes \cdots \otimes b_{r+1}.$$

One checks that $k_{r+1} \circ d_{r+1} + d_r \circ k_r = \mathrm{id}$ (see also section 1.3), from which the exactness follows. □

We are now ready to prove that morphisms of schemes can be defined fpqc-locally (that is, theorem 12.5). Before doing so, recall the following lemma (which we have essentially already seen):

Lemma 12.9. *A flat, local map of local rings $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is faithfully flat.*

Proof. By proposition 8.20 it suffices to prove that the image of the map $\operatorname{Spec} B \to \operatorname{Spec} A$ contains the maximal spectrum of A, and this is obvious since by assumption the contraction of $\mathfrak{n}$ is $\mathfrak{m}$. □

Proof of theorem 12.5. Setting $Y = \sqcup_{i \in I} U_i$ we can reduce to the case of a single flat, surjective map $\phi : Y \to X$ which is quasi-compact. Suppose $h : Y \to Z$ is a morphism such that $h \circ p_1 = h \circ p_2$, where p_i is the i-th canonical projection from $Y \times_X Y$ to Y. We wish to prove the existence and uniqueness of a morphism $g : X \to Z$ such that $g \circ \phi = h$.

1. We first prove uniqueness. Suppose g_1, g_2 are two such maps. Since ϕ is surjective as a map of topological spaces, it follows that g_1, g_2 agree on the underlying topological spaces, so we just need to check that they agree on stalks. Hence, let $x \in X, z = g_1(x) = g_2(x) \in Z$ and pick $y \in Y$ such that $\phi(y) = X$. Then we have induced maps

$$\mathcal{O}_{Z,z} \rightrightarrows \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$$

where the first pair of maps is given by $(g_1^{\#}, g_2^{\#})$, the second map is $\phi^{\#}$, and the compositions agree and equal $h^{\#}$. Since Y is flat over X and local flat maps are faithfully flat by lemma 12.9, it follows that $\phi^{\#}$ is injective, and thus $g_1^{\#} = g_2^{\#}$. It follows that g_1, g_2 induce the same maps of stalks at every point, and hence they are the same.

2. We have already noticed that the desired g exists as a map of sets. Using that ϕ is an fpqc covering one can show[17] that $U \subseteq X$ is open if and only if $\phi^{-1}(U) \subseteq Y$ is. This clearly implies that g is continuous.

3. By the uniqueness above, it suffices to work locally on X. Thus take $x \in X$, and consider $y \in Y$ such that $\phi(y) = x$ and $h(y) = z$. Now let Z' be an affine open neighbourhood of z, let $X' = g^{-1}(Z')$ and $Y' = \phi^{-1}g^{-1}(Z')$. As we can work locally on X, replace X, Y, Z by X', Y', Z' to reduce to the case that X, Y, Z are all affine.

4. Write $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} C$. The sequence

$$\operatorname{Hom}(X,\ Z) \to \operatorname{Hom}(Y,\ Z) \rightrightarrows \operatorname{Hom}(Y_J \times_X Y, Z)$$

can be rewritten as

$$\operatorname{Hom}(C, A) \to \operatorname{Hom}(C, B) \rightrightarrows \operatorname{Hom}(C, B \otimes_A B).$$

Since the Hom functor is left-exact, the main lemma of fpqc descent (lemma 12.7) implies[18] that this sequence is exact. Thus, there exists a unique map $g : X \to Z$ inducing $h : Y \to Z$, which is what we needed to prove.

□

Remark 12.10. One can shorten the proof a little. For example, using lemma 12.1, we can reduce immediately to the case of X, Y being affine. At the moment, however, I don't see how to 'abstractly' reduce to the case of Z also being affine.

For completeness, we state without proof another incarnation of fpqc descent, in which we consider quasi-coherent sheaves on schemes.

Definition 12.11. Let S be a scheme. Let $\{f_i : S_i \to S\}_{i \in I}$ be a family of morphisms with target S. For every product $S_i \times_S S_j \times_S S_k$ denote by π_0, π_1, π_2 the three canonical projections, and by $\pi_{01}, \pi_{02}, \pi_{12}$ the canonical projections on pairs of factors.

- A **descent datum** $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{S_i \times S_j}$-modules $\varphi_{ij} : \pi_0^*\mathcal{F}_i \to \pi_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{tikzcd}
\pi_0^*\mathcal{F}_i \arrow[rr, "\pi_{02}^*\varphi_{ik}"'] \arrow[dr, "\pi_{01}^*\varphi_{ij}"'] & & \pi_2^*\mathcal{F}_k \\
& \pi_1^*\mathcal{F}_j \arrow[ur, "\pi^*\varphi_{jk}"'] &
\end{tikzcd}$$

of $\mathcal{O}_{S_i \times_S S_j \times_S S_k}$-modules commutes. This is called the *cocycle condition.*

- A descent datum is called **effective** if it comes by pullback from a single quasi-coherent sheaf $\mathcal{F}$ on S.

The main theorem of fpqc descent for quasi-coherent sheaves is the statement that the local descent data always glue together to give a globally defined sheaf; formally, we have the following result:

[17] first reduce to the case of ϕ being faithfully flat and quasi-compact (which an fpqc morphism, despite its name, needs not be!). Then reduce to the affine case, where the desired property follows from the fact that flat ring maps satisfy going-down. See [Vis08, Proposition 2.35 (vi)] and [Sta19, Lemma 02JY] for details.

[18] exercise: fill in the details, paying attention to the difference between *ring* homomorphisms and *A-module* homomorphisms. Here is the key observation: if $\beta = \varphi^\# \circ \gamma$, both β and $\varphi^\sharp$ are ring homomorphisms, and $\varphi^\#$ is injective, then γ is also a ring homomorphism.

Theorem 12.12. *Let $(S_i \to S)_{i \in I}$ be an fpqc covering. Every descent datum for quasi-coherent sheaves with respect to this family is effective.*

Remark 12.13. We give a topological interpretation of the cocycle condition. On the one hand, interpreting fibre products as intersections, the isomorphisms φ_{ij} encode the (obvious) requirement that, if $\mathcal{F}_i$ is to come by pullback from a single sheaf $\mathcal{F}$, $\mathcal{F}_i|_{S_i \cap S_j}$ should be isomorphic to $\mathcal{F}_j|_{S_i \cap S_j}$. On the other hand, these isomorphisms should be compatible in the sense that if we further restrict to a triple intersection $S_i \cap S_j \cap S_k$ and we compose the restriction of $\varphi_{ij} : \mathcal{F}_i|_{S_i \cap S_j} \cong \mathcal{F}_j|_{S_i \cap S_j}$ to $S_i \cap S_j \cap S_k$ with the restriction of $\varphi_{jk} : \mathcal{F}_j|_{S_j \cap S_k} \cong \mathcal{F}_k|_{S_j \cap S_k}$ to $S_i \cap S_j \cap S_k$, then we should get the restriction of $\varphi_{ik} : \mathcal{F}_i|_{S_i \cap S_k} \cong \mathcal{F}_k|_{S_i \cap S_k}$ to $S_i \cap S_j \cap S_k$.

Finally, *affine* schemes also descend along fpqc coverings:

Theorem 12.14. *Let $Y \to X$ be faithfully flat. Suppose $Z' \to Y$ is an affine scheme, and let $\varphi : Z' \times_X Y \to Y \times_X Z'$ be an isomorphism of $Y \times_X Y$-schemes. Denote by $\varphi_{01} : Z' \times_X Y \times_X Y \to Y \times_X Z' \times_X Y$ the obvious map induced by φ, and define similarly*

$$\varphi_{02} : Z' \times_X Y \times_X Y \to Y \times_X Y \times_X Z', \quad \varphi_{12} : Y \times_X Z' \times_X Y \to Y \times_X Y \times_X Z'.$$

Suppose that $\varphi_{02} = \varphi_{12} \circ \varphi_{01}$. Then (up to isomorphism) there exists a unique pair (Z, ψ) consisting of an affine X-scheme Z and an isomorphism $\psi : Z \times_X Y \to Z'$ such that, under the identification induced by ψ, the map φ on $Z' \times_X Y \cong Z \times_X Y \times_X Y \to Y \times_X Z \times_X Y$ becomes simply $(z, y_1, y_2) \mapsto (y_1, z, y_2)$.

Remark 12.15. Theorem 12.14 is stated for a single fpqc morphism $Y \to X$ because the result is easier to formulate in this situation. However, the case of a general fpqc covering is not very different: one can simply replace a fpqc covering family $(Y_i \to X)_{i \in I}$ with the single morphism $\coprod Y_i \to X$.

12.2.2 Proof of theorems 12.3 and 12.4

We are now ready to finish the proof of theorems 12.3 and 12.4:

Proof of theorem 12.3. By lemma 12.1 we need to check that $\mathcal{F}$ is a Zariski sheaf and that for every (fpqc) cover $U \to V$ with U, V affine the sequence

$$\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is exact.

- $\mathcal{F}$ is a Zariski sheaf. This amounts to saying that for every $f : U \to S$ and for every jointly surjective collection of open immersions $(U_i \hookrightarrow U)_{i \in I}$ we have

  $$\mathcal{F}(U) = H^0\left((U_i \to U)_{i \in I}, \mathcal{F}\right),$$

  that is,

  $$\Gamma(U, f^*\mathcal{G}) = \left\{ (s_i) \in \prod_{i \in I} f^*\mathcal{G}(U_i) \mid s_i|_{U_i \times U_j} = s_j|_{U_i \times U_j} \right\}.$$

  But this is simply the fact that $f^*\mathcal{G}$ is a sheaf (in the usual sense) on U.

- $\mathcal{F}$ satisfies the sheaf condition for affine fpqc coverings. Write $U = \operatorname{Spec} A, V = \operatorname{Spec} B$ and let $A \to B$ be a faithfully flat ring morphism.

  As U, V are affine, the pullback of $\mathcal{G}$ to U is of the form $\widetilde{M}$, and its pullback to V is $\widetilde{M \otimes_A B}$. Similarly, $V \times_U V = \operatorname{Spec}(B \otimes_A B)$ and the pullback of $\mathcal{G}$ to $V \times_U V$ is $M \otimes_A \widetilde{B \otimes_A} B$. Taking global sections on U, V and $V \times_U V$ we see that we need to prove exactness of

  $$M \to B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M;$$

notice that the two maps $B \otimes_A M \to B \otimes_A B \otimes_A B$ are $(b \otimes m) \mapsto (b \otimes 1 \otimes m)$ and $(b \otimes m) \mapsto (1 \otimes b \otimes m)$. Thus in order to prove exactness of this sequence it is enough to prove the exactness of

$$0 \to M \to B \otimes_A M \xrightarrow{d_0} B \otimes_A B \otimes_A B,$$

where $d_0(b \otimes m) = (b \otimes 1 - 1 \otimes b) \otimes m$. But this is precisely the content of the main lemma of fpqc descent (lemma 12.8) since $A \to B$ is faifhully flat by assumption.

□

As already mentioned, theorem 12.4 is essentially equivalent to theorem 12.5:

Proof of theorem 12.4. We check directly that h_X is an fpqc sheaf. Let $\mathcal{U} = (X_i \to U)_{i \in I}$ be an fpqc covering of an S-scheme U. Then we need to check that

$$h_X(U) \to H^0(\mathcal{U}, h_X)$$

is an isomorphism, that is, that a morphism $U \to X$ is determined by its restrictions to $U_i \to U \to X$ (which we checked in part (1) of the proof of theorem 12.5), and that given a compatible collection of morphisms $\varphi_i : U_i \to X$ with $\varphi_i|_{U_i \times_X U_j} = \varphi_j|_{U_i \times_X U_j}$ there is a (unique) morphism $\varphi : U \to X$ such that $U_i \to U \to X$ coincides with φ_i – but this is precisely the statement of theorem 12.5. □

12.3 Some important examples of étale sheaves

In this section we list several simple but fundamental examples of representable étale sheaves. The fact that they are in fact étale sheaves follows at once from theorem 12.4.

12.3.1 Additive group $\mathbb{G}_a$

Consider the scheme $\mathbb{G}_{a,S} := \mathrm{Spec}(\mathbb{Z}[t]) \times_{\mathrm{Spec}\,\mathbb{Z}} S$. On an S-scheme T, the sheaf represented by $\mathbb{G}_{a,S}$ takes the value

$$\begin{aligned}\mathrm{Hom}_S(T, \mathbb{G}_{a,S}) &= \mathrm{Hom}_{\mathbb{Z}}(T, \mathrm{Spec}\,\mathbb{Z}[t]) \\ &= \mathrm{Hom}(\mathbb{Z}[t], \Gamma(T, \mathcal{O}_T) \\ &= \Gamma(T, \mathcal{O}_T).\end{aligned}$$

12.3.2 Multiplicative group $\mathbb{G}_m$

Let $\mathbb{G}_{m,S} := \mathrm{Spec}(\mathbb{Z}[t, 1/t]) \times_{\mathrm{Spec}\,\mathbb{Z}} S$. On an S-scheme T, the sheaf represented by $\mathbb{G}_{m,S}$ takes the value

$$\begin{aligned}\mathrm{Hom}_S(T, \mathbb{G}_{m,S}) &= \mathrm{Hom}_{\mathbb{Z}}(T, \mathrm{Spec}\,\mathbb{Z}[t, 1/t]) \\ &= \mathrm{Hom}(\mathbb{Z}[t, 1/t], \Gamma(T, \mathcal{O}_T) \\ &= \Gamma(T, \mathcal{O}_T)^\times \\ &= \Gamma(T, \mathcal{O}_T^\times).\end{aligned}$$

Thus $\mathbb{G}_{m,S}$ is a subsheaf of sets of $\mathbb{G}_{a,S}$ (but certainly not an abelian subsheaf!).

12.3.3 Roots of unity

Let n be a positive integer. Define the fpqc sheaf

$$\mu_{n,S} := \ker(\mathbb{G}_{m,S} \xrightarrow{\cdot^n} \mathbb{G}_{m,S}).$$

Proposition 12.16 (Kummer sequence). *If n is invertible on S (that is, if $n \in \Gamma(S, \mathcal{O}_S)^\times$) then*

$$0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{\cdot^n} \mathbb{G}_{m,S} \to 0$$

*is an exact sequence of abelian sheaves on $S_{\text{ét}}$, called the n-**Kummer sequence**.*

Proof. The only non-obvious part of the statement is surjectivity of $\mathbb{G}_{m,S} \xrightarrow{\cdot^n} \mathbb{G}_{m,S}$. According to our general characterisation of surjective maps between abelian sheaves on topoi (theorem 11.30), we need to prove that for every[19] S-scheme U and every section $s \in \mathbb{G}_m(U)$ there exists an étale covering $(U_i \to U)_{i \in I}$ such that for every $i \in I$ the section $s|_{U_i}$ is in the image of $\mathbb{G}_{m,S}(U_i) \xrightarrow{\cdot^n} \mathbb{G}_{m,S}(U_i)$. We shall construct such a covering consisting of a single scheme V.

The fact that n is invertible on S implies that n is invertible on U, and a section $s \in \mathbb{G}_{m,S}(U)$ is simply an element of $\Gamma(U, \mathcal{O}_U)^\times$. Cover U with affine opens $\operatorname{Spec} A_j \xrightarrow{\iota_j} U$ and let $B_j = A_j[t]/(t^n - \iota_j^\#(s))$. Then the collection $\operatorname{Spec}(B_j)$ glues to give a scheme $V \to U$; we claim that the map $V \to U$ is étale. Since being étale is a (Zariski-)local property, it suffices to check that $\operatorname{Spec} A_j[t]/(t^n - \iota_j^\#(s)) \to \operatorname{Spec} A_j$ is étale, which follows from the Jacobian criterion since n is invertible on U (hence a fortiori on A_j). On the other hand, it's clear that t gives a section of $\mathbb{G}_{m,S}(V)$ with the property that $t^n = s$, so taking $V \to U$ as our étale covering we have that $s|_V$ is in the image of $\cdot^n : \mathbb{G}_{m,S}(V) \to \mathbb{G}_{m,S}(V)$. □

12.3.4 Constant sheaves

Let C be an abelian group. The Zariski sheafification of the constant presheaf[20] on S_{Zar} with value C is the sheaf

$$C_S : U \mapsto C^{\pi_0(U)}.$$

Lemma 12.17. *The following hold:*

1. *C_S is representable by the (group) scheme $S \times_{\operatorname{Spec} \mathbb{Z}} C$, hence it is an fpqc sheaf (and therefore also an étale sheaf). Here (by abuse of notation) we denote by C the constant group scheme attached to C, that is, the disjoint union of copies of* $\operatorname{Spec} \mathbb{Z}$ *indexed by elements of C.*

2. *For an arbitrary abelian sheaf $\mathcal{F}$ on $S_{\text{ét}}$ we have*

$$\operatorname{Hom}_{\mathbf{Ab}}(C, \mathcal{F}(S)) = \operatorname{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(C_S, \mathcal{F}).$$

   In other words, the functors that sends an abelian group C to the constant sheaf with value C and the evaluation functor $\mathcal{F} \mapsto \mathcal{F}(S)$ are an adjoint pair of functors between $\mathbf{Ab}(X_{\text{ét}})$ and $\mathbf{Ab}$.

3. *The constant sheaf $(\mathbb{Z}/n\mathbb{Z})_S$ is isomorphic (as an étale sheaf) to μ_n if and only if there exists a primitive root of* 1 *on S, that is, an element of $\mu_n(S)$ which has exact order n on each connected component of S.*

Proof. 1. For every S-scheme U we have

$$\operatorname{Hom}_S(U, S \times_{\operatorname{Spec} \mathbb{Z}} C) = \operatorname{Hom}_{\mathbb{Z}}(U, C) = \operatorname{Hom}_{\mathbb{Z}}\left(\coprod U_i, C\right) = \prod \operatorname{Hom}_{\mathbb{Z}}(U_i, C),$$

where U_i are the connected components of U. As U_i is connected and C is a disjoint union of copies of $\operatorname{Spec} \mathbb{Z}$, any homomorphism from U_i to C factors through a single copy of $\operatorname{Spec} \mathbb{Z}$ (and these are indexed by C); furthermore, for any U_i there is a unique morphism of schemes $U_i \to \operatorname{Spec} \mathbb{Z}$, so we get

$$\operatorname{Hom}_S(U, S \times_{\operatorname{Spec} \mathbb{Z}} C) = \prod C = C^{\pi_0(U)}.$$

[19] this actually depends on whether we are working with the small or big étale topos: in the former case, we may assume that $U \to S$ is étale, but this does not change the rest of the proof

[20] that is, the presheaf that takes any nonempty scheme U to the abelian group C

2. Denote by $\mathbf{P}C$ the constant *pre*sheaf with value C. By the universal property of sheafification we have
$$\mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}\left(C_S,\mathcal{F}\right)=\mathrm{Hom}_{\mathbf{PAb}(X_{\text{ét}})}\left(\mathbf{P}C,\mathcal{F}\right), \tag{12}$$
and a homomorphism of presheaves $\mathbf{P}C\to\mathcal{F}$ induces in particular a homomorphism of abelian groups $C=\mathbf{P}C(S)\to\mathcal{F}(S)$. On the other hand, given a homomorphism $\varphi : C\to\mathcal{F}(S)$, for every object $U\in S_{\text{ét}}$ we can construct $C=C(U)\to\mathcal{F}(U)$ by simply putting
$$C(U)\cong C\cong C(S)\xrightarrow{\varphi}\mathcal{F}(S)\xrightarrow{\cdot|_U}\mathcal{F}(U);$$
it is clear that these morphisms glue to give a homomorphism $\mathbf{P}C\to\mathcal{F}$, and by commutativity of the diagram
$$\begin{tikzcd} C(S) \arrow[r, "\varphi"] \arrow[d, "|_U"'] & \mathcal{F}(S) \arrow[d, "|_U"] \\ C(U) \arrow[r] & \mathcal{F}(U) \end{tikzcd}$$
this is the *only* morphism extending $\varphi : C(S)\to\mathcal{F}(S)$. It follows that
$$\mathrm{Hom}_{\mathbf{PAb}(X_{\text{ét}})}\left(\mathbf{P}C,\mathcal{F}\right)=\mathrm{Hom}_{\mathbf{Ab}}(C,\mathcal{F}(S)),$$
which combined with (12) gives the result.

3. Thanks to the adjunction in (2) we have
$$\mathrm{Hom}_{\mathbf{Ab}(S_{\text{ét}})}((\mathbb{Z}/n\mathbb{Z})_S,\mu_{n,S})=\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z},\mu_{n,S}(S))=\{x\in\Gamma(S,\mathcal{O}_S)^\times : x^n=1\},$$
so homomorphisms are in bijection with n-th roots of unity x on S. By restricting to every connected component S_i of S, it is clear that the map corresponding to x can only be an isomorphism if $x|_{S_i}$ has order n for every i. Conversely, if $x|_{S_i}$ has order n for every i, we want to show that the corresponding map is an isomorphism. By theorem 11.30 this is equivalent to the fact that for any object $U\in\mathbf{Sch}/S$ the induced map $(\mathbb{Z}/n\mathbb{Z})_S(U)\to\mu_{n,S}(U)$ is an isomorphism.

   We claim that for any U we have $\mu_{n,S}(U)\cong(\mathbb{Z}/n\mathbb{Z})^{\pi_0(U)}$: indeed since $\mu_{n,S}$ is a sheaf we have that $\mu_{n,S}(U)$ injects into $\prod_i\mu_{n,S}(U_i)$, and as U_i is connected $\{x\in\Gamma(U_i,\mathcal{O}_{U_i}^\times)\mid x^n=1\}$ is a subgroup of (a group isomorphic to) $\mathbb{Z}/n\mathbb{Z}$. Conversely, U_i maps to some connected component S_j of S, and the induced map of rings $\Gamma(S_j,\mathcal{O}_{S_j})\to\Gamma(U_i,\mathcal{O}_{U_i})$ is injective on units (since it cannot be the zero map), so (since by assumption $\mu_{n,S}(S_j)\cong\mathbb{Z}/n\mathbb{Z}$) we have that the n-torsion in $\Gamma(U_i,\mathcal{O}_{U_i}^\times)$ has order n. This proves the claim.

   Now to check that $(\mathbb{Z}/n\mathbb{Z})_S(U)=(\mathbb{Z}/n\mathbb{Z})^{\pi_0(U)}\to\mu_{n,S}(U)\cong(\mathbb{Z}/n\mathbb{Z})^{\pi_0(U)}$ is an isomorphism, it is enough to restrict to every connected component U_i of U and notice that $1\in(\mathbb{Z}/n\mathbb{Z})_S(U_i)$ is carried to an element of exact order n (the restriction of x to U_i, which is of order n by the argument above).

□

12.4 Exercises

Exercise 12.18. Let $\varphi : Z\to X$ be a morphism of schemes and let $\psi : Y\to X$ be a faithfully flat morphism. Let $\varphi_Y : Y\times_X Z\to Y$ be the base change of φ along ψ. Prove that φ is étale if and only if φ_Y is. Deduce that étaleness is an fpqc-local property: if $(U_i\xrightarrow{f_i}X)_{i\in I}$ is an fpqc covering, then φ is étale iff $\varphi\circ f_i$ is étale for every $i\in I$.

Exercise 12.19. A sheaf of abelian groups $\mathcal{F}\in\mathbf{Ab}(S_{\text{ét}})$ is said to be **locally constant** if there exists an étale covering $\{U_i\to S\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf (that is, it is isomorphic to the sheafification of the constant presheaf $U\mapsto C$ for some fixed abelian group C). A sheaf of abelian groups is said to be **finite locally constant** if it is locally constant and the values are finite abelian groups. Suppose that S is connected and that $\mathcal{F} \in \mathbf{Ab}(S_{\text{ét}})$ is a finite locally constant sheaf. Prove that

1. the group C in the definition above is the same for every U_i in the covering.
2. there is a finite étale morphism $U \to X$ such that $\mathcal{F} = h_U$.

Exercise 12.20. Let X be a scheme. Prove that for every n the (Kummer) sequence of sheaves $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m \to 0$ is exact **in the fppf topology**, that is, in $\mathbf{Ab}\left((\mathbf{Sch}/X)_{\mathrm{fppf}}\right)$. Show with an example that this needs not be true in the étale topology.

Exercise 12.21. Consider the presheaf

$$\begin{array}{rccc} \mathcal{F}: & (\mathbf{Sch}/S)^{\mathrm{opp}} & \to & \mathbf{Ab} \\ & T & \mapsto & \Gamma(T, \Omega_{T/S}). \end{array}$$

Prove that $\mathcal{F}$ is a sheaf for the étale topology but not for the fpqc topology (hint for the counterexample: work in characteristic p and consider a non-étale cover of the affine line).

13 Stalks of étale sheaves

13.1 Rigidity of sections of étale maps

As we have seen in multiple circumstances, étale maps are the algebraic equivalent of topological covering maps; just like topological coverings, they also satisfy strong rigidity properties of their sections. All the properties we need are in fact already true for unramified separated morphisms:

Proposition 13.1. *1. Any section of an unramified morphism is an open immersion.*

2. Any section of a separated morphism is a closed immersion.

3. Any section of an unramified separated morphism is open and closed.

Proof. Fix a base scheme S. If $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $\Delta_{X/S} : X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$. If $g : X \to S$ is separated (resp. unramified) then the diagonal is a closed immersion (resp. open immersion) by definition (resp. by lemma 7.8). Hence so is Γ_f, because being an closed/open immersion is stable under base change. In the special case $X' = S$, we obtain (1), resp. (2). Part (3) follows on combining (1) and (2). □

Lemma 13.2. *Let X be a connected scheme and let $f : Y \to X$ be Unramified and separated. Let s be a section of f. Then s is an isomorphism onto an open connected component of Y. In particular, s is known if its value at a single points is known.*

Proof. By the previous lemma s is open and closed. Hence s is an isomorphism onto its image, which is both open and closed, hence a connected component. In other words, sections of f are in bijection with those open and closed subschemes Z of Y such that Z induces an isomorphism $Z \to X$. □

Lemma 13.3. *Consider a diagram*

$$\begin{tikzcd} Y \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

where Y is connected and $X \to S$ is unramified and separated. Suppose that:

1. there exists $y \in Y$ such that $f(y) = g(y) = x$;

2. the maps $\kappa(x) \to \kappa(y)$ induced by f, g coincide.

Then $f = g$.

Proof. The maps $f, g : Y \to X$ define maps $f', g' : Y \to X_Y = Y \times_S X$ which are sections of the structure map $X_Y \to Y$. Note that $f = g$ if and only if $f' = g'$. The structure map $X_Y \to Y$ is the base change of π and hence unramified and separated. Thus by the previous lemma (see the last sentence in the proof) it suffices to prove that f' and g' pass through the same point of X_Y, which is guaranteed by hypotheses (2) and (3) (namely we have $f'(y) = g'(y) \in X_Y$). □

13.2 Étale neighbourhoods and stalks

Definition 13.4. Let S be a scheme and s a point of the topological space $|S|$.

1. An étale neighbourhood of (S, s) is an étale morphism $(U, u) \to (S, s)$ that carries u to s.

2. If $\overline{s} : \operatorname{Spec} k^s \to X$ is a geometric point of S with topological image s, an étale neighbourhood of $\overline{s}$ is a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k^s \arrow[ur, "\overline{u}"] \arrow[r, "\overline{s}"'] & S \end{tikzcd}$$

where $\varphi : (U, u) \to (S, s)$ is an étale neighbourhood of (S, s). We write $(U, \overline{u}) \to (S, \overline{s})$ to denote an étale neighbourhood of $\overline{s}$.

3. Morphisms of étale neigbourhoods are defined in the obvious way.

Definition 13.5 (Stalks)**.** Let $\mathcal{F} \in \mathbf{PSh}(S_{\text{ét}})$. The **stalk** of F at s is the set

$$\mathcal{F}_{\overline{s}} := \varinjlim_{(U,\overline{u})\to(S,\overline{s})} \mathcal{F}(U),$$

where the colimit is taken over the (co)filtered system of étale neighbourhoods of $(S, \overline{s})$.

As the colimit is filtered we have the usual description of elements of $\mathcal{F}_{\overline{s}}$ as equivalence classes: an element of $\mathcal{F}_{\overline{s}}$ is a pair $((U, \overline{u}), s)$ with $s \in \mathcal{F}(U)$, and two pairs $((U_1, \overline{u}_1), s_1)$ and $((U_2, \overline{u}_2), s_2)$ represent the same element if there are morphisms of étale neighbourhoods $(V, \overline{v}) \to (U_1, \overline{u}_1)$ and $(V, \overline{v}) \to (U_2, \overline{u}_2)$ such that the images of s_1, s_2 in $\mathcal{F}(V)$ are the same.

Also notice that since taking the stalk of a sheaf corresponds to taking a cofiltered colimit it commutes both with colimits and with any finite limit, so it is exact. We record this fact as a lemma:

Lemma 13.6. *For any geometric point $\overline{s}$ the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact and commutes with arbitrary colimits.*

The following is also easy to check:

Lemma 13.7. *Let $\mathcal{F}$ be a presheaf on $S_{\text{ét}}$. Then for every geometric points $\overline{s}$ of S we have $(\mathcal{F}^{\#})_{\overline{s}} = \mathcal{F}_{\overline{s}}$.*

Theorem 13.8. *Let S be a scheme. The sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \in \mathbf{Ab}(S_{ét}) \tag{13}$$

is exact if and only if for every geometric point $\overline{s}$ of S the sequence of abelian groups

$$0 \to \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}} \to 0 \tag{14}$$

is exact.

Proof. We have already seen that $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is an exact functor, so for any exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ in $\mathbf{Ab}(S_{ét})$ and any geometric point $\overline{s}$ of S the sequence $0 \to \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}} \to 0$ is exact. Let us prove the other implication.

Let $\alpha : \mathcal{G} \to \mathcal{H}$ be a map of sheaves such that $\mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}}$ is surjective for all geometric points. Fix $U \in |S_{\text{ét}}|$ and $h \in \mathcal{H}(U)$. For every $u \in U$ choose some $\overline{u} \to u$ lying over u and an étale neighbourhood $(V_u, \overline{v_u}) \to (U, \overline{u})$ such that $h|_{V_u} = \alpha(g_{V_u})$ for some $g_{V_u} \in \mathcal{G}(V_u)$. This is possible since α is surjective on stalks. Then $\{V_u \to U\}_{u\in U}$ is an étale covering on which the restrictions of h are in the image of the map α, hence (since this holds for any h) α is surjective.

Now for injectivity: suppose that $\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is injective for every geometric point $\overline{s}$. Let $f \in \mathcal{F}(U)$ be a section that maps to 0 in $\mathcal{G}(U)$; we want to prove that f is zero. Suppose by contradiction that it is not. Let $(U_i \to U)_{i\in I}$ be a Zariski covering of U by open affines; as $\mathcal{F}$ is in particular a Zariski sheaf, there is some i such that $f|_{U_i} \neq 0$. We claim that f cannot be zero in all the stalks corresponding to the geometric points of U that factor via U_i. Indeed, by definition, if f

is zero in the stalk at $\overline{u}$ then it is zero in some (étale) neighbourhood of $\overline{u}$. Clearly the topological images of these étale neighbourhoods cover U_i, thus giving an étale cover $(V_{ij} \to U_i)_{j\in J}$ such that $f|_{V_{ij}} = 0$. But since $\mathcal{F}$ is a sheaf this implies $f = 0$ as desired.

Combining all of the above we see that a morphism of sheaves $\mathcal{F}_1 \to \mathcal{F}_2$ is an isomorphism if and only if $\mathcal{F}_{1,\overline{s}} \to \mathcal{F}_{2,\overline{s}}$ is an isomorphism for every $\overline{s}$. Applying this to $\mathrm{Im}(\mathcal{F} \to \mathcal{G})$ and $\ker(\mathcal{G} \to \mathcal{H})$ we obtain that exactness of (13) at $\mathcal{G}$ is equivalent to exactness of (14) at $\mathcal{G}_{\overline{s}}$ for every geometric point $\overline{s}$. Notice that since the stalk functors $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ are exact they commute with kernels and cokernels, so we have

$$\ker(\mathcal{G} \to \mathcal{H})_{\overline{s}} = \ker(\mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}})$$

and

$$(\mathrm{Im}(\mathcal{F} \to \mathcal{G}))_{\overline{s}} = \mathrm{Im}(\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}})$$

since images are kernels of cokernels. Thus (in the light of the results above) checking the isomorphism $\ker(\mathcal{G} \to \mathcal{H}) \cong \mathrm{Im}(\mathcal{F} \to \mathcal{G})$ boils down precisely to the exactness of (14). $\square$

13.3 Henselian rings

We now need to recall the notion of **Henselian rings**. These rings turn out to play a role in étale topology that is similar to that of local rings in the usual Zariski setting.

Definition 13.9 (Henselian rings)**.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. For polynomials $f \in R[t]$ we denote by $\overline{f} \in \kappa[t]$ the reduction of f modulo $\mathfrak{m}$.

1. R is said to be **henselian** if given a monic polyomial $f \in R[t]$ and an element $a_0 \in \kappa$ such that $\overline{f}(a_0) = 0$ and $\overline{f'}(a_0) \neq 0$ there exists a unique $a \in R$ that reduces to a_0 and is such that $f(a) = 0$.
2. R is said to be **strictly henselian** if moreover κ is separably closed.

The notion of coprimality for polynomials over a general ring is clearly more subtle than the corresponding notion for polynomials with coefficients in a field; the following lemma allows us to somewhat blur the distinction in our special case of a local ring:

Lemma 13.10. *Let $(R, \mathfrak{m})$ be a local ring and $f, g \in R[t]$ be polynomials with f monic. If the reductions $\overline{f}, \overline{g}$ are coprime, then $(f, g) = (1)$ in $R[t]$.*

Proof. Let $M = R[t]/(f, g)$. As f is monic, this is a finitely generated R-module. Furthermore, the hypothesis $(\overline{f}, \overline{g}) = (1)$ gives

$$(f, g) + \mathfrak{m}R[t] = R[t],$$

and quotienting out by (f, g) we obtain $\mathfrak{m}M = M$. By Nakayama's lemma, this implies $M = 0$. $\square$

Definition 13.11. Two polynomials $f, g \in R[t]$ are said to be **strictly coprime** if $(f, g) = (1)$.

We now give some useful equivalent characterisations of henselian rings:

Theorem 13.12. *The following are equivalent:*

1. *R is henselian.*
2. *if $f \in R[t]$ is a monic polynomial and $\overline{g}, \overline{h} \in \kappa[t]$ are monic polynomials satisfying $(\overline{g}, \overline{h}) = 1$ and $\overline{f} = \overline{g} \cdot \overline{h}$, there exist polynomials $g, h \in R[t]$ with $f = gh$, $\deg g = \deg \overline{g}$, and such that the reductions modulo $\mathfrak{m}$ of g, h are $\overline{g}, \overline{h}$.*
3. *any finite extension of R is a finite product of local rings.*
4. *for any étale morphism $R \to S$ and $\mathfrak{q} \in \operatorname{Spec} S$ lying over $\mathfrak{m}$ with $\kappa(\mathfrak{q}) = \kappa$ there exists a section $\tau : S \to R$ of $R \to S$ such that $\tau^{-1}(\mathfrak{m}) = \mathfrak{q}$.*

The proof of this theorem is quite involved, and relies on some tools from commutative algebra that we haven't discussed (in particular, Zariski's main theorem). As a consequence, we shall omit some of the more technical steps of the argument, but we hope to give the reader a good idea of how the result is proven:

Proof. We also consider the following multi-dimensional version of (1):

(5) Let $f_1, \dots, f_n \in R[t_1, \dots, t_n]$. If there exists $(a_1, \dots, a_n) \in \kappa^n$ such that $\overline{f_i}(a_1, \dots, a_n) = 0$ and $\det \left(\frac{\partial \overline{f_i}}{\partial t_j}(a_1, \dots, a_n) \right)_{i,j} \neq 0$, then there exists $(b_1, \dots, b_n) \in R^n$ such that $f_i(b_1, \dots, b_n) = 0$ for $i = 1, \dots, n$ and $(\overline{b_1}, \dots, \overline{b_n}) = (a_1, \dots, a_n)$.

Clearly (5) implies (1). We now focus on the other implications:

$(5 \Rightarrow 2)$ Write $f(t) = t^n + a_{n-1}t^{n-1} + \dots + a_0$. Let r, s be the degrees of $\overline{g}, \overline{h}$ respectively. In order to find a factorisation of f we want to solve the system of equations corresponding to the polynomial equality

$$t^n + a_{n-1}t^{n-1} + \dots + a_0 = (t^r + X_{r-1}t^{r-1} + \dots + X_0)(t^s + Y_{s-1}t^{s-1} + \dots + Y_0).$$

Explicitly,

$$\begin{cases} X_0Y_0 = a_0 \\ X_0Y_1 + X_1Y_0 = a_1 \\ \vdots \\ X_{r-1} + Y_{s-1} = a_{n-1} \end{cases}$$

This is a system of $n = r + s$ equations in the $r + s$ unknowns X_i, Y_j. We also know of a solution $(b_{r-1}, \dots, b_0, c_{s-1}, \dots, c_0)$ in κ^n, given by the factorisation $\overline{f} = \overline{g}\overline{h}$ we started from. In order to apply (5) we just need to check that the determinant of the Jacobian of these equations is nonzero in κ when evaluated at $(b_{r-1}, \dots, b_0, c_{s-1}, \dots, c_0)$. However, the Jacobian in question is

$$\begin{pmatrix} Y_0 & 0 & \cdots & \cdots & X_0 & 0 & 0 & 0 & \cdots \\ Y_1 & Y_0 & \cdots & \cdots & X_1 & X_0 & 0 & 0 & \cdots \\ Y_2 & Y_1 & Y_0 & \cdots & X_2 & X_1 & X_0 & 0 & \cdots \\ \vdots & \vdots & & & & \vdots & & & \end{pmatrix},$$

so its determinant is the resultant of the polynomials $t^r + X_{r-1}t^{r-1} + \dots + X_0$ and $t^s + Y_{s-1}t^{s-1} + \dots + Y_0$. When we evaluate at $(b_{r-1} \dots, b_0, c_{s-1}, \dots, c_0)$ we therefore obtain the resultant of $\overline{g}$ and $\overline{h}$, which is nonzero by assumption (the resultant of two polynomials $\overline{g}$ and $\overline{h}$ vanishes if and only if they have a common factor).

$(2 \Rightarrow 3)$ According to the going-up theorem, any maximal ideal of S lies over $\mathfrak{m}$. Thus S is local if and only if $S/\mathfrak{m}S$ is local.

Assume first that S is of the form $S = R[t]/(f)$ with $f(t)$ monic. If f is a power of an irreducible polynomial, then $S/\mathfrak{m}S = \kappa[t]/(\overline{f})$ is local and S is local. If not, then (2) implies that $f = gh$ where g and h are monic, strictly coprime, and of degree ≥ 1 (because such a factorisation exists over the residue field). Then $S \cong R[t]/(g) \times R[t]/(h)$ (by the Chinese Remainder Theorem: notice that $(g, h) = (1)$ by lemma 13.10), and this process may be continued to get the required splitting (and since $R \to S$ is finite, it also terminates).

Now let S be an arbitrary finite R-algebra. If S is not local, then there is a $b \in S$ such that $\overline{b}$ is a nontrivial idempotent in $S/\mathfrak{m}S$. Let f be a **monic** polynomial such that $f(b) = 0$; this exists since S is finite. Let $C = R[t]/(f)$, and let $\phi : C \to S$ be the map that sends t to b. Since C is of the form studied above, we know[21] that there is an idempotent $c \in C$ such that $\overline{\phi(c)} = b$. Now $\phi(c) = e$ is a nontrivial idempotent in B; $B = Be \times B(1 - e)$ is a non-trivial splitting, and the process may be continued (and terminates).

[21]first lift $\overline{b}$ to an idempotent in $C \otimes \kappa$ and then, using the first half of the proof, lift this to an idempotent in C

(3 ⇒ 4) The nontrivial difficulty lies in reducing from the general étale case to the **finite** étale case. This reduction relies on Zariski's main theorem, and we will not carry it out here. Once $f : R \to S$ is assumed to be finite as well as étale, (3) applies to show that we can assume S to be local with the same residue field as R. But then S, considered as an R-module, is finitely generated and flat (by definition of étale) and therefore free (since R is local). Since $S/\mathfrak{q}S = S \otimes_R R/\mathfrak{m}$ is isomorphic to $R/\mathfrak{m}$, this implies that S is in fact free of rank 1, so that $R \cong S$ and a section exists trivially.

(4 ⇒ 5) Consider the algebra $S = R[t_1, \ldots, t_n]/(f_1, \ldots, f_n)$. The hypothesis implies that there is a prime $\mathfrak{q}$ of S such that the Jacobian determinant is a unit in $S_\mathfrak{q}$. Hence we can choose a suitable $g \notin \mathfrak{q}$ for which S_g is étale over R; applying (4) to $R \to S_g$ and the prime $\mathfrak{q}$ we get a section $\tau : S_g \to R$, and it suffices to take $b_i = \tau(t_i)$.

Thus we see that 2, 3, 4, 5 are all equivalent, and as already remarked 5 implies 1. It now suffices to show that 1 implies 4 to complete the proof.

(1 ⇒ 4) Let $R \to S$ be étale and let $q \subset S$ be a prime with residue field κ. By theorem 10.25 we can find $g \in S$ such that $g \notin q$ and $R \to S_g$ is standard étale. Replacing S with S_g we may therefore write $S = R[t]_g/(f(t))$ with $f(t)$ monic and $g \in R[t]$. The prime q of S (which contains $\mathfrak{m}$ by assumption) has residue field κ, so it is of the form $q = (\mathfrak{m}, t - a)$ with $\overline{a}$ a root of $\overline{f}$. Furthermore, a is not a root of $\overline{g}$ (otherwise $g \in q$) and $\overline{a}$ is not a root of f' either (because of the definition of a standard étale map). It follows from 1 that $f(t)$ has a root a in R, and – noticing that $g(a)$ is not in $\mathfrak{m}$, so is a unit in R – the universal property of localisation yields the existence of a map $R[t]_g \to R$ that sends t to a. By definition, $f(a) = 0$, so this map factors through $R[t]_g/(f(t)) = S$ and gives the desired section

□

Morally, the content of this lemma is that (as long as one only considers étale extensions) a henselian ring is 'equivalent' to its residue field. We shall now turn this vague principle into formal statements:

Lemma 13.13. *Let $(R, \mathfrak{m})$ be a local ring with residue field κ. Let κ' be a finite separable extension of κ. There exists a local étale map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ of local rings such that the residue field of S is κ'.*

Proof. Write $\kappa' = \kappa[t]/(\overline{f}(t))$ with $\overline{f}(t)$ monic and separable. Fix any monic lift $f(t) \in R[t]$ of $\overline{f}(t)$: then $S = R[t]/(f(t))$ is étale over R and the ideal $\mathfrak{m}R'$ is maximal with residue field κ'. □

We now make precise the fact that étale extensions of henselian rings are determined by the corresponding extension of residue fields[22]

Lemma 13.14. *Let $(R, \mathfrak{m}, \kappa)$ be a local henselian ring. Reduction modulo $\mathfrak{m}$ establishes an equivalence of categories between the category of finite étale extensions $R \to S$ and the category of finite étale extensions $\kappa \to S/\mathfrak{m}S$.*

Proof. By theorem 13.12 it suffices to consider local extensions. The canonical map

$$\mathrm{Hom}_R(S_1, S_2) \to \mathrm{Hom}_\kappa(S_1 \otimes \kappa, S_2 \otimes \kappa)$$

is injective by lemma 13.3. Furthermore, notice that given a map $\overline{g} : S_1 \otimes \kappa \to S_2 \otimes \kappa$ we obtain an R-morphism $g : S_1 \to S_1 \otimes \kappa \to S_2 \otimes \kappa$, hence an R-morphism

$$\begin{array}{rccc} \rho : & S_2 \otimes S_1 & \to & S_2 \otimes \kappa \\ & b_2 \otimes b_1 & \mapsto & b_2 g(b_1) \end{array}$$

[22]this is familiar to all number theorists: the unramified extensions of $\mathbb{Q}_p$ (that is to say, the étale extensions of $\mathbb{Z}_p$) are in bijection with the finite extensions of the residue field $\mathbb{F}_p$

Consider now the étale map $S_2 \to S_1 \otimes S_2$. Since the map ρ is clearly a surjection, it identifies a maximal ideal $\mathfrak{q}$ of $S_1 \otimes S_2$ whose contraction in S_2 contains (hence coincides with) its maximal ideal. By theorem 13.12 we get a splitting $S_1 \otimes S_2 \to S_2$, which we can then compose with the canonical map $S_1 \to S_1 \otimes S_2$ to get a morphism $S_1 \to S_2$. We are thus in the situation of the following (commutative) diagram:

$$\begin{tikzcd} & S_2 \arrow[d] \\ S_1 \arrow[r, "\iota_1"] \arrow[d, "\pi"'] & S_2 \otimes_R S_1 \arrow[u, bend left, "\tau"] \arrow[d, "\pi"] \\ \kappa \otimes S_1 \arrow[r, "\overline{g}"'] & \kappa \otimes S_2 \end{tikzcd}$$

By diagram chasing one sees that the morphism $\tau \circ \iota_1$ induces $\overline{g}$ at the level of residue fields. This implies that the functor we are studying is fully faithful. Essential surjectivity follows from lemma 13.13. $\square$

The next result, that at this point is easy to prove, is fundamental: it states that (the spectra of) strictly henselian rings are the "true points" of the étale theory, in the sense that all their higher cohomology vanishes.

Lemma 13.15. *Let $(R, \mathfrak{m}, \kappa)$ be a local strictly henselian ring, let $S := \operatorname{Spec} R$, and let $\overline{s}$ denote a geometric point of S. Finally let $\mathcal{F}$ be an étale abelian sheaf on $S_{\text{ét}}$. Then $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$, and as a consequence $\Gamma(S, -)$ is an exact functor, so that $H^i_{\text{ét}}(S, \mathcal{F}) = 0$ for every étale sheaf on S and every $i > 0$.*

Proof. Take any étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$: by refining it we may assume that it is affine and local, hence by theorem 13.12 we see that there is a section $S \to U$ (notice that the residue field of both $\mathfrak{m}$ and of any prime lying over $\mathfrak{m}$ is necessarily κ, as κ is separably closed). This implies that any étale neighbourhood is dominated by $\mathrm{id} : (S, \overline{s}) \to (S, \overline{s})$, so this morphism is cofinal among étale neighbourhoods of $\overline{s}$. Thus the colimit defining the stalk at $\overline{s}$ coincides with taking sections over S itself,

$$\mathcal{F}_{\overline{s}} = \varinjlim_{(U,\overline{u}) \to (S,\overline{s})} \mathcal{F}(U) = \mathcal{F}(S).$$

Since taking stalks is exact (lemma 13.6) this implies that the functor $\Gamma(S, -)$ is also exact, hence that is derived functors are all zero. $\square$

13.4 Henselisation and strict henselisation

Definition 13.16. Let R be a local ring with maximal ideal $\mathfrak{m}$. A local homomorphism $h : R \to R^h$ is called the **henselisation** of R if it is universal among henselian extensions, that is, if for every local homomorphism $R \to S$ with S henselian there exists a unique way to make the following diagram commutative:

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "h"'] & S \\ & R^h \arrow[u, dotted] \end{tikzcd}$$

The **strict henselisation** of R is a local morphism $sh : (R, \mathfrak{m}) \to (R^{sh}, \mathfrak{q})$ such that for every local map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ with S strictly henselian and for every map $R^{sh}/\mathfrak{q} \to S/\mathfrak{n}$ there is a unique map φ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "sh"'] & S \\ & R^{sh} \arrow[u, dotted, "\varphi"'] \end{tikzcd}$$

and such that the map on residue fields is the given one.

The key to the existence of henselisations (which we will prove in the next section) is the following lemma:

Lemma 13.17. *Let $R \to S$ be a ring map with S local and henselian. Given*

1. *an étale ring map $R \to A$,*
2. *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,*
3. *a $\kappa(\mathfrak{p})$-algebra map $\tau : \kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,*

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$ and $f \bmod \mathfrak{q} = \tau$.

Proof. Consider $A \otimes_R S$. This is an étale algebra over S (since it's the base change of an étale map). Moreover, the kernel

$$\mathfrak{q}' = \ker(A \otimes_R S \to \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{m}_S) \to \kappa(\mathfrak{m}_S))$$

of the map induced by the map given in (3) is a prime ideal lying over $\mathfrak{m}_S$ with residue field equal to the residue field of S. Hence by theorem 13.12 there exists a unique splitting $\sigma : A \otimes_R S \to S$ with $\sigma^{-1}(\mathfrak{m}_S) = \mathfrak{q}'$. Set f equal to the composition $A \to A \otimes_R S \to S$. Uniqueness follows from lemma 13.3. □

13.5 Local rings for the étale topology

As already anticipated, the reason we are interested in Henselian rings is their intimate connection with the (étale) stalks of the structure sheaf of a scheme. Recall that the Zariski sheaf $\mathcal{O}_X$ can be considered as an incarnation of the étale sheaf $\mathbb{G}_a$; stalks of $\mathbb{G}_a$ correspond to strict henselisations:

Proposition 13.18. *The ring $\varinjlim_{(U,\overline{u})\to(X,\overline{x})} \mathcal{O}_{U,u}$, which is the stalk at $\overline{x}$ of the étale sheaf $\mathbb{G}_a$, is the strict henselisation $\mathcal{O}_{X,x}^{sh}$ of $\mathcal{O}_{X,x}$.*

Proof. Let's temporarily denote the ring $\varinjlim_{(U,\overline{u})\to(X,\overline{x})} \mathcal{O}_{U,u}$ by B. As a first step let's show that B is isomorphic to the colimit $\varinjlim_{(U,\overline{u})} \Gamma(U, \mathcal{O}_U)$ (which is by definition the stalk of $\mathbb{G}_a$ at $\overline{x}$). By definition we have

$$B = \varinjlim_{(U,\overline{u})\to(X,\overline{x})} \varinjlim_{\overline{u}\in V\subseteq U} \Gamma(V, \mathcal{O}_V),$$

where the colimit in V is taken over all the open *Zariski* neighbourhoods of $\overline{u}$ in U. On the other hand,

$$(\mathbb{G}_a)_{\overline{x}} = \varinjlim_{(U,\overline{u})\to(X,\overline{x})} \Gamma(U, \mathcal{O}_U).$$

It is then easy to construct morphisms $B \to (\mathbb{G}_a)_{\overline{x}}$ (indeed, it suffices to construct maps

$$\varinjlim_{\overline{u}\in V\subseteq U} \Gamma(V, \mathcal{O}_V) \to \varinjlim_{(U,\overline{u})\to(X,\overline{x})} \Gamma(U, \mathcal{O}_U),$$

and these certainly exist since the neighbourhoods $(V, \overline{u})$ are among those considered in the colimit) and also a morphism in the opposite direction, since for every $(U, \overline{u})$ we have a canonical morphism $\Gamma(U, \mathcal{O}_U) \to \varinjlim_{\overline{u}\in V\subseteq U} \Gamma(V, \mathcal{O}_V) \to B$. One checks that these two maps are inverse to each other. We now need to prove the following:

1. B is local;
2. the residue field of B is separably closed;

3. B is henselian;
4. for any local ring homomorphism $\mathcal{O}_{X,x} \to S$ with S strictly henselian, there is an extension to B which is furthermore unique if we specify the induced map on residue fields.

Let $\operatorname{Spec} R$ be a (Zariski) open neighbourhood of x in X and let $\mathfrak{p}$ be the prime of R that corresponds to the point x. Observe that we have a canonical identification of $R_{\mathfrak{p}}$ with $\mathcal{O}_{X,x}$ and of $\kappa(\mathfrak{p})$ with $\kappa(x)$. Consider étale neighbourhoods $(U, \overline{u}) \to (X, \overline{x})$ that factor through $\operatorname{Spec} R$. These are cofinal in the system of all étale neighbourhoods, because given any étale $U \to X$ we can consider the fibre product

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec} R \arrow[r, hook] & X \end{tikzcd}$$

which gives an étale map $Y \to R \hookrightarrow X$. Replacing U with a (Zariski) open neighbourhood of u we can also assume that $U = \operatorname{Spec} A$ is affine. Thus we see that we can equivalently write B as the colimit $\varinjlim_{(A,\mathfrak{n},\phi)} A$ over triples $(A, \mathfrak{n}, \phi)$, where A is an étale extension of R, $\mathfrak{n}$ is a prime ideal of A lying over $\mathfrak{p}$, and $\phi : \kappa(\mathfrak{n}) \to \kappa(x)^s$ is a morphism of $\kappa(\mathfrak{p})$-algebras. Notice that in order to take into account the role of the geometric point $\overline{u} \to U$ one would need a $\kappa(\mathfrak{p})$-algebra map $\kappa(\mathfrak{n})^s \to \kappa(x)^s$, but two triples $(A, \mathfrak{n}, \phi)$ and $(A, \mathfrak{n}, \phi')$ where ϕ, ϕ' coincide on $\kappa(\mathfrak{n})$ can be identified in the colimit $\varinjlim_{(A,\mathfrak{n},\phi)} A$: indeed, the object is A for both triples, and the map $A \to A$ is the identity. We now show that B satisfies the four properties above.

1. Locality. By definition we have $B = \varinjlim_{(U,\overline{u}) \to (X,\overline{x})} \mathcal{O}_{U,u}$, where each ring in the limit is local, with maximal ideal $\mathfrak{m}_{U,u}$. One checks without difficulty that $\varinjlim \mathfrak{m}_{U,u}$ is the unique maximal ideal $\mathfrak{m}_B$ in B. Indeed, if an element $b \in B$ is not in $\mathfrak{m}_B$ then it is represented by an element s in one of the rings $(\mathcal{O}_{U,u}, \mathfrak{m}_{U,u})$, and from $b \notin \mathfrak{m}_B$ we deduce $s \notin \mathfrak{m}_{U,u}$, whence $s \in \mathcal{O}_{U,u}^\times$ and therefore $b \in B^\times$.

2. The residue field of B is separably closed. From now on we use the representation of B as $\varinjlim_{(A,\mathfrak{n},\phi)} A$.

   The residue field of B is $\varinjlim_{(A,\mathfrak{n},\phi)} \kappa(\mathfrak{n})$, and each field $\kappa(\mathfrak{n})$ is a separable extension of $\kappa(x) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ since $R \to A$ is étale. Conversely, for each finite separable extension $\mathbb{F}$ of $\kappa(x)$ we can construct a triple $(A, \mathfrak{n}, \phi)$ as above in such a way that $A/\mathfrak{n} \cong \mathbb{F}$. Indeed, let $\overline{g}(t) \in \kappa(x)[t]$ be a monic, separable, irreducible polynomial such that $\kappa(x)[t]/(g(t)) \cong \mathbb{F}$. Fix a monic lift $g(t) \in R[t]$ of $\overline{g}(t)$ and let $A = R[t]/(g(t))$. Also denote by $\mathfrak{n}$ the prime ideal $(\mathfrak{p}A, t)$ of A, whose contraction to R is $\mathfrak{p}$. The map $R \to A$ is étale at $\mathfrak{n}$, because we can localise at $\mathfrak{n}$ and apply exercise 7.12: we are then reduced to checking that $(g, g') = 1$ in $A_{\mathfrak{n}}$, which is easily seen to be the case (we shall need a similar calculation for part 3 below, and we give more details there). Since the étale locus is open (Theorem 10.23), there exists $g \in A$ such that $R \to A_g$ is étale. Finally, we let ϕ be any $\kappa(\mathfrak{p})$-algebra map $\kappa(\mathfrak{n}A_g) \to \kappa(x)^s$: such a map exists because $\kappa(\mathfrak{n}A_g) \cong \mathbb{F}$ is separable over $\kappa(x)$ by assumption. This shows that the residue field of B contains the union of all finite separable extensions of $\kappa(x)$, and since it cannot be larger than this it must coincide with the separable closure of $\kappa(x)$ as desired.

3. B is henselian. Consider a monic polynomial $f(t) \in B[t]$ and a root $a_0 \in B/\mathfrak{m}_B$ of $\overline{f}$ with the property that $\overline{f}'(a_0) \neq 0$. Let $a \in B$ be an element whose class in the residue field is a_0. Given our description of B, the coefficients of $f(t)$ and a are all represented by elements of a certain ring A from a triple $(A, \mathfrak{n}, \phi)$ as above. By a little abuse of notation, we will denote by $a \in A$ an element representing $a \in B$, and by a_0 its class in $A/\mathfrak{n}$. Notice that $A/\mathfrak{n}$ is in general only an integral domain, and not necessarily a field.

   Let $C = A[t]/(f(t))$ and $\mathfrak{q} = (\mathfrak{n}C, t - a)$: this is a prime ideal of C, because
$$\frac{C}{\mathfrak{q}} \cong \frac{A[t]/(f(t))}{\mathfrak{q}} \cong \frac{A[t]}{(f(t), t-a, \mathfrak{n})} \cong \frac{(A/\mathfrak{n})[\overline{t}]}{(\overline{f}(t), \overline{t} - a_0)} \cong A/\mathfrak{n}$$

is an integral domain. Notice that this shows $\kappa(\mathfrak{q}) = \kappa(\mathfrak{n})$, and notice furthermore that the contraction of $\mathfrak{q}$ in A is simply $\mathfrak{n}$.

We claim that the map $A \to C$ is étale at $\mathfrak{q}$. To see this we can apply exercise 7.12: localising at $\mathfrak{q}$ (which we can do, because we are checking étaleness at that point) it suffices to show that $(f, f') = 1$ in $C_{\mathfrak{q}}$, that is, we need to prove that $(f(t), f'(t))$ is not contained in $\mathfrak{q}$. If it were, we would have $(f(t), f'(t), \mathfrak{n}C, t-a) \subseteq \mathfrak{q}$; but this is not the case, because $A[t]/(f(t), f'(t), \mathfrak{n}C, t-a) = (A/\mathfrak{n})[\bar{t}]/(\overline{f}(t), \overline{f'}(t), \bar{t}-a_0) = (0)$ since by assumption $\overline{f'}(t)$ and $\bar{t} - a_0$ are relatively prime. Finally, we also have a map $\kappa(\mathfrak{q}) \to \kappa(x)^s$ because as we have already seen we have a canonical identification $\kappa(\mathfrak{q}) = \kappa(\mathfrak{n})$, so we can use the map ϕ from the triple $(A, \mathfrak{n}, \phi)$. We have thus defined a triple $(C, \mathfrak{q}, \phi)$ such that $A \to C$ is étale at $\mathfrak{q}$, and since the composition of étale maps is étale we have $R \to C$ étale at $\mathfrak{q}$. Since the étale locus is open by theorem 10.23, there exists a localisation C_g of C (for some $g \in C, g \notin \mathfrak{q}$) such that the map $R \to C_g$ is étale. The triple $(C_g, \mathfrak{q}C_g, \phi)$ is then an element in the colimit defining B, and by definition the image in C_g of $t \in A[t]$ is a root of the polynomial f. Since we have a map $C_g \to B$, this shows that f also has a root in B as desired.

4. Universal property. It suffices to construct compatible R-algebra maps $A \to S$ for every $(A, \mathfrak{n}, \phi)$ as above, and we do so by applying lemma 13.17. Let us specify exactly how; notice that we already know that the residue field of B is separably closed, so it makes sense to ask that we are given a $\kappa(x)$-algebra map $\chi : B/\mathfrak{m}_B \to S/\mathfrak{m}_S$ between the separably closed residue fields.

   We have a natural map $R \to R_{\mathfrak{p}} = \mathcal{O}_{X,x}$, hence given a local ring homomorphism $\mathcal{O}_{X,x} \to S$ we get an induced map $R \to S$. We also have a map $R \to A$, which is étale by definition. The fact that $\mathcal{O}_{X,x} \to S$ is local implies that $\mathfrak{p} = R \cap \mathfrak{m}_S$, and by definition of the triples $(A, \mathfrak{n}, \phi)$ we have a prime $\mathfrak{n}$ of A lying over $\mathfrak{p}$. Finally, we are also given a $\kappa(\mathfrak{p})$-algebra map $\tau : \kappa(\mathfrak{q}) \xrightarrow{\phi} \kappa(x)^s = B/\mathfrak{m}_B \xrightarrow{\chi} S/\mathfrak{m}_S$, where the second arrow in this composition is the given map on (separably closed) residue fields. We are then in a position to apply lemma 13.17, which gives as desired the existence of a unique map $f_A : A \to S$ such that $f^{-1}(\mathfrak{m}_S) = \mathfrak{n}$ and with $f_A \bmod \mathfrak{n} = \tau$. Uniqueness of f_A implies that the maps we build for different A are all compatible, hence we get as desired a unique map $B = \varinjlim_{(A,\mathfrak{n},\phi)} A \to S$.

$\square$

13.6 The étale-local rings of k-varieties

In this last short section we show (corollary 13.21) that the local rings of smooth k-varieties for the étale topology only depend on the dimension of the variety. This is yet another indication of the fact that we've recovered a good notion of local isomorphism; in fact, just as in the topological/smooth case, we shall show that all these local rings are isomorphic to the corresponding local ring for the 'model' d-dimensional k-variety, namely $\mathbb{A}^d_k$.

Proposition 13.19. *If $\varphi : Y \to X$ is étale at y, then the map $\mathcal{O}_{X,\overline{\varphi(y)}} \to \mathcal{O}_{Y,\overline{y}}$ induced by φ is an isomorphism.*

Proof. After replacing Y by an open neighbourhood of y, we may suppose that it is étale over all of X (because the locus where a map is étale is open, theorem 10.23). Then every étale neighbourhood of y is in particular also an étale neighbourhood of x, and such neighbourhoods are cofinal in the set of all étale neighbourhoods of x. It follows that the two direct limits are canonically isomorphic. $\square$

Recall the following statement (theorem 10.16):

Theorem 13.20. *Let X be a k-variety and let x be a smooth point. There exists an open set U containing x and an étale morphism $U \to \mathbb{A}^n_k$.*

Combining these two facts we obtain:

Corollary 13.21. *Let k be an algebraically closed field and let X be a smooth variety over k of dimension d. Then all the étale-local rings $\mathcal{O}_{X,\overline{x}}$ are isomorphic to the étale-local ring of $\mathbb{A}^d$ at the origin.*

13.7 Exercises

Exercise 13.22.
1. Let $f : R \to S$ be a faithfully flat ring map. Prove that f is universally injective, that is, for any R-module N the map $N \to N \otimes_R S$ is injective.
2. Deduce that if $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is étale and local, then f is injective.
3. Prove that the henselisation map of a local ring is faithfully flat and therefore universally injective.
4. Prove that if $R \to S$ is faithfully flat and S is Noetherian, then R is also Noetherian (consider an ascending chain of ideals in R...). In particular, if R^h is Noetherian then so is R.
5. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Assume (or prove) that the $\mathfrak{m}$-adic completion of R, denoted by $\widehat{R}$, is faithfully flat over the henselisation of R (hint if you want to prove it: show that the completion of a local ring A is faithfully flat over A and that this this statement passes to the limit). Recall (or prove) that $\widehat{R}$ is Noetherian. Deduce that R^h is Noetherian.
6. Using that $R \to R^h$ is flat and that $R/\mathfrak{m} \to R^h/\mathfrak{m}R^h$ is an isomorphism, show that $R/\mathfrak{m}^n \to R^h/\mathfrak{m}^n R^h$ is an isomorphism for every n. Deduce that the $\mathfrak{m}$-adic completion of R is isomorphic to the $\mathfrak{m}R^h$-adic completion of R^h, hence that if R is Noetherian there is an embedding $R^h \hookrightarrow \widehat{R}$.

Exercise 13.23. Let A be a noetherian local ring, and let B be the intersection of all local Henselian rings $(H, \mathfrak{m}_H)$ with[23]

$$A \subseteq H \subseteq \widehat{A}, \quad \mathfrak{m}_A \subseteq \mathfrak{m}_H \subseteq \mathfrak{m}_{\widehat{A}} :$$

then B is Henselian, and $A \to B$ is the henselisation of A.

Exercise 13.24 (Henselisation of a DVR)**.** Let R be a DVR with fraction field K. Let

$$B = \{x \in \widehat{R} \mid x \text{ separable over } K\}.$$

1. Prove that B is Henselian (it can be useful to remember that the usual version of Hensel's lemma proves that the completion of a DVR is Henselian).
2. Prove that B is the henselisation of R.

[23]these inclusions make sense thanks to part (6) of the previous exercise

14 Spectral sequences

We start by recalling several facts we will use multiple times.

1. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be a pair of adjoint functors between Abelian categories. Then:
   - F is right exact and takes inductive limits to inductive limits
   - if G is exact, then F carries projective objects to projective objects
   - G is left exact and takes projective limits to projective limits
   - if F is exact, then G takes injective objects to injective objects
2. Projective limits are left exact and inductive limits are right exact. **Filtered** inductive limits of modules, sheaves of sets, sheaves of modules, sets (categories in which one can reason with elements) are exact.
3. In the category of presheaves, 'all constructions are trivial':
$$\varinjlim_{\mathbf{PSh}} \mathcal{F}_i(U) := \varinjlim \mathcal{F}_i(U), \quad \varprojlim_{\mathbf{PSh}} \mathcal{F}_i(U) := \varprojlim \mathcal{F}_i(U),$$
and for example the cohomology of a complex $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is simply given by the quotients $\frac{\ker \mathcal{G}(U) \to \mathcal{H}(U)}{\operatorname{imm} \mathcal{F}(U) \to \mathcal{G}(U)}$ the same is *not* true in the category of sheaves, where for example
$$\varinjlim_{\mathbf{Sh}} \mathcal{F}_i = \varinjlim_{\mathbf{PSh}} \mathcal{F}_i, \quad \varprojlim_{\mathbf{Sh}} \mathcal{F}_i = \left(\varprojlim_{\mathbf{PSh}} \mathcal{F}_i \right)^{\#},$$
4. As we have seen, there is an adjoint pair ${}^{\#} : \mathbf{PSh} \rightleftarrows \mathbf{Sh} : i$ with ${}^{\#}$ exact. It follows in particular that the forgetful functor i takes injective objects to injective objects

14.1 Spectral sequences: general idea

Let $X^\bullet$ be a complex (for example in the category of modules, or sheaves, or any concrete abelian category). Suppose that this complex is equipped with a filtration (by subcomplexes)
$$\cdots \supset F^{p-1}X^\bullet(X) \supset F^pX^\bullet(X) \supset F^{p+1}X^\bullet(X) \supset \cdots$$
The aim of spectral sequences is to relate the cohomology $H^n(X)$ with the cohomology of the quotients, namely the cohomology groups $H^q(F^p/F^{p+1})$. More precisely consider the filtration of $H^\bullet(X)$, given by
$$F^pH^n(X^\bullet) = \operatorname{imm}\left(H^n(F^pX) \to H^n(X^\bullet)\right).$$
By functoriality, we will have
$$\cdots \supset F^{p-1}H^n \supset F^pH^n \supset F^{p+1}H^n \supset \cdots$$
and our objective will be to describe the quotients $F^pH^n/F^{p+1}H^n$.

Lemma 14.1.
$$\frac{F^pH^n}{F^{p+1}H^n} = \frac{\{x \in F^pX^n : dx = 0\} + F^{p+1}X^n}{\{x \in F^pX^n : x = dy, y \in X^{n-1}\} + F^{p+1}X^n}$$

Proof. We need to describe
$$\begin{aligned}\frac{\operatorname{imm} H^n(F^p) \to H^n(X^\bullet)}{\operatorname{imm} H^n(F^{p+1}) \to H^n(X^\bullet)} &= \frac{\{x \in F^pX^n : dx = 0\} + B^nX/B^nX}{\{x \in F^{p+1}X^n : dx = 0\} + B^nX/B^nX} \\ &= \frac{\{x \in F^pX^n : dx = 0\} + B^nX}{\{x \in F^{p+1}X^n : dx = 0\} + B^nX};\end{aligned}$$

notice that in general if $C \subseteq A$ we have

$$\frac{A}{C + (B \cap A)} \cong \frac{A+B}{C+B},$$

and applying this in our case we obtain

$$\frac{\operatorname{imm} H^n(F^p) \to H^n(X^\bullet)}{\operatorname{imm} H^n(F^{p+1}) \to H^n(X^\bullet)} = \frac{\{x \in F^p X^n : dx = 0\}}{\{x \in F^{p+1} X^n : dx = 0\} + \{x \in F^p X^n : x = dy\}}.$$

The same remark again (with $B = F^{p+1}X^n$) then yields

$$\frac{\operatorname{imm} H^n(F^p) \to H^n(X^\bullet)}{\operatorname{imm} H^n(F^{p+1}) \to H^n(X^\bullet)} = \frac{\{x \in F^p X^n : dx = 0\} + F^{p+1}X^n}{\{x \in F^p X^n : x = dy\} + F^{p+1}X^n}.$$

□

We now try to *approximate* this quotient: instead of requiring that $dx = 0$, we ask that it belongs to $F^{p+r}X^n$ (the idea being that since F^pX 'goes to zero' with p this is a finite-level approximation of the condition $dx = 0$), and – to make things work on a formal level – for the denominator we only take $x = dy$ with $y \in F^{p-r+1}X$.

Definition 14.2. We set

$$\widehat{Z}^p_r := \{x \in F^p : dx \in F^{p+r}\}, \quad \tilde{Z}^p_r := \widehat{Z}^p_r + F^{p+1}, \quad Z^p_r = \tilde{Z}^p_r / F^{p+1},$$

and similarly

$$\widehat{B}^p_r = \{x \in F^p : \exists y \in F^{p-r+1}X \text{ with } dy = x\}, \quad \tilde{B}^p_r = \widehat{B}^p_r + F^{p+1}, \quad B^p_r = \widehat{B}^p_r / F^{p+1}.$$

We then have a chain of inclusions

$$Z^p_r \supset Z^p_{r+1} \supset \cdots \supset B^p_{r+1} \supset B^p_r \supset \cdots$$

Define

$$Z^p_\infty = \bigcap Z^p_r \supseteq \frac{\{x \in F^p : dx = 0\} + F^{p+1}}{F^{p+1}}$$

and

$$B^p_\infty = \bigcup B^p_r \subseteq \frac{\{x \in F^p : x = dy\} + F^{p+1}}{F^{p+1}}.$$

Definition 14.3. We say that the spectral sequence (which we haven't defined yet...) *weakly converges* if the following hold:

- $Z^p_\infty = \frac{\{x \in F^p : dx = 0\} + F^{p+1}}{F^{p+1}}$
- $B^p_\infty = \frac{\{x \in F^p : x = dy\} + F^{p+1}}{F^{p+1}}$
- $\bigcup F^pX = X$
- $\bigcap F^pX = (0)$

Further define $E^p_\infty = \frac{Z^p_\infty}{B^p_\infty}$; we have maps, in general neither injective nor surjective,

$$E^p_\infty \to E^p_r = \frac{Z^p_r}{B^p_r};$$

the 'interesting' situations will be those for which these maps are isomorphisms, at least for $r \gg 0$.

Remark 14.4. Notice that all the objects we defined depend on $X^\bullet$, and in particular are graded (with the grading induced from that of $X^\bullet$).

Definition 14.5. We say that the spectral sequence *converges* to the cohomology of X if it converges weakly and moreover for all fixed degree n we have

- $F^p H^n = H^n$ for n small enough and $F^p H^n = 0$ for p big enough, so that at every degree the filtration of the cohomology has a finite number of steps;
- for every p we have $Z^p_\infty X^n = Z^p_r X^n$, $B^p_\infty X^n = Z^p_r X^n$, $Z^p_\infty X^n = Z^p_r X^n$ for r big enough.

In this case we write

$$E^p_r \Rightarrow H(X)$$

Notice that these definitions are not so standard.

The reason why this construction can be interesting is that the various E^p_r are somewhat easier to compute (at least inductively) than the cohomology of the original complex $X^\bullet$. We have

$$E^p_r = \frac{Z^p_r}{B^p_r} = \frac{\widehat{Z}^p_r + F^{p+1}}{\widehat{B}^p_r + F^{p+1}} = \frac{\widehat{Z}^p_r}{\widehat{B}^p_r + (F^{p+1} \cap \tilde{Z}^p_r)},$$

where the intersection in the denominator equals $\{x \in F^{p+1} : dx \in F^{p+r}\}$. In particular, we have a map $Z^p_r \to F^{p+r}$ induced by d. The inductive construction of the E^r_p is captured by the following lemma:

Lemma 14.6. *The following hold:*

1. *d induces a map $E^p_r \to E^{p+r}_r$*
2. *$d^p_r \circ d^{p-r}_r = 0$*
3. *$E^p_{r+1} = \frac{\ker d^p_r}{\operatorname{imm} d^{p-r}_r}$*

The proof consists of straightforward verifications, which we leave as an exercise to the reader.

Exercise 14.7. Prove the Lemma.

The initial steps of this induction are easy to compute. Indeed for $r = 0$ we have:

$$\widehat{Z}^p_0 = \{x \in F^p : dx \in F^p\} = F^p$$

and

$$\widehat{B}^p_0 = \{x \in F^p : x = dy, y \in F^{p+1}\},$$

so that

$$E^p_0 = \frac{\widehat{Z}^p_0 + F^{p+1}}{\widehat{B}^p_0 + F^{p+1}} = \frac{F^p}{d(F^{p+1}) + F^{p+1}} = \frac{F^p}{F^{p+1}}.$$

In particular, the exact sequence of complexes

$$0 \to F^{p+1} \to F^p \to E^p_0 \to 0$$

induces $H(F^{p+1}) \to H(F^p) \to E^p_1 \to H(F^{p+1}) \to H(F^p)$ – recall that E^p_1 *is* the cohomology of E^p_0.

14.2 Double complexes

Let $A^{p,q}$ be a double complex, ie a large commutative diagram

$$\begin{tikzcd}
A^{p,q+1} \arrow[r, "d_O^{p,q+1}"] & A^{p+1,q+1} \\
A^{p,q} \arrow[u, "d_V^{p,q}"] \arrow[r, "d_O^{p,q}"'] & A^{p+1,q} \arrow[u, "d_V^{p+1,q}"']
\end{tikzcd}$$

with $d_O \circ d_O = 0$ whenever it makes sense (that is, the rows are complexes) and $d_V \circ d_V = 0$ whenever it makes sense (that is, the columns are complexes). We also assume that the complex is bounded from below and from the left, that is, $A^{p,q} = 0$ if $p < p_0$ or $q < q_0$. For simplicity we shall work with first-quadrant complexes, that is, $p_0 = q_0 = 0$.

Definition 14.8. The **total complex** of A is $T^n = \bigoplus_{p+q=n} A^{p,q}$ with differential

$$d_T^n = \sum_{p+q=n} d_T^{p,q},$$

where $d_T^{p,q} = d_O^{p,q} + (-1)^p d_V^{p,q}$. It is indeed a complex.

We now define a 'filtration by half-planes', namely

Definition 14.9. We set

$${}'F^pT^n = \bigoplus_{\substack{i \geq p \\ i+j=n}} A^{i,j}$$

and

$${}''F^pT^n = \bigoplus_{\substack{j \geq p \\ i+j=n}} A^{i,j}.$$

We will concentrate now on the first filtration, of course everything can be repeated for the second one. In this case we can write the graded modules Z^p, B^p slightly more explicitly. Consider the component of degree $p+q$ of the module Z^p. We have

$$Z_r^pT^{p+q} = \frac{\tilde{Z}_r^pT^{p+q}}{F^{p+1}T^{p+q}} \subset \frac{F^pT^{p+q}}{F^{p+1}T^{p+q}} = A^{p,q},$$

where the inclusion is given by mapping the class of $(x_{p,q}, x_{p+1.q-1}, \dots)$ in Z^pT^{p+q} in $x_{p,q}$. Under this identifications we will write $Z_r^p = \bigoplus Z_r^{p,q}$ with $Z_r^{p,q} \subset A^{p,q}$ and similarly $B_r^p = \bigoplus_q B_r^{p,q}$ with $B_r^{p,q} \subseteq A^{p,q}$, and $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$.

Remark 14.10 (Reinterpretation of the spectral sequence)**.** For every $r \geq 0$ we have a *page* of modules $Z_r^{p,q}$, $B_r^{p,q}$ and $E_r^{p,q}$. We think of these as being organized in a book of sorts, with the page indexed by r.

Lemma 14.11. *With the notation introduced above, we have the following description of $Z_r^{p \cdot q}$ and $B_r^{p \cdot q}$:*

$$Z_r^{p,q} = \left\{ x \in A^{p,q} : \exists\, x_{p+1}, \dots, x_{p+r-1} \text{ with } x_{p+i} \in A^{p+i,q-i}, \begin{array}{c} d_V(x) = 0 \\ d_V(x_{p+1}) = d_O(x) \\ d_V(x_{p+2}) = d_O(x_{p+1}) \\ \vdots \\ d_V x_{p+r-1} = d_O x_{p+r-2} \end{array} \right\}$$

$$B_r^{p,q} = \left\{ x \in A^{p,q} : \exists\, y_{p-r+1}, \dots, y_{p-1}, \text{ with } y_{p-i} \in A^{p-i,q+i-1} \begin{array}{c} d_V y_{p-r+1} = 0 \\ d_V y_{p-r+2} = d_O y_{p-r+1} \\ \vdots \\ d_O y_{p-1} = x \end{array} \right\}.$$

Proof. We have

$$Z_r^p T^{p+q} = \{\tilde{x} = (x_p, x_{p+1}, x_{p+2}, \cdots) : x_i \in A^{i,p+q-i}, d\tilde{x} \in F^{p+r}\} + F^{p+1}/F^{p+1}$$

The condition $d\tilde{x} \in F^{p+r}$ works out to conditions of the form $d_V(x_{i+1}) = d_O(x_i)$; quotienting out by F^{p+1} amounts to then only remembering x_p. The proof for $B_r^{p,q}$ is similar. □

The boundary map $d_r^p : E_r^p \to E_r^{p+r}$ can also be described explicitly: for $x \in Z_r^{p,q} \subset A_r^{p,q}$ and $x_{p+1}, \ldots, x_{p+r-1}$ as in description of $Z_r^{p,q}$ the boundary is induced by

$$dx = \pm d_O x_{p+r-1},$$

with the sign given by

$$\begin{cases} (-1)^{\lfloor (r-1)/2 \rfloor}, p \text{ odd} \\ (-1)^{\lfloor (r-2)/2 \rfloor}, p \text{ even} \end{cases}$$

To see why this is the boundary map, simply notice that when taking the boundary of something of the form $(x_p, x_{p+1}, \cdots, x_{p+r-1})$ everything cancels out (by definition) or is quotiented out by F^{p+2}, so the only relevant part of the differential is indeed $d_O x_{p+r-1} \in A^{p+r,q-r+1}$.

In particular, on the r-th page the differentials move from one diagonal to the next, and moves to the right by r steps.

Example 14.12. On the second page there is a differential

$$d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}.$$

Theorem 14.13 (Double complex spectral sequence)**.** *Let $A^{i,j} = 0$ for $i < 0$ or $j < 0$. Then the spectral sequence converges to $H(T)$. More precisely:*

1. *$F^0 T = T$, $F^0 H = H$ and $F^{n+1} T^n = 0$, hence $F^{n+1} H^n = 0$. In particular, we have a finite filtration of $H^n T$.*
2. *$Z_\infty^{p,q} = Z_{p+q+2}^{p+q}$, $B_\infty^{p,q} = B_{p+q+2}^{p,q}$, and therefore $E_\infty^{p,q} = E_{p+q+2}^{p,q}$. Even more,*

$$\{z \in F^p T : dz = 0\} + F^{p+1}/F^{p+1} = Z_\infty^{p,q},$$

so the spectral sequence weakly converges, and

$$E_\infty^{p,q} = E_{p+q+2}^{p,q} = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

Proof. Point 1 is trivial. For point 2 notice first that for $r > p+q+1$, the differentials $d_r^{p,q}$ and $d_r^{p-r,q+r}$ are zero (since the target or the source of this morphism are outside the first quadrant) hence $Z_r^{p,q} = Z_{r+1}^{p,q} = Z_\infty^{p,q}$ and similarly for $B^{p,q}$ and $E_r^{p,q}$. Moreover for $r > p+q+1$

$$Z_r^{p,q} = \{x \in A^{p,q} : \exists\, (x, x_{p+1,\ldots}) \in F^p T^{p+q} \text{ such that } d_T x = 0\}$$
$$B_r^{p,q} = \{x \in A^{p,q} : \exists\, (y, x_{p+1,\ldots}) \in F^{p-r+1} T^{p+q-1} \text{ such that } d_T x = 0\}$$

In particular the spectral sequence converges weakly. □

Example 14.14 (The first pages of the spectral sequence)**.** We now compute $E_r^{p,q}$ for $r = 0, 1, 2$. For $r = 0$ we had already seen that (in the general setting) $E^p = F^p/F^{p+1}$, and the differential d_0^p is the global differential induced on the quotient complex, which in the double context setting means

$$E_0^{p,q} = A^{p,q} \quad \text{and } d_0^{p,q} = (-1)^p d_V^{p,q}.$$

in other words, the zero-th page of the spectral sequence is

$$\begin{tikzcd}
A^{0,2} & A^{1,2} & A^{2,2} . \\
A^{0,1} \arrow[u, "d_V"] & A^{1,1} \arrow[u, "-d_V"] & A^{2,1} \arrow[u, "d_V"] \\
A^{0,0} \arrow[u, "d_V"] & A^{1,0} \arrow[u, "-d_V"] & A^{2,0} \arrow[u, "d_V"]
\end{tikzcd}$$

We denote the complexes give by the column of this page by $A^{p,\bullet}$. The horizontal differentials induce a morphism of complexes $d_0^{p,\bullet} : A^{p,\bullet} \longrightarrow A^{p+1,\bullet}$ and

$$E_1^{p,q} = H^q(A^{p,\bullet}) \quad \text{and } d_1^{p,q} = H^q(d_O^{p,\bullet}).$$

This module is also denote by $H_V^{p,q}(A)$ and we notice that, for a fixed q, the modules $H_V^{\bullet,q}(A)$ with the differentials $H^q(d_O^{p,\bullet})$ form a complex. Hence the $r = 1$ page of the spectral sequence is given by

$$H_V^{0,2}(A) \xrightarrow{H(d_O)} H_V^{1,2}(A) \xrightarrow{H(d_O)} H_V^{2,2}(A)$$

$$H_V^{0,1}(A) \xrightarrow{H(d_O)} H_V^{1,1}(A) \xrightarrow{H(d_O)} H_V^{2,1}(A)$$

$$H_V^{0,0}(A) \xrightarrow{H(d_O)} H_V^{1,0}(A) \xrightarrow{H(d_O)} H_V^{2,0}(A)$$

Hence we can compute $E_2^{p,q}$ by taking the (horizontal) cohomology of these complexes:

$$E_2^{p,q} = H^p\big(H_V^{\bullet,q}(A)\big).$$

which is often written as $H_O^p\big(H_V^{\bullet,q}(A)\big)$.

Example 14.15 (H^0 and H^1 from the spectral sequence)**.** We now use the spectral sequence above to compute H^0T and H^1T. For $n = 0$ we have $F^1H^0 = (0)$ and $F^0H^0 = H^0$, so

$$H^0 = \frac{F^0H^0}{F^1H^0} = E_\infty^{0,0}.$$

Notice that for $r \geq 2$ all the boundary maps coming into $E_2^{0,0}$ or going out of it respectively come from (0) and go to (0), so $E_\infty^{0,0} = E_2^{0,0}$. Hence

$$H^0 = E_\infty^{0,0} = E_2^{0,0} = H_O^0\left(H_V^0\left(A^{0,\bullet}\right)\right).$$

Let's now consider the case $n = 1$. We have

$$E_\infty^{1,0} = \frac{F^1H^1}{F^2H^1} = E_\infty^{1,0}, \quad E_\infty^{0,1} = \frac{F^0H^1}{F^1H^1};$$

an argument similar to the above says that $E_\infty^{1,0} = E_2^{1,0}$ (because all the boundary maps either come from or go to modules that vanish by assumption), while

$$E_\infty^{0,1} = E_3^{0,1} = \ker\left(E_2^{0,1} \to E_2^{2,0}\right).$$

From the previous example we obtain

Proposition 14.16 (Low degree exact sequence)**.** *The following sequence is exact:*

$$0 \to E_2^{1,0} \to H^1T \to E_2^{0,1} \to E_2^{2,0} \to H^2T$$

Remark 14.17 ($n = 2$)**.** We have $F^3 = 0 \subset F^2 \subset F^1 \subset F^0$, with $F^2H^2 = E_\infty^{2,0} \subseteq H^2$. One sees that $E_\infty^{2,0} = E_3^{2,0} = \operatorname{coker}\left(E_2^{0,1} \to E_2^{2,0}\right)$, so the low degree exact sequence can be rewritten as

$$0 \to E_2^{1,0} \to H^1T \to E_2^{0,1} \to E_2^{2,0} \to F^2H^2 \to 0.$$

14.3 Composing derived functors

Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be left exact, additive functors between abelian categories. Suppose that $\mathcal{A}, \mathcal{B}$ have enough injectives and that F maps injective objects to G-acyclic[24] objects.

Lemma 14.18.

$$R(G \circ F)(X) = RG(RF(X))$$

Proof. To compute derived functors one replaces $X^\bullet$ by an injective resolution, so we may assume that $X^\bullet$ consists of injective objects. Then $G(F(X^\bullet))$ is $R(G \circ F)(X)$. On the other hand, $RF(X) = F(X)$ because X consists of injective objects, and now $F(X)$ by assumption consists of G-acyclic objects, so it computes the derived functor of G, so

$$RG(RF(X)) = RG(F(X)) = G(F(X)) = R(G \circ F)(X)$$

□

14.3.1 Cartan-Eilenberg resolutions

Let $\mathcal{A}$ be an abelian category with sufficiently many injectives, let $X^\bullet$ be a complex in $\mathcal{A}$ with $X^n = 0$ for $n < 0$ (or more generally bounded below).

Theorem 14.19 (Cartan-Eilenberg resolution)**.** *In this setting, there is a double complex $I^{p,q}$ such that:*

1. $I^{p,q} = 0$ *for* $p < 0$ *or* $q < 0$*;*
2. $I^{p,q}$ *injective;*
3. *there is a map*
$$\varepsilon^p : X^p \to I^{p,0} \xrightarrow{d_V^{p,0}} I^{p,1} \to \cdots$$
such that this is an injective resolution of X^p*;*
4. *Consider the commutative diagram*

$$\begin{tikzcd}
I^{0,1} \arrow[r] & I^{1,1} \arrow[r] & I^{2,1} \arrow[r] & \cdots \\
I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
X^0 \arrow[r, "d_X^0"] \arrow[u, "\varepsilon_0"] & X^1 \arrow[r, "d_X^1"] \arrow[u, "\varepsilon_1"] & X^2 \arrow[r, "d_X^2"] \arrow[u, "\varepsilon_2"] & \cdots
\end{tikzcd}$$

then:

(a) $\ker d_O^{p,\bullet} : I^{p,\bullet} \to I^{p+1,\bullet}$ *is an injective resolution of* $\ker d_X^p$*;*

[24] that is, $R^iG(X) = 0$ for $i > 0$

(b) $\operatorname{imm} d^{p-1,\bullet} : I^{p-1,\bullet} \to I^{p,\bullet}$ *is an injective resolution of* $\operatorname{imm} d_X^p$*;*

(c) $H_O^{p,\bullet}(I)$ *is an injective resolution of* $H^p(X)$*.*

For the proof, we recall the following lemma:

Lemma 14.20. *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence of objects in* A*. Let* I_X *be an injective resolution of* X*,* I_Z *an injective resolution of* Z*. Then there exists an injective resolution* J *of* Y *and a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_X \arrow[r] & J \arrow[r] & I_Z \arrow[r] & 0 \end{tikzcd}$$

Proof. By induction; the first step consists in looking at

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d, "\varepsilon_X"'] & Y \arrow[r, "\pi"] \arrow[d, dotted, "\varepsilon_Y"'] & Z \arrow[r] \arrow[d, "\varepsilon_Z"'] & 0 \\ 0 \arrow[r] & I_X^0 \arrow[r] & I_X^0 \oplus I_Z^0 \arrow[r] & I_Z^0 \arrow[r] & 0 \end{tikzcd}$$

Using the injectivity of I_X^0, we have a map $\varphi : Y \to I_0^X$ such that $\varphi\alpha = \varepsilon_X$. We can define $\varepsilon_Y :\to I_X^0 \oplus I_Z^0$ by

$$\varepsilon_Y(y) = \big(\varphi(y), \varepsilon_Z \circ \pi(y)\big)$$

Now do the same for the cokernel of these maps and proceed by induction. □

Proof of theorem 14.19. We split the complex $0 \to X^0 \to X^1 \to X^2 \to \cdots$ in many exact sequences,

$$0 \to H^0 = Z^0 \to X^0 \to B^1 \to 0,$$
$$0 \to B^1 \to Z^1 \to H^1 \to 0,$$
$$0 \to Z^1 \to X^1 \to B^2 \to 0;$$

we fix injective resolutions I_B and I_H of B^i and of H^i for every i (including $H^0 = Z^0$), and (using the previous lemma) we construct injective resolutions I_Z and I_X of Z^i and X^i. By construction we have sort exact sequences among these complexes

$$0 \longrightarrow I_{Z^p} \xrightarrow{\alpha} I_{X^p} \xrightarrow{\beta} I_{B^{p+1}} \longrightarrow 0$$

$$0 \longrightarrow I_{B^p} \xrightarrow{\gamma} I_{Z^p} \xrightarrow{\delta} I_{H^p} \longrightarrow 0.$$

Now if we set $I^{p,q} = I^q_{X^p}$ the vertical differential d_V equal to the differential of the complex I_{X^p} and the horizontal differential equal to the map $\alpha \circ \gamma \circ \beta$, it is easy to check that it as all the required properties. □

Corollary 14.21. *Let* $I^{p,q}$ *be a Cartan-Eilenberg resolution and let* $T^\bullet$ *be the corresponding total complex. The map* $X^p \to T^p$ *induced by* $X^p \to I^{p,0} \to T^p$ *is a quasi-isomorphism.*

Proof. See exercise 14.24. □

14.3.2 Grothendieck's spectral sequence for derived functors

Theorem 14.22. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives and let $X^\bullet$ be a complex bounded below. There exists a spectral sequence*

$$E_2^{p,q} = R^p F\left(H^q(X)\right) \Rightarrow R^{p+q}F(X^\bullet).$$

Proof. Consider a Cartan-Eilenberg resolution of X, let's call it $I^{p,q}$. Let $T^\bullet$ be the total complex; we know that $X^\bullet$ and $T^\bullet$ are quasi-isomorphic, so $RF(X^\bullet) = F(T^\bullet)$. Now $F(T^\bullet)$ is the total complex corresponding to $F(I^{p,q})$. Thus: we want to compute $R^nF(X^\bullet)$, that is, $H^n(F(T^\bullet))$. By the analogue of Theorem 14.13 in the case of the filtration $''F^p = \bigoplus_{j\geq p} F(I^{i,j})$ there is a spectral sequence converging to $H^n(F(T^\bullet))$. We have

$$''E_0^{a,b} = F(I^{a,b})$$

and we have maps $d_0 : E_0^{a,b} \to E^{a,b+1}$. Thus

$$''E_1^{a,b} = H_O^a\left(F(I^{\bullet,b})\right) = H_O^{a,b}\big(F(I)\big),$$

which – by the properties of Cartan-Eilenberg resolutions – is equal to $F\big(H_O^{a,b}(I)\big)$. To see this, notice that to the diagram

$$\begin{tikzcd}
 & Z^{a-1}X \arrow[d, hook] & & & B^{a+1}X & \\
X^{a-2} \arrow[r, "d_X^{a-2}"] & X^{a-1} \arrow[rr, "d_X^{a-1}"] \arrow[dr, two heads] & & X^a \arrow[ur, two heads] \arrow[rr, "d_X^a"] & & X^{a+1} \\
 & & B^aX \arrow[r, hook] & Z^aX \arrow[u, hook] \arrow[r, two heads] & H^aX &
\end{tikzcd}$$

where all short exact sequences are exact, correposponds (by the properties of the Cartan Eilenberg resolution) a diagram of their resolutions, where all short sequences are also exact.

$$\begin{tikzcd}
 & I^\bullet_{Z^a} \arrow[d, hook] & & & I^\bullet_{B^{a+1}} & \\
I^{a-2,\bullet} \arrow[r, "d_O^{a-2,\bullet}"] & I^{a-1,\bullet} \arrow[rr, "d_O^{a-1,\bullet}"] \arrow[dr, two heads] & & I^{a,\bullet} \arrow[ur, two heads] \arrow[rr, "d_O^{a,\bullet}"] & & I^{a+1,\bullet} \\
 & & I^\bullet_{B^a} \arrow[r, hook] & I^\bullet_{Z^a} \arrow[u, hook] \arrow[r, two heads] & I^\bullet_{H^a} = H_O^{a,\bullet}(I) &
\end{tikzcd}$$

Since all the objects are injective, the short exact sequences stay exact after application of F, which gives the desired equality. In particular $H_O^{a,\bullet}(I)$ is a resolution of $H^a(X)$, so at the next stage we have

$$''E_2^{a,b} = R^bF\left(H^a(X)\right)$$

and

$$d_r^{a,b} : E_r^{a,b} \to E_r^{a-r+1,b+r};$$

notice that the indices are swapped with respect to the statement, but this is because we are considering the 'unusual' direction of the filtration; reorganizing the modules in the usual order, we get a spectral sequence as in the statement. □

Corollary 14.23 (Grothendieck spectral sequence)**.** *Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be left-exact additive functors of abelian categories with sufficiently many injectives; suppose that F carries injectives to G-adapted objects. There is a second-page spectral sequence*

$$E_2^{p,q} = R^pG\left(R^qF(X)\right) \Rightarrow R^{p+q}(G \circ F)(X).$$

Proof. Exercise. □

14.4 Exercises

Exercise 14.24. Let $A^{p,q}$ be a double complex. Suppose that all columns are exact, except at most in 0 (that is, $d_V^{p,0} : A^{p,0} \to A^{p,1}$ is not necessarily injective). Define $B^p = \ker d_V^{p,0}$. The double complex maps induce $d_0^p : B^p \to B^{p+1}$. We have maps $T^\bullet \to T^\bullet$ given by the fact that $B^p \subseteq A^{p,0} \subseteq T^p$. Prove that $B^\bullet \to T^\bullet$ is a quasi-isomorphism, that is, $H^n(B^\bullet) = H^n(T^\bullet)$.

Exercise 14.25. Prove corollary 14.23.[Apply Theorem 14.22 to $RF(X)$.]

15 Čech cohomology in the étale setting

As for coherent sheaves, or topological manifolds, it is possible to develop Čech cohomology also for étale topology exactly in the same way. Let X be a scheme, $\mathcal{U}$ an étale covering of X, and $\mathcal{F}$ an étale abelian (pre)sheaf on X. Consider the complex

$$\prod_{i_0} \mathcal{F}(U_{i_0}) \xrightarrow{d^0} \prod_{i_0,i_1} \mathcal{F}(U_{i_0} \times_X U_{i_1}) \xrightarrow{d^1} \prod_{i_0,i_1,i_2} \mathcal{F}(U_{i_0} \times_X U_{i_1} \times_X U_{i_2}) \to \cdots$$

We denote by $\check{C}^p$ the p-th term in the above complex. The differential $d^p : \check{C}^p \to \check{C}^{p+1}$ is

$$(d^p(\sigma))_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}|_{U_{i_0}\times_X \cdots \times_X U_{i_{p+1}}}.$$

Definition 15.1 (Čech cohomology)**.**

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{C}^\bullet).$$

We explain now the connection between Čech cohomology and étale cohomology. If $\mathcal{F}$ is a sheaf on X we introduce the presheaves $\underline{H}^p(\mathcal{F})$ given by

$$\underline{H}^p(\mathcal{F})(U) = H^p_{\text{ét}}(U, \mathcal{F}).$$

One could interpret $H^p_{\text{ét}}(U, \mathcal{F})$ in two different ways:

1. $H^p_{\text{ét}}(U, \mathcal{F}) = R^p\Gamma(U, -)$ and $\Gamma(U, -) : \mathbf{Ab}_X \longrightarrow \mathbf{Ab}$; this is how we think about it at the moment;

2. or we could consider the obvious map of small étale sites $U_{\text{ét}} \to X_{\text{ét}}$; clearly $\mathcal{F}$ restricts to a sheaf on $U_{\text{ét}}$, and we could compute its cohomology with respect to the étale site of U.

We will prove they are the same in the next lecture (see Corollary 16.8). Notice also that if $V \longrightarrow U \longrightarrow X$ are étale morphism over X then the restriction $\Gamma(U, \mathcal{F}) \longrightarrow \Gamma(V, \mathcal{F})$ induces a morphism

$$H^p_{\text{ét}}(U, \mathcal{F}) \longrightarrow H^p_{\text{ét}}(V, \mathcal{F})$$

that we will be the restrition map for the presheaf $\underline{H}^p(\mathcal{F})$. We will see soon that for $p > 0$ the associated sheaf is zero, see 15.8.

The relation between Čech cohomology and étale cohomology will be given by a spectral sequence associated to the following compositions of functors

$$\mathbf{Ab}_X \xrightarrow{\ \iota\ } \mathbf{PSh}_X \xrightarrow{\ \check{H}^0\ } \mathbf{Ab}$$

where ι is the forgetful functors. To apply the result of the previous section we need the following two lemmas.

Lemma 15.2. *Let $\iota : \mathbf{Ab}_X \to \mathbf{PAb}_X$ be the forgetful functor. Then*

1. *ι carries injectives to injectives;*

2. *$(R^p\iota\mathcal{F}) = \underline{H}^p_{\text{ét}}(\mathcal{F})$.*

Lemma 15.3. *Consider the functor $\check{H}^0 : \mathbf{PAb}_X \to \mathbf{Ab}$. We have*

$$R^p\check{H}^0 = \check{H}^p.$$

As a consequence we have the following relation between Čech cohomology and étale cohomology.

Theorem 15.4. *Let $\mathcal{U}$ be an étale covering of X. There is a spectral sequence $E_r^{p,q}$ with non zero terms only for $p \geq 0$ and $q \geq 0$ which converges to $H_{\text{ét}}^{p+q}(X, \mathcal{F})$ given by*

$$E_2^{p,q} = \check{H}^p\left(\mathcal{U}, \underline{H}^q(\mathcal{F})\right) \Rightarrow H_{\text{ét}}^{p+q}(X, \mathcal{F})$$

where $\underline{H}^q(\mathcal{F})$ is the presheaf given by $U \mapsto H^{\text{ét}}(U, \mathcal{F})$ defined above.

Proof of Theorem 15.4. Consider the functors

$$\mathbf{Ab}_X \xrightarrow{\iota} \mathbf{PSh}_X \xrightarrow{\check{H}^0} \mathbf{Ab}$$

By the sheaf condition, the composition is $H^0(X, -)$, and since by lemma 15.2 the functor ι carries injectives to injectives we can apply corollary 14.23 to get a spectral sequence

$$E_2^{p,q} = R^p\check{H}^0\left(\mathcal{U}, R^q\iota\mathcal{F}\right) \Rightarrow H_{\text{ét}}^{p+q}(X, \mathcal{F}).$$

Finally by lemma 15.3 we have $R^p\check{H}^0 = \check{H}^p$ and by lemma 15.2 we have $R^q\iota\mathcal{F} = \underline{H}^q(\mathcal{F})$, proving the claim. ☐

We leave te proof of lemma 15.2 as an exercise (exercise 15.11) and we prove lemma 15.3.

Proof of lemma 15.3. We show that:

i) given a short exact sequence of presheaves, the various $\check{H}^p$ fit naturally into a corresponding long exact sequence.

ii) $\check{H}^p(\mathcal{U}, I) = 0$ for every $p > 0$ if I is injective.

By the long exact sequence of derived functors, and induction, these two facts taken together imply the claim of the lemma (Exercise 15.10).

To prove i) take an exact sequence of presheaves,

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0;$$

for every $U \in X_{\text{ét}}$ we then obtain

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0. \tag{15}$$

The definition of the Čech cochains then easily implies that

$$0 \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to 0,$$

because (in each degree) the maps of cochains are obtained by taking products of exact sequences of the form (15). From the exact sequence of Čech complexes we obtain a long exact sequence in cohomology as desired.

To prove ii) we need to show that

$$0 \to \check{C}^0\left(\mathcal{U}, I\right) \to \check{C}^1\left(\mathcal{U}, I\right) \to \check{C}^2\left(\mathcal{U}, I\right) \to \cdots$$

is exact except in degree 0. We construct a complex of sheaves

$$\cdots \to Z^3 \to Z^2 \to Z^1 \to Z^0 \to 0,$$

exact except in degree 0, such that $\check{C}^\bullet(\mathcal{F}) = \operatorname{Hom}_{\mathbf{PAb}_X}(Z^\bullet, \mathcal{F})$ for every presheaf $\mathcal{F}$. If $\mathcal{F} = I$ is injective, taking $\operatorname{Hom}(-, I)$ is exact, and this will give the exactness of the complex of Čech cochains, except at most at 0. Given $U \in X_{\text{ét}}$, we define

$$P\mathbb{Z}_U(V) = \mathbb{Z}[\operatorname{Hom}_X(V, U)]$$

Notice that we can rewrite $P\mathbb{Z}_U(V)$ in the slightly more exotic form $\mathbb{Z}[h_U(V)]$, where as usual $h_U(-)$ is the functor $\mathrm{Hom}_X(-,U)$. For any presheaf $\mathcal{F}$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PAb}_X}(P\mathbb{Z}_U,\mathcal{F}) &= \{h_U(V)\to\mathcal{F}(V) \text{ with compatibility conditions}\}\\ &= \mathrm{Hom}(h_U,\mathcal{F}) = \mathcal{F}(U),\end{aligned}$$

where the last equality follows from Yoneda's lemma. We can now construct the complex $Z^\bullet$:

$$\cdots \to \bigoplus_{i_0,i_1,i_2} P\mathbb{Z}_{U_{i_0}\times_X U_{i_1}\times_X U_{i_2}} \to \bigoplus_{i_0,i_1} P\mathbb{Z}_{U_{i_0}\times_X U_{i_1}} \to \bigoplus_{i_0} P\mathbb{Z}_{U_{i_0}} \to 0.$$

Notice that if $V\to U$ is a map in $X_{\text{ét}}$, then there is an obvious map natural morphism given by composition

$$inc_U^V : P\mathbb{Z}_V \to P\mathbb{Z}_U.$$

This allows us to construct the boundary map in the complex: given

$$\sigma \in \mathrm{Hom}_X(W, U_{i_0}\times_X \times \times_X U_{i_p})$$

we set

$$d\sigma = \sum_{j=0}^{p}(-1)^j\,\mathrm{inc}_{U_{i_0}\times_X\cdots\times_X\widehat{U_{i_j}}\times_X\times_X U_{i_p}}^{U_{i_0}\times_X\cdots\times_X U_{i_p}}\circ\sigma.$$

Since $\mathrm{Hom}(\mathbb{Z}_U,\mathcal{F})=\mathcal{F}(U)$, we see immediately that $\mathrm{Hom}(Z^p,\mathcal{F})=\check{C}^p(\mathcal{U},\mathcal{F})$. It remains to see that the complex $Z^\bullet$ is exact; since we are working with presheaves, we need to check exactness on each étale open $\varphi : V \longrightarrow X$. This amounts to showing exactness of

$$\cdots \to \bigoplus_{i_0,i_1}\mathbb{Z}[\mathrm{Hom}_X(V,U_{i_0}\times_X U_{i_1})] \to \bigoplus_{i_0}\mathbb{Z}[\mathrm{Hom}_X(V,U_{i_0})] \to 0;$$

one sees without difficulty that this is the same as

$$\cdots \to \mathbb{Z}[S_\varphi\times S_\varphi\times S_\varphi] \to \mathbb{Z}[S_\varphi\times S_\varphi] \to \mathbb{Z}[S_\varphi] \to 0, \tag{16}$$

where

$$S_\varphi = \{(j,\alpha) \mid \alpha : V\to U_j, \mathrm{inc}\circ\alpha = \varphi\}.$$

We have already used multiple times that (16) is exact (morally, because it computes the cohomology of the point: see section 1.3). ☐

Remark 15.5. We are working with abelian groups, but the same argument applies for arbitrary R-modules.

Remark 15.6. The presheaf $P\mathbb{Z}_U$ is not a sheaf. We will meet the associated sheaf $\mathbb{Z}_U$ in section 16.2.

Remark 15.7. Notice the analogy wih the topological case: if the only maps are the inclusions, then $P\mathbb{Z}_U(V) = \begin{cases}\mathbb{Z}, & \text{if } V\subseteq U\\ 0, & \text{otherwise.}\end{cases}$

Finally we prove that the sheaf associated to $\underline{H}^p(\mathcal{F})$ is zero. This result will be used in the next sections.

Lemma 15.8. *Let $\mathcal{F}$ be an étale sheaf on X. For any $\xi\in\underline{H}^q(U,\mathcal{F})$, with $q>0$, there exists a covering $(U_i\to U)_{i\in I}$ of U such that $\mathrm{Res}^U_{U_i}(\xi)=0\quad\forall i$. In particular $\underline{H}^q(\mathcal{F})=0$.*

Proof. Let $\mathcal{F}\to I^0\to I^1\to I^2\to\cdots$ be an injective resolution of $\mathcal{F}$. The element ξ can be lifted to an element $\tilde{\xi}\in I^p(U)$ such that $\partial\tilde{\xi}=0$. The exactness of the sequence *as a sequence of sheaves* shows that $\tilde{\xi}$ comes from I^{p-1} if we restrict to a sufficiently fine covering: $\xi|_{U_i}=\partial\eta_i$; but then $\tilde{\xi}$ is a coboundary, hence is zero, when restricted to this fine covering. ☐

15.1 Étale cohomology of quasi coherent sheaves

We come to our first computation of étale cohomology. Recall from section 12.2 that given a quasi coherent sheaf $\mathcal{F}$ on a sheme X, the presheaf which assign to an fpqc open set $f : V \longrightarrow X$ the group $\Gamma(V, f^*\mathcal{F})$ define an fpqc sheaf on X and in particular an étale sheaf. The following theorem computes its cohomology.

Theorem 15.9. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then*

$$H^p_{\text{ét}}(X, \mathcal{F}) = H^p(X, \mathcal{F}),$$

where the right hand side is the usual sheaf cohomology.

Proof. For $p = 0$ there is nothing to prove. Suppose that we proved that for X affine and $p > 0$ both sides are zero (this is well-known for the right-hand side).

This implies the claim: indeed, if $\mathcal{U}$ is a covering of X by affines Zarisky open subsets. The Čech-to-cohomology spectral sequence (theorem 15.4) gives

$$\check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

However, since $H^q(U_i, \mathcal{F}) = 0$ for every i and every $q > 0$, we have that $H^{p+q}(X, \mathcal{F})$ is filtered with a single object, namely $H^p_{\text{ét}}(X, \mathcal{F}) = \check{H}^p(\mathcal{U}, \mathcal{F})$. But the very same exact sequence exists also in the case of usual sheaf cohomology, so $H^p(X, \mathcal{F}) = \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p_{\text{ét}}(X, \mathcal{F})$.

So it suffices to treat the affine case. We make the following preliminary remarks:

1. given an open covering $\mathcal{U}$ of X, with X affine, we can refine $\mathcal{U}$ to a covering $\mathcal{U}'$ of the form $U_1 \coprod U_2 \coprod \cdots \coprod U_n$ with U_i affine
2. set $V = U_1 \coprod U_2 \coprod \cdots \coprod U_n$; it is an étale cover of X with a single affine (a finite disjoint union of affines is affine). Then we have $\check{C}^p(\mathcal{V}, \mathcal{F}) = \check{C}^p(\mathcal{U}', \mathcal{F})$, where $\mathcal{V}$ is the covering consisting of the single morphis $V \to U$.

We now proceed by induction on p, starting with $p = 1$. Let $\xi \in H^1(X, \mathcal{F})$, we want to prove that it is zero. By lemma 15.8 there exists an étale covering $\mathcal{U} = (U_i \to U)_{i \in I}$ (with I finite and every U_i affine) such that $\xi|_{U_i} = 0$ for every i. The Čech-to-cohomology spectral sequence (theorem 15.4) gives a filtration of H^1 such that

$$F^1 \cong \check{H}^1(\mathcal{U}, \mathcal{F}), \quad H^1/F^1 = E_3^{0,1} \subseteq E_2^{0,1} = \check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F})).$$

Notice that $\check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F})) \subseteq \prod H^1(U_i, \mathcal{F})$, so the image of ξ in $\check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$ is zero by our choice of U_i. This implies that ξ is in $\check{H}^1(\mathcal{U}, \mathcal{F}) = \check{H}^1(\mathcal{V}, \mathcal{F})$, where $\mathcal{V}$ is the étale covering consisting of the single morphism $V = \coprod U_i \to U$. Notice that $V \to X$ is faithfully flat.

Let $V = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, $A \to B$ étale and faithfully flat, and write $\mathcal{F} = \tilde{M}$ for some A-module M. Now the pullback of $\mathcal{F}$ to B is $\widetilde{B \otimes_A M}$, and in order to compute the Čech cohomology of $\mathcal{F}|_V$ we just have to consider the complex

$$M \otimes_A B \to M \otimes_A B \otimes B \to M \otimes_A \otimes B \otimes B \otimes B \to \cdots,$$

which we know to be exact in positive degree by the fundamental lemma of fpqc descent (lemma 12.8). Now proceed by induction: groups of the form $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanish by the same argument as above, and groups of the form $\check{H}^p(\mathcal{U}, \underline{H}^q\mathcal{F})$ with $q > 0$ vanish by the inductive hypothesis. □

15.2 Exercises

Exercise 15.10. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor between two abelian categories and assume that $\mathcal{A}$ has enough injectives. Let $K^p : \mathcal{A} \longrightarrow \mathcal{B}$ be a sequence of additive functors defined for $p \geq 0$ such that

i) $K^0 = F$,

ii) $K^p(I) = 0$ for all $p > 0$ and all injective object I in $\mathcal{A}$

iii) given a short exact sequence of objects $0 \to X \to Y \to Z \to 0$ there is a long exact sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to K^1(X) \to K^1(Y) \to K^1(Z) \to K^2(X) \cdots .$$

Prove that $K^p(X) \simeq R^pF(X)$ for all X. Moreover if the long exact sequence is natural in the given short exact sequence we have a natural equivalence of functors $K^p \simeq R^pF$.

Exercise 15.11. Prove lemma 15.2. Notice that we are computing cohomologies in the category of *pre*sheaves, which is easier than working in the category of sheaves!

Exercise 15.12. Complete the inductive steps in the proof of Theorem 15.9.

16 Inverse and direct images

16.1 Inverse and direct image

Let $f : X \to Y$ be a morphism of schemes and let $\alpha : V \longrightarrow Y$ be an étale map. Then $f^{-1}V := V \times_Y X \to X$ is étale, and if $\mathcal{V} = (V_i \to V)$ is an étale covering of Y, then $f^{-1}\mathcal{V} = \{f^{-1}V_i \to X\}$ is an étale covering of X. Moreover, if $\varphi : V \longrightarrow V'$ is a morphism of étale morphisms over X we define $f^{-1}\varphi = \varphi \times_Y \mathrm{id}_Y : f^{-1}V \longrightarrow f^{-1}V'$.

We will assume that the schemes are locally noetherian since we proved all result we need on étale morphisms (and in particular Exercise 9.9) under this assumption. However almost all constructions and results we are going to discuss can be given without this hypothesis. We will make some further comments about notherianity in the last section where this assumption will be the reason of one extra difficulty.

Definition 16.1. Let $f : X \to Y$ be a morphism of locally noetherian schemes. We now define three functors corresponding to f; the definitions are formally identical to those for topological spaces, except that in the étale case $f^{-1}V$ is an étale map and not an open subset.

- The *direct image* functor $f_* : \mathbf{Ab}_X \longrightarrow \mathbf{Ab}_Y$ (or from presheaves to presheaves). If $\mathcal{F}$ is a sheaf (or a presheaf) on X we define

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V).$$

- The *inverse image of presheaves* functor $f_P^{-1} : \mathbf{PAb}_Y \to \mathbf{PAb}_X$. If $\mathcal{G}$ is a presheaf on Y we define

$$f_P^{-1}\mathcal{G}(U) = \varinjlim_{V} \mathcal{G}(V)$$

  where the colimit is taken over diagrams of the form

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & f^{-1}V \arrow[dl] \\ & X & \end{tikzcd}$$

  If $U' \longrightarrow U$ is an étale map then any diagram as above induces a similar digram for U', hence the limit defining $f^{-1}P\mathcal{G}(U)$ is on a smaller set than the limit defining $f^{-1}P\mathcal{G}(U')$. In particular we have a natual map from $f^{-1}P\mathcal{G}(U)$ to $f^{-1}P\mathcal{G}(U')$ which makes $f^{-1}P\mathcal{G}$ into a presheaf.

- The *inverse image* (of sheaves) functor $f^{-1} : \mathbf{Ab}_Y \to \mathbf{Ab}_X$. If $\mathcal{G}$ is a sheaf we define

$$f^{-1}\mathcal{G} = \left(f_P^{-1}\mathcal{G}\right)^{\#}.$$

These definitions extend to morphisms in the natural way. Notice also that similar constructions work for sheaves of sets, R-modules, etc.

The definition of the presheaf $f_P^{-1}\mathcal{G}$ is not standard, however it will be useful for us to have a notation for this object in some proofs.

Remark 16.2 (Conventions for sheaves over a separably closed field)**.** If K is a separably closed field and $S = \operatorname{Spec} K$, then every étale maps $U \longrightarrow S$ is a disjoint union of copies of S, Proposition 7.2. Hence $\mathcal{F} \mapsto \mathcal{F}(S)$ is an equivalence from the category of sheaves on S to the category of abelian groups (or from the category of sheaves in sets on S and the category of sets). We will often identify sheaves on S with an abelian group using this morphism. In this case we also have that the stalk of $\mathcal{F}$ over the geometric point $\overline{s} : \mathrm{id} : S \longrightarrow S$ satisfies

$$\mathcal{F}_{\overline{s}} \simeq \mathcal{F}(S).$$

Lemma 16.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of locally noetherian schemes. Then*

1) $(g \circ f)_* = g_* \circ f_*$;

2) let $j_{\overline{x}} : \overline{x} \to X$ be a geometric point. Then $j_{\overline{x}}^{-1}\mathcal{F} = \mathcal{F}_{\overline{x}}$ (as in remark 16.2 we identify a sheaf on the spectrum of a separably closed field with its global sections);

3) (f^{-1}, f_*) is an adjoint pair;

4) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$;

5) for every geometric point $\overline{x} \longrightarrow X$ of X we have $(f^{-1}\mathcal{F})_{\overline{x}} = \mathcal{F}_{f(\overline{x})}$, where $f(\overline{x})$ is the composition $\overline{x} \to X \xrightarrow{f} Y$;

6) f^{-1} is exact, while f_* is left exact and carries injectives to injectives.

Proof. This is very similar to the topological case.

1. We have $g^{-1}f^{-1}(V) = (g \circ f)^{-1}(V)$, and the claim follows.

2. Sheafification is not relevant since we are working with sheaves on a point. Hence it suffices to prove that the stalk of $\mathcal{F}$ at $\overline{x}$ is canonically isomorphic to $(j_{\overline{x}})_P^{-1}\mathcal{F}(\overline{x})$, which is true because (by definition) both are equal to $\varinjlim \mathcal{F}(U)$, where the colimit is taken over all the étale neighbourhoods of $\overline{x}$.

3. We only describe the bijection on Homs. Recall that since sheafification is left adjoint to the forgetful functor from sheaves to presheaves we have

$$\mathrm{Hom}_{\mathbf{Ab}_X}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{PAb}_X}(f_P^{-1}\mathcal{G}, \mathcal{F}).$$

We will give a bijection between $\mathrm{Hom}_{\mathbf{PAb}_X}(f_P^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Hom}_{\mathbf{PAb}_Y}(\mathcal{G}, f_*\mathcal{F})$. Given $\varphi \in \mathrm{Hom}_{\mathbf{PAb}_X}(f_P^{-1}\mathcal{G}, \mathcal{F})$ we construct $\psi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ for every $V \to Y$ étale. The data of φ determines in particular a map

$$\varphi_{f^{-1}V} : f_P^{-1}\mathcal{G}(f^{-1}V) \to \mathcal{F}(f^{-1}V).$$

Recall that $f_P^{-1}\mathcal{G}(f^{-1}V)$ is a certain colimit over $\mathcal{G}(W)$, where W ranges over the diagrams of the form

$$\begin{tikzcd} f^{-1} \arrow[rr] \arrow[dr] & & f^{-1}W \arrow[dl] \\ & X. & \end{tikzcd}$$

In particular, one of the objects in the colimit is V itself, so we get a map $\mathcal{G}(V) \to f_P^{-1}G(f^{-1}V) \to \mathcal{F}(f^{-1}V)$, which we define to be ψ_V.

Conversely, given $\psi \in \mathrm{Hom}_{\mathbf{PAb}_Y}(\mathcal{G}, f_*\mathcal{F})$, for all $U \longrightarrow X$ étale, and for all $V \longrightarrow Y$ étale such that the map from U to X factors through $f^{-1}V$, the compositions

$$\mathcal{G}(V) \xrightarrow{\psi_V} \mathcal{F}(f^{-1}V) \xrightarrow{Res_U^{f^{-1}V}} \mathcal{F}(U)$$

form a compatible system of maps which induces a map $\varphi_U : f_P^{-1}\mathcal{G}(U) \longrightarrow \mathcal{F}(U)$.

The two constructions are inverse to each other.

4. Both functors are adjoint to $(g \circ f)_*$, and the adjoint is unique.

5. Follows from the what we have just seen:

$$(f^{-1}\mathcal{F})_{\overline{x}} = j_{\overline{x}}^{-1}f^{-1}\mathcal{F} = (f \circ j_{\overline{x}})^{-1}\mathcal{F} = \mathcal{F}_{f(\overline{x})}.$$

6. f^{-1} is exact: exactness of a sequence of abelian sheaves can be checked on stalks, and f^{-1} induces the identity on stalks. Finally, f_* is a right adjoint (hence left exact), and since its adjoint is exact it carries injective objects to injective objects.

□

Remark 16.4 (Again on stalks at geometric points)**.** Let $K \subset L$ be two separably closed fields and let $f : \mathrm{Spec}(L) = T \longrightarrow \mathrm{Spec}(K) = S$ be the induced map. Then $f_*\mathcal{F}(T) = \mathcal{F}(S)$ and $f^{-1}\mathcal{G}(S) = \mathcal{G}(T)$.

In particular, if X is a scheme and $j : S \longrightarrow x$ is a geometric point whose topological image is the point x, then the stalk does not depend on the separably closed field K containing $k(x)$ but only on x. The standard choice is to take as K the separable closure of $k(x)$, but occasionally different choices can also prove useful.

If $x \in X$ we will denote with $\overline{x}$ any geometric point whose image is equal to x.

Remark 16.5 (Adjunctions 1)**.** Let $f : X \to Y$ be a map of schemes, $\mathcal{F}$ a sheaf on X, and let $\mathcal{G}$ be a sheaf on Y. As a particular case of part 3 of Lemma 16.3 we have adjunctions

$$\mathrm{adj}_{\mathcal{F}} : f^{-1}f_*\mathcal{F} \to \mathcal{F} \quad \text{and} \quad \mathrm{adj}_{\mathcal{G}} : \mathcal{G} \to f_*f^{-1}\mathcal{G}$$

We describe them once again since they appear in many contexts.

To describe $\mathrm{adj}_{\mathcal{F}}$ it is enough to give a map $f_P^{-1}f_*\mathcal{F} \to \mathcal{F}(U)$. This is induced by the fact that $f_P^{-1}f_*\mathcal{F}(U)$ is the limit of objects of the form $\mathcal{F}(f^{-1}V)$, where $U \to f^{-1}V$ is an étale refinement. In particular, we have a restriction map $\mathcal{F}(f^{-1}V) \to \mathcal{F}(U)$ for every V over which we take the limit, and therefore a map $f_P^{-1}f_*\mathcal{F}(U) \to \mathcal{F}(U)$ as desired.

The adjunction $\mathrm{adj}_{\mathcal{G}}$ corresponding to an étale map $V \longrightarrow Y$ is given by the composition

$$\mathcal{G}(V) \xrightarrow{\varphi} (f_P^{-1}\mathcal{G})(f^{-1}V) \xrightarrow{\theta^2} f^{-1}\mathcal{G}(f^{-1}V)$$

where φ is given by the fact that $\mathcal{G}(V)$ appears in the definition of the middle object as a colimit, while θ^2 is the natural map from the sections of a presheaf to the sections of its associated sheaf.

Remark 16.6 (Adjunctions 2)**.** As above let $f : X \to Y$ be a map of schemes, $\mathcal{F}$ be a sheaf on X, and $\mathcal{G}$ be a sheaf on Y. We have natural maps at the level of cohomology:

$$H^p(Y, \mathcal{G}) \to H^p(X, f^{-1}\mathcal{G}) \qquad \text{and} \qquad H^p(Y, f_*\mathcal{F}) \to H^p(X, \mathcal{F}).$$

In particular, when $\mathcal{G}$ is a constant sheaf we have maps between the cohomology of Y with values in a given abelian group and and the cohomology of X with values in the same group (see exercise 16.30) as for singular or Čech cohomology.

We explain how this canonical maps are constructed. Let $\mathcal{G} \to \mathcal{I}^\bullet$ be an injective resolution in $\mathbf{Ab}(Y)$ and $f^{-1}\mathcal{G} \longrightarrow \mathcal{J}^\bullet$ an injective resolution in $\mathbf{Ab}_X$. By properties of injective complexes in an abelian category there exists a morphism of complexes $f^{-1}\mathcal{I}^\bullet \longrightarrow \mathcal{J}^\bullet$, unique up to homotopy, such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} f^{-1}\mathcal{G} \arrow[r, "\text{quasi-iso}"] \arrow[d, equal] & f^{-1}\mathcal{I}^\bullet \arrow[d, "h"] \\ f^{-1}\mathcal{G} \arrow[r, "\text{quasi-iso}"'] & \mathcal{J}^\bullet. \end{tikzcd}$$

This diagram induces

$$\Gamma(Y, \mathcal{I}^\bullet) \xrightarrow{adj_{\mathcal{I}}} \Gamma(X, f^{-1}\mathcal{I}^\bullet) \xrightarrow{h_X} \Gamma(X, J^\bullet)$$

and by passing to cohomology we get the desired map $H^p(Y, \mathcal{G}) \to H^p(X, f^{-1}\mathcal{G})$.

For the second map let now $\mathcal{F} \longrightarrow \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ in $\mathbf{Ab}_X$ and $f_*\mathcal{F} \longrightarrow \mathcal{J}^\bullet$ be an injective resolution of $f_*\mathcal{F}$ in $\mathbf{Ab}_Y$. Notice that $f_*\mathcal{I}^\bullet$ is a complex of injective objects, hence by properties of injective complexes there exists a map of complexes $\ell : \mathcal{J}^\bullet \longrightarrow f_*\mathcal{I}$, unique up to homotopy, such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} f_*\mathcal{F} \arrow[r, "\text{quasi-iso}"] \arrow[d, equal] & \mathcal{J}^\bullet \arrow[d, "\ell"] \\ f_*\mathcal{F} \arrow[r] & f_*\mathcal{I}^\bullet. \end{tikzcd}$$

This diagram induces

$$\Gamma(Y, \mathcal{J}^\bullet) \xrightarrow{\ell_Y} \Gamma(Y, f_*\mathcal{I}^\bullet) = \Gamma(X, \mathcal{I}^\bullet)$$

and by passing to cohomology we get the desired map $H^p(Y, f_*\mathcal{F}) \to H^p(X, \mathcal{F})$.

We now use the morphism f_* to prove that the category of abelian sheaves for the étale site has enough injectives.

Theorem 16.7. *The category of abelian sheaves on the étale site of a locally noetherian scheme X has sufficiently many injectives.*

Proof. We first construct a family of injective sheaves. For every $x \in X$ let $\overline{x} = \operatorname{Spec} k(x)^{\mathrm{sep}}$ be a corresponding geometric point; denote by $j_{\overline{x}}$ the inclusion of $\overline{x}$ in X. For every $x \in X$ let I_x be an injective $\mathbb{Z}$-module. Denote by

$$\mathcal{I} = \prod_x (j_{\overline{x}})_* I_x \tag{17}$$

By adjunction we have

$$\operatorname{Hom}_{\mathbf{Ab_X}}(\mathcal{F}, \mathcal{I}) = \prod_x \operatorname{Hom}_{\mathbb{Z}}\left(\mathcal{F}_{\overline{x}}, I_{\overline{x}}\right),$$

and this implies easily that $\mathcal{I}$ is injective. In particular, if we want to embed a sheaf $\mathcal{F}$ in an injective sheaf $\mathcal{I}$, it suffices to embed every $\mathcal{F}_{\overline{x}}$ in a corresponding injective $\mathbb{Z}$-module $I_{\overline{x}}$ and run the construction above with this specific choice of $I_{\overline{x}}$. Then the collection of maps $\mathcal{F}_{\overline{x}} \to I_{\overline{x}}$ gives a map $\mathcal{F} \to \mathcal{I}$, which (being an injection on every stalk) is an injection of sheaves. □

16.2 Inverse image for étale morphisms and extension by zero

We begin with a small remark concerning notation. Given a sheaf $\mathcal{F}$ on X and an étale map $\alpha : U \longrightarrow X$, we have thus far written $\mathcal{F}(U)$ for the sections of $\mathcal{F}$ on $\alpha : U \to X$. This was in the interest of having a more compact notation, and does not usually give rise to any confusion. In this section, however, it will occasionally be convenient to stress the role of the map α, and we will accordingly write $\mathcal{F}(\alpha : U \longrightarrow X)$.

Assume that $f : X \longrightarrow Y$ is an étale map of locally noetherian schemes. In this case any étale open subset $\alpha : U \longrightarrow X$ of X induces, by composition, an étale open subset of Y, namely $f \circ \alpha : U \longrightarrow Y$. Describing the inverse image of a sheaf $\mathcal{G}$ on Y is simpler than in the general case. Indeed in the limit defining $f_P^{-1}\mathcal{F}(\alpha : U \longrightarrow X)$ we have a final object, given by $f \circ \alpha$, hence we obtain

$$f^{-1}\mathcal{F}(\alpha : U \longrightarrow X) = f_P^{-1}\mathcal{F}(\alpha : U \longrightarrow X) = \mathcal{F}(f \circ \alpha : U \longrightarrow Y).$$

For this reason it is also common to write $\mathcal{F}|_U$ in this situation. From the construction of the injective sheaves in 16.7 we deduce the following corollary.

Corollary 16.8. *Let $U \to X$ be an étale morphism and $\mathcal{F}$ be a sheaf on a locally noetherian scheme X. Then we can describe $H^p(U, \mathcal{F})$ in two ways: either as $R^p\Gamma(U, -)$ (considered as a derived functor from the category of abelian sheaves on X to the category of abelian groups) or as $H^p(U, \mathcal{F}|_U)$ (considered as a derived functor from the category of abelian sheaves on U to the category of abelian groups). These two descriptions coincide.*

Proof. Fix a resolution of $\mathcal{F}$ by injective sheaves $\mathcal{I}^\bullet$ constructed as in the proof of theorem 16.7. Upon restriction to U, every $\mathcal{I}^p$ is still a sheaf obtained as the product of sheaves of the form $(j_{\overline{x}})_* I_{\overline{x}}$, hence is injective. Thus the *same* resolution can be used to compute the two derived functors, which therefore coincide. $\square$

When f is étale we can define a third functor, $f_! : \mathbf{Ab}_X \longrightarrow \mathbf{Ab}_Y$, called the *extension by zero.* This is a particular case of the direct image with compact support which can be defined for any morphism of schemes, but we will not see the reason for this second name. Let $\mathcal{F}$ be an abelian sheaf on X. If $\alpha : V \longrightarrow Y$ is étale and $\beta : V \longrightarrow X$ is such that $\alpha = f \circ \beta$ then β is étale and it makes sense to compute $\mathcal{F}(\beta : V \longrightarrow X)$. We define the presheaf $f_!^P \mathcal{F}$ as follows

$$f_!^P \mathcal{F}(\alpha : V \longrightarrow Y) = \bigoplus_{\beta : f \circ \beta = \alpha} \mathcal{F}(\beta : V \longrightarrow X).$$

Define $f_! \mathcal{F}$ as the associated sheaf. Similarly we define $f_!$ for morphisms.

Remark 16.9. We have given the definition for an abelian sheaf. The definition for a sheaf of sets is formally similar, but there is a subtlety. Namely, let $\iota : \mathbf{Ab}_X \to \mathbf{Sh}_X$ be the forgetful functor from sheaves of abelian groups to sheaves of sets, and denote by the same letter the forgetful functor $\iota : \mathbf{Ab}_Y \to \mathbf{Sh}_Y$. Then $\iota(f_! \mathcal{F})$ is *not* the same as $f_! \iota(\mathcal{F})$, because coproducts of abelian groups and of sets do not agree.

Part d) of the next Proposition gives a reason for the name *extension by zero.*

Proposition 16.10. *Let $f : X \longrightarrow Y$ be an étale morphism of locally noetherian shemes and let $\mathcal{F}$ be a sheaf on X.*

a) $(f_!, f^{-1})$ is an adjoint pair.

b) Let $g : Y' \longrightarrow Y$ be a morphism of schemes and let $f' : X' \longrightarrow X$ and $g' : X' \longrightarrow Y'$ be the morphisms obtained by pull back. There is a natural equivalence

$$(f')^{-1} \circ g'_* \simeq g_* \circ f^{-1}.$$

c) With the same notation, we have a natural equivalence

$$f'_! \circ (g')^{-1} \simeq g^{-1} \circ f_!.$$

d) If $\bar{y}$ is a geometric point of Y we have

$$(f_! \mathcal{F})_{\bar{y}} = \bigoplus_{\bar{x} \in X_{\bar{y}}} \mathcal{F}_{\bar{x}}$$

e) f^{-1} sends injectives to injectives and $R^p f_! = 0$ for $p > 0$.

The proof is left as an exercise. As a Corollary we notice that part e) gives another proof of Corollary 16.8.

Notice that we have already encountered a sheaf of the form $f_!^P \mathcal{F}$. Indeed the presheaf $P\mathbb{Z}_U$ introduced in the proof of lemma 15.3 is the extension by zero presheaf of the constant presheaf on U. The associated sheaf is very important in many constructions.

Definition 16.11 (Constant sheaves and constant sheaves supported on an open "subset")**.** If X is a scheme and A is abelian group, the *constant sheaf* on X with coefficients in A is the sheaf associated with the presheaf $V \longrightarrow A$ for all non empty étale $V \longrightarrow X$.

Moreover, if $f : V \longrightarrow X$ is an étale map, the *constant sheaf supported on* V with coefficients in A, denoted by A_V, is the sheaf on X given by $f_! C$, where C is the constant sheaf on V with coefficients in A. Notice that we denote this sheaf by A_V even though it is a sheaf on X; this notation, while not perfect, is standard.

When $A = \mathbb{Z}$, A_V is the sheaf associated to the presheaf $P\mathbb{Z}_V$, and (as we have seen in the proof of lemma 15.3) it satisfies the important property

$$\mathrm{Hom}_{\mathbf{Ab}_X}(\mathbb{Z}_V, \mathcal{F}) = \mathcal{F}(V). \tag{18}$$

We point out a consequence of this construction. In the topological situation, every injective sheaf $\mathcal{I}$ is also *flabby*, that is, for $U \subset V$ the restriction map $\mathcal{I}(V) \longrightarrow \mathcal{I}(U)$ is surjective. This is no longer true for étale maps $V \longrightarrow U$ and for injective sheaves in the étale topology; however, a weaker statement still holds:

Lemma 16.12. *Let U be an open Zariski subset of a locally noetherian scheme X. Then*

a) there is a natural adjunction map $\mathbb{Z}_U \longrightarrow \mathbb{Z}_X$ that is injective;

b) if $\mathcal{I}$ is an injective sheaf then the restriction map $\mathcal{I}(X) \longrightarrow \mathcal{I}(U)$ is surjective.

Proof. There is an obvious map $P\mathbb{Z}_U \longrightarrow P\mathbb{Z}_X$ of presheaves which is injective by construction. Since sheafication is exact the induced map at the level of sheaves is injective. Part b) now follows from a) and equation (18). □

16.3 Direct image for closed immersions and support of a section

The next Proposition describes the behaviour of direct images along closed immersions. We will generalise this result in the next sections, first to finite morphisms (Theorem 16.24) and then to arbitrary maps (Theorem 16.26), so that at the end we will have given three proofs of this result that are increasingly more difficult. The third proof, although technically more complicated, will share some arguments with the particular case of closed immersions.

Proposition 16.13. *Let $i : Z \longrightarrow Y$ be a closed immersion of locally noetherian schemes and let $\mathcal{F}$ be a sheaf on Z. Then for every $y \in Y$*

$$(i_*\mathcal{F})_{\overline{y}} = \begin{cases} \mathcal{F}_{\overline{y}} & \text{if } y \in X; \\ 0 & \text{otherwise.} \end{cases}$$

In particular i_* is exact.

Proof. If $y \notin Z$ then the set of étale neighbourhoods $U \to Y$ of y whose image is contained in the complement of Z is cofinal in the limit computing $(i_*\mathcal{F})_{\overline{y}}$, and $\mathcal{F}(i^{-1}U) = 0$ for any such neighbourhood, proving the claim in this case.

If $y \in Z$ then the claim follows from the following Lemma, which intuitively says that every neighbourhood of a point $x \in Z$ is the intersection of a neighbourhood of x in Y with Z. Indeed this implies that the limit defining $(i_*\mathcal{F})_{\overline{y}}$ is the same as the limit computing $\mathcal{F}_{\overline{y}}$. □

Lemma 16.14. *Let $i : Z \longrightarrow Y$ be a closed immersion af locally noetherian schemes and let $x \in Z$. Let $(U, \overline{u}) \longrightarrow (Z, \overline{x})$ be an affine étale neighbourhood of $\overline{x}$. Then there exists an étale neighbourhood $V \longrightarrow Y$ of $\overline{x}$ in Y such that $i^{-1}V$ refines U.*

Proof. The question is local on Y so we can assume that $Y = \operatorname{Spec} A$, $Z = \operatorname{Spec} B$ and $B = A/I$. We can also assume that $U = \operatorname{Spec} C$ with $B \longrightarrow C$ étale. We will prove that there exists an étale neighbourhood V of $\overline{x}$ in Y such that $i^{-1}V = U$.

Let P be the prime of A corresponding to x, let $\mathfrak{p} = P/I$ be the prime of B corresponding to x and $\mathfrak{q}$ be the prime ideal corresponding to u (so that in particular $\mathfrak{q}^c = \mathfrak{p}$). Then, by Exercise 9.9 (the assumption of Notherianity is mainly used here), there exists an isomorphism

$$C \simeq \frac{B[t_1, \dots, t_n]}{(\bar{f}_1, \dots, \bar{f}_n)}$$

such that the determinant $\bar{\Delta}$ of the Jacobian matrix is invertible in C. Let $f_1, \ldots, f_n$ be lifts of $\bar{f}_1, \ldots, \bar{f}_n$ to $A[t_1, \ldots, t_n]$ and consider the ring

$$D_0 = \frac{A[t_1, \ldots, t_n]}{(f_1, \ldots, f_n)}.$$

There is a natural projection from D_0 to C; let Q be the inverse image of the ideal $\mathfrak{q}$. The determinant Δ of the associated Jacobian matrix is not in Q. Hence $D = (D_0)_\Delta$ is étale over A and its spectrum is a neighbourhood of $\overline{x}$ in Y. Finally $D \otimes_A B = C$ since $\bar{\Delta}$ is already invertible in C. □

Notice that when i is a closed immersion the functor i_* shares some similarities with the functor $f_!$ for an étale map; in particular, it is exact, and it makes sense to look for a right adjoint to i_*. In this section we will construct such an adjoint.

Definition 16.15 (Support of a section and of a sheaf)**.** Let $\mathcal{F}$ be a sheaf on Y and let $\sigma \in \mathcal{F}(Y)$ be a section. Let U be the union of all Zariski open subsets V of Y such that $\sigma|_V = 0$. Since $\mathcal{F}$ is a sheaf, we have $\sigma|_U = 0$, hence U is the maximal Zariski open subset of Y such that $\sigma|_U = 0$. We call $\operatorname{supp}(\sigma) = Y \setminus U$ the *support of the section* σ. We also define the *support of the sheaf* $\mathcal{F}$ as the set

$$\operatorname{supp}(\mathcal{F}) = \{y \in Y : \mathcal{F}_{\overline{y}} \neq 0\}.$$

If $i : Z \longrightarrow Y$ is a closed immersion and $\mathcal{F}$ is a sheaf on Y then, by the previous Lemma, $\operatorname{supp} i_*\mathcal{F} = \operatorname{supp} \mathcal{F} \subset Z$. Conversely, if $\mathcal{F}$ is a sheaf on Y whose support is contained in Z, the adjunction $\operatorname{adj}_\mathcal{F} : \mathcal{F} \longrightarrow i_*i^{-1}\mathcal{F}$ is an isomorphism on stalks, hence it is an isomorphism. Similarly, if $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves supported on Z, then $\beta = i_*i^{-1}\alpha : i_*i^{-1}\mathcal{F} \longrightarrow i_*i^{-1}\mathcal{G}$ satisfies $\operatorname{adj}_\mathcal{G} \circ \beta = \beta \circ \operatorname{adj}_\mathcal{F}$. We have just proved the following lemma

Lemma 16.16. *Let $i : Z \longrightarrow Y$ be a closed immersion of locally noetherian schemes. Then*

$$i_* : \mathbf{Ab}_Z \longrightarrow \mathbf{Ab}_Y$$

is a fully faithful functor, whose image is the subcategory of sheaves on Y whose support is contained in Z.

We now define the right adjoint to i_*. Such a functor exists.

Definition 16.17 (Sheaf of sections supported on a closed subscheme)**.** Let $i : Z \longrightarrow Y$ be a closed immersion of locally noetherian schemes and let $\mathcal{F}$ be an étale sheaf on Y. Let $U = Y \setminus Z$. For every étale map $\alpha : V \longrightarrow Y$ we define the set of sections of V supported on Z, denoted by $\Gamma_Z(V, \mathcal{F})$, as the kernel of the restriction map

$$\Gamma(V, \mathcal{F}) \longrightarrow \Gamma(\alpha^{-1}U, \mathcal{F}).$$

If V' refines V, then the restriction map $\Gamma(V, \mathcal{F}) \longrightarrow \Gamma(V', \mathcal{F})$ induces a map $\Gamma_Z(V, \mathcal{F}) \longrightarrow \Gamma_Z(V', \mathcal{F})$. In this way we construct a subsheaf of $\mathcal{F}$ that we will denote by $\Gamma_Z(\mathcal{F})$. Finally we will denote by $i_0^!\mathcal{F}$ the sheaf $i^{-1}\Gamma_Z(\mathcal{F})$.

The derived functors of $\mathcal{F} \mapsto \Gamma_Z(Y, \mathcal{F})$ (which is left exact) from $\mathbf{Ab}_Y$ to $\mathbf{Ab}$ will be denoted by $R\Gamma_Z(Y, \mathcal{F})$ while its associated cohomology groups by $H^i_Z(Y, \mathcal{F})$. The derived functor of the functor $\mathcal{F} \mapsto i_0^!\mathcal{F}$ (which is also left exact) from $\mathbf{Ab}_Y$ to $\mathbf{Ab}_Z$ is denoted by $i^!\mathcal{F}$. The strange notation for $i_0^!$ and $i^!$ is due to the existence of similar functors in the general case; in particular, the notation $i^!$ is standard, while $i_0^!$ is not.

Proposition 16.18. *Let $i : Z \longrightarrow Y$ be a closed immersion of locally noetherian schemes. We consider the functor $i_* : \mathbf{Ab}_Z \longrightarrow \mathbf{Ab}_Y$ and $i_0^! : \mathbf{Ab}_Y \longrightarrow \mathbf{Ab}_Z$. Then*

a) $(i_*, i_0^!)$ are adjoint functors;

b) $i_0^!$ is left exact and takes injective sheaves to injective sheaves.

Proof. b) follows from a) by general nonsense. We prove a). Let $\mathcal{F}$ be an abelian sheaf on Z and $\mathcal{G}$ be an abelian sheaf on Y. We want to prove

$$\mathrm{Hom}_{\mathbf{Ab}_Z}(\mathcal{F}, i_0^!\mathcal{G}) = \mathrm{Hom}_{\mathbf{Ab}_Y}(i_*\mathcal{F}, \mathcal{G}).$$

By the previous Lemma we have

$$\mathrm{Hom}_{\mathbf{Ab}_Z}(\mathcal{F}, i_0^!\mathcal{G}) = \mathrm{Hom}_{\mathbf{Ab}_Y}(i_*\mathcal{F}, i_*i_0^!\mathcal{G}) = \mathrm{Hom}_{\mathbf{Ab}_Y}(i_*\mathcal{F}, \Gamma_Z(\mathcal{G})).$$

We now construct a bijection between $\mathrm{Hom}_{\mathbf{Ab}_Y}(i_*\mathcal{F}, \Gamma_Z(\mathcal{G}))$ and $\mathrm{Hom}_{\mathbf{Ab}_Y}(i_*\mathcal{F}, \mathcal{G})$ as follows. Notice that by construction we have an inclusion $s : \Gamma_Z(\mathcal{G}) \hookrightarrow \mathcal{G}$. Now given $\varphi : i_*\mathcal{F} \longrightarrow \Gamma_Z(\mathcal{G})$ we define $\psi : i_*\mathcal{F} \longrightarrow \mathcal{G}$ as $\psi = s \circ \varphi$. Conversely given ψ it is enough to check that the image of ψ_V in $\mathcal{G}(V)$ is contained in $\Gamma_Z(V, \mathcal{F})$. This follows from the fact that the stalks of $i_*\mathcal{F}$ over $U := Y \setminus Z$ are zero. □

16.4 Exact sequence associated to an open subset and its complement

Let Y be a locally noetherian scheme, $j : U \longrightarrow Y$ the immersion of a Zariski open subset and $i : Z \longrightarrow Y$ be the closed immersion of the complement[25] of U in Y. In this section we study some relations between sheaves on Y and sheaves on Z and U, and between the cohomology of a sheaf on Y and its on Z and U.

Proposition 16.19. *Let $\mathcal{F}$ be a sheaf on the locally noetherian scheme Y.*

1. *There is an exact sequence*

$$0 \longrightarrow j_!j^{-1}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_*i^{-1}\mathcal{F} \longrightarrow 0$$

2. *there is an exact sequence*

$$0 \longrightarrow i_*i_0^!\mathcal{F} = \Gamma_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*j^{-1}\mathcal{F}$$

   If furthermore $\mathcal{F}$ is injective, the last map is surjective.

3. *we have a long exact sequence*

$$H^n_Z(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}|_U) \longrightarrow H^{n+1}_Z(X, \mathcal{F}) \qquad \cdots$$

4. *in the derived category $D^+(\mathbf{Ab}_Y)$ there exists a distinguished triangle*

$$i_*i^!\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_*j^{-1}\mathcal{F} \longrightarrow i_*i^!\mathcal{F}[1]$$

Proof. We have to collect the results obtained so far. We will not give many details.

The morphisms of the first two sequences are constructed by adjunction. The exactness of the first sequence follows from the computation of the stalks of the various terms.

The exactness of the second sequence is just the definition of $\Gamma(\mathcal{F})$. The fact that the last map is surjective for $\mathcal{F}$ an injective sheaf follows from lemma 16.12.

To prove (3) take $\mathcal{I}^*$ to be an injective resolution of $\mathcal{F}$. Then by (2) the sequence of complexes

$$0 \longrightarrow i_*i_0^!\mathcal{I}^* \longrightarrow \mathcal{I}^* \longrightarrow j_*j^{-1}\mathcal{I}^* \longrightarrow 0$$

is exact. Applying the functor $R\Gamma(X, \cdot)$ we obtain the long exact sequence:

$$H^n(X, i_*i_0^!\mathcal{I}^*) \longrightarrow H^n(X, \mathcal{I}^*) \longrightarrow H^n(X, j_*j^{-1}\mathcal{I}^*) \longrightarrow H^{n+1}(X, i_*i_0^!\mathcal{I}^*) \qquad \cdots$$

[25] The algebraic structure on Z is not important for what we are going to say.

Now $i_*i_0^!\mathcal{I}$ is an injective complex, hence $H^n(X, i_*i_0^!\mathcal{I}^*)$ is the cohomology of the complex

$$\Gamma(X, i_*i_0^!\mathcal{I}^*) = \Gamma_Z(X, \mathcal{I})$$

and hence it is equal to $H_Z^n(X, \mathcal{F})$. The term $H^n(X, \mathcal{I}^*)$ is equal to $H^n(X, \mathcal{F})$ by definition. Finally j^{-1} and j_* take injectives to injectives, so – arguing as for the first term – we see that its cohomology is equal to the cohomology of the complex $\Gamma(X, j_*j^{-1}\mathcal{I}^*) = \Gamma(U, \mathcal{I}^*)$, hence it is equal to $H(U, \mathcal{F})$.

Finally, to prove (4) take again an injective resolution $\mathcal{I}^*$ of $\mathcal{F}$. Then we obtain a short exact sequence of complexes as in the proof of part (3) and hence a distinguished triangle. Finally since $i^!\mathcal{F} = i_0^!\mathcal{I}^*$ by the definition of a derived functor and since i_* is exact we have $i_*i_0^!\mathcal{I}^* = i_*i^!\mathcal{F}$. Similarly $j^{-1}\mathcal{I}$ is an injective resolution of $j^{-1}\mathcal{F}$ and $j_*j^{-1}\mathcal{I}^* = Rj_*j^{-1}\mathcal{F}$ by definition of derived functors. □

Remark 16.20. There is one more exact sequence that one could add to the list above. Namely, taking cohomology with compact support (which we did not define) of the first exact sequence in the previous lemma one obtains the following long exact sequence:

$$H_c^n(U, j^{-1}\mathcal{F}) \longrightarrow H_c^n(X, \mathcal{F}) \longrightarrow H_c^n(Z, i^{-1}\mathcal{F}) \longrightarrow H_c^{n+1}(U, j^{-1}\mathcal{F}) \qquad \cdots$$

16.5 Direct image for finite morphisms

We now study the direct image functor for a finite morphism $f : X \longrightarrow Y$ of locally noetherian schemes. We will generalise Proposition 16.13 proving that the direct image functor is exact and we will give a description of the stalks. This will follow from a description of the inverse image of a "small enough" étale open "subset" of Y. We start by briefly discussing the underlying geometric intuition in the context of complex varieties with the usual topology; in this setting, the basic idea is quite simple.

Let $y \in Y$ and let $x_1, \ldots, x_n$ be the inverse images of y in X. Then there is a neighbourhood U of y such that the inverse image of U is the disjoint union $U_1 \sqcup \cdots \sqcup U_n$, where each U_i a neighbourhood of the corresponding x_i, and conversely, given neighbourhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$, there exists a neighbourhood of y whose inverse image is contained in $U_1 \sqcup \cdots \sqcup U_n$. We will prove an analogue of this result in the context of the étale topology.

Lemma 16.21. *Let A be a noetherian ring. Let $f : A \longrightarrow B$ be a finite extension and let $\mathfrak{p}$ be a prime of A with residue field equal to $\Bbbk$. Let $\overline{\Bbbk}$ be a separable closure of $\Bbbk$ and let $(A^{sh}, \mathfrak{m}^{sh})$ be the strict henselisation of $A_\mathfrak{p}$. Define $\overline{B} = \overline{\Bbbk} \otimes_A B$ and $C = A^{sh} \otimes_A B$. Finally let $\bar{\mathfrak{q}}_1, \ldots, \bar{\mathfrak{q}}_n$ be the primes of $\overline{B}$ and set $\overline{B}_i = \overline{B}_{\bar{\mathfrak{q}}_i}$. Then*

a) $\overline{B}$ is an artinian ring and $\overline{B} = \overline{B}_1 \times \cdots \times \overline{B}_n$;

b) $\overline{B} = C/\mathfrak{m}^{sh}C$ and C has exactly n maximal ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, where $\mathfrak{q}_i/\mathfrak{m}^{sh}C = \bar{\mathfrak{q}}_i$. If $C_i = C_{\mathfrak{q}_i}$ then C_i is strictly henselian and $C = C_1 \times \cdots \times C_n$.

c) There exists an étale neighbourhood $(A', \mathfrak{p}')$ of $\mathfrak{p}$ such that the ring $B' = A' \otimes_A B$ has exactly n ideals $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_n$ over $\mathfrak{p}'$ with $\mathfrak{q}'_i$ the contraction of the prime $\mathfrak{q}_i$ through the map $B' \longrightarrow C$ induced by the natural map $A' \longrightarrow A^{sh}$. Finally $B' = B'_1 \times \cdots \times B'_n$ with $\mathfrak{q}'_i$ of the form $B'_1 \times \cdots \times \mathfrak{r}_i \times \cdots \times B_n$.

Proof. Part a) follows from the fact that $\overline{B}$ is a finite extension of $\overline{\Bbbk}$. The first claim of part b) follows from the fact that the residue field of A^{sh} is $\overline{\Bbbk}$. Since C is a finite extension of A^{sh}, every maximal ideal of C contracts to $\mathfrak{m}^{sh}$, hence is the inverse image of a maximal ideal in $C/\mathfrak{m}^{sh}C$. Moreover, as A^{sh} henselian, by Theorem 13.12 we see that C is a product of local rings, which must be the localisations of C at its maximal ideals. The residue field of C_i is equal to the residue field of $\overline{B}_i$, and in particular is separably closed. Finally C is finite over A^{sh}, hence also C_i is finite over A^{sh}. So if D is a finite extension of C_i then it is also a finite extension of A^{sh}, hence it splits as a product of local rings. By Theorem 13.12 this implies that C_i is henselian (and in fact strictly henselian since we have already checked that the residue field of C_i is separably closed).

To prove c) let e_i be the obvious system of orthogonal idempotents in $C = C_1 \times \cdots \times C_n$. These objects exist in the limit, so there exists an étale neighborhood $(A', \mathfrak{p}')$ of $\mathfrak{p}$ [26] such that the e_i belong to $A' \otimes_A B$. Up to going further in the filtered system that defines the colimit, we can assume that there are $\varepsilon_i \in A' \otimes_A B$ such that $[\varepsilon_i] = e_i$, $\varepsilon_i^2 = \varepsilon_i$ and $\sum \varepsilon_i = 1$. Notice that we are implicitly using the fact that

$$C = A^{sh} \otimes_A B = (\varinjlim A')f \otimes_A B = \varinjlim(A' \otimes_A B)$$

since tensor products commute with colimits. Then $B' := A' \otimes_A B = \prod_{j=1}^n B'\varepsilon_j = \prod_j B'_j$. By construction the image of $B'\varepsilon_i$ in C is non zero and contained in C_i. In particular, the contractions of the ideals $\mathfrak{q}'_i$ are distinct, decompose as a product as claimed, and each of them contracts to $\mathfrak{p}'$ in A'.

Finally let $\mathbb{k}'$ be the residue field of $\mathfrak{p}'$. Then any ideal of B' which contracts to $\mathfrak{p}'$ is an ideal of the fiber $\mathbb{k}' \otimes_A B \subset \bar{\mathbb{k}} \otimes_A B$ (this is an inclusion by flatness), so we can have at most n such ideals which must be the ideals $\mathfrak{q}'_i$ constructed above. □

We now refine this Lemma[27].

Lemma 16.22. *We keep the notation of the previous Lemma.*

a) Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_\ell$ be the prime ideals of B over $\mathfrak{p}$. Then for every maximal ideal $\mathfrak{q}_i \subset C$ the contraction of $\mathfrak{q}_i$ in B is equal to $\mathfrak{r}_j$ for some j and C_j is the strict henselisation of $B_{\mathfrak{r}_j}$.

b) for any collection of étale neighbourhoods $(D_i, \mathfrak{s}_i)$ of $\bar{\mathfrak{q}}_i$ (these are the points in the geometric fibre $\mathbb{k} \otimes_A B$ over $\mathfrak{p}$ introduced in the previous Lemma) there exists a neighbourhood $(A', \mathfrak{p}')$ of $\mathfrak{p}$ such that $B' = A' \otimes_A B =$ splits as a product $B'_1 \times \cdots \times B'_n$, where the factor B'_i has a prime $\mathfrak{q}'_i$ over $\mathfrak{p}$ and $(B_i, \mathfrak{q}'_i)$ is a refinement of the neighbourhood $(D_i, \mathfrak{s}_i)$ of $\bar{\mathfrak{q}}_i$.

Proof. If we choose A' as in part c) of the previous Lemma and apply base change we can reduce to the situation where B has only one prime ideal over $\mathfrak{p}$ and $\bar{B}$ and C have only one maximal ideal. Indeed the strict henselisation of $A'_{\mathfrak{p}'}$ is equal to A^{sh} and for A', B' each of the statements considered in both a) and b) split into n separate statements, one for each B'_i.

So let $\mathfrak{q}$ be the only maximal ideal of C and $\overline{\mathfrak{q}}'$ be the only maximal ideal of $\overline{B}$ and $\mathfrak{r}$ the only maximal ideal of B over $\mathfrak{p}$. We must have $\mathfrak{q}^c = \mathfrak{r}$ since $\mathfrak{q}$ contracts to $\mathfrak{p}$ in A. We already know that $C = A^{sh} \otimes_A B$ is strictly henselian. Notice also that $B_{\mathfrak{p}}$ has only one maximal ideal, hence $B_{\mathfrak{p}} = B_{\mathfrak{r}}$. The natural map from B to C extends to a natural map from $B_{\mathfrak{r}}$ to C. We want to prove that this map has the universal property that characterises the strict henselisation.

Since the elements in $B \setminus \mathfrak{r} \subset C \setminus \mathfrak{q}$ are already invertible in C, we have $C = A^{sh} \otimes_A B_{\mathfrak{r}} = A^{sh} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{r}} = A^{sh} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$.

As in the previous lemma we denote by $\bar{\mathbb{k}}$ the separable closure of the residue field of $A_{\mathfrak{p}}$. Then $\bar{\mathbb{k}}$ it is the residue field of A^{hs}. Moreover the residue field of $C_{\mathfrak{q}}$ is a separable finite etension of $\bar{\mathbb{k}}$ hence it is equal to $\mathbb{k}$.

Let now $\varphi : B_{\mathfrak{r}} \longrightarrow E$ be a local morphism into a strictly henselian ring and fix a map $\lambda : \bar{k} \longrightarrow \kappa(E)$ which extends $\bar{\varphi} : B_{\mathfrak{r}}/\mathfrak{r}B_{\mathfrak{r}}$. The morphism φ induces a local morphim ψ from $A_{\mathfrak{p}}$ to E. By the definition of stric henselisation there exists a unique extension of ψ to A^{hs} such that the induced morpshim of residue field is equal to λ. Finally by property of the tensor product there exists a unique morphism ϕ from C to E such that the induced morpshim of residue field is equal to λ the universal property of the tensor product also a morphism from C. This proves a).

Let now $(D, \mathfrak{s})$ be an étale neighbourhood of $\bar{\mathfrak{q}}$. We have a natural map $\eta : D \longrightarrow C$ since C is the colimit of all such rings by part a). Moreover, D must be finitely presented as a B-algebra.

[26] By which we mean that $(\operatorname{Spec} A', \mathfrak{p}')$ is an étale neighbourhood of $\mathfrak{p}$.

[27] il lemma che segue è la parte che mancava a lezione, serve per dimostrare il punto b) nella proposizione 16.23

Since $C = A^{sh} \otimes_A B$, as in the proof of part c) of the previous lemma there exists an étale neighbourhood $(A', \mathfrak{p}')$ of $\mathfrak{p}$ such that η factors through

$$D \longrightarrow B' = A' \otimes_A B \longrightarrow C.$$

In particular A' has the desired property. □

We can restate these two lemmas in a more geometric way:

Proposition 16.23. *Let $f : X \longrightarrow Y$ be a finite map between locally noetherian schemes and let $y \in Y$. Let $\overline{x}_1, \dots, \overline{x}_n$ be the points in the geometric fibre $X_{\overline{y}}$.*

1. *There exists an étale neighbourhood (V, y') of $\overline{y}$ such that $f^{-1}V$ splits as a disjoint union $U_1 \sqcup \dots \sqcup U_n$, where U_i is an étale neighborhood of $\overline{x}_i$.*
2. *For all collections $U_1, \dots, U_n$ where each U_i an étale neighbourhood of $\overline{x}_i$, there exists an étale neighbourhood V of $\overline{y}$ such that $f^{-1}V$ is an étale neighbourhood of $X_{\overline{y}}$ which refines $U_1 \cup \dots \cup U_n$.*

Proof. Part (1) is Lemma 16.21 c), while (2) is the content of part b) of Lemma 16.22. □

As an immediate consequence of this proposition we obtain the following theorem.

Theorem 16.24. *Let $f : X \to Y$ be a finite morphism between locally noetherian schemes. We have*

1. $$(f_*\mathcal{F})_{\overline{y}} = \bigoplus_{\overline{x} \in X_{\overline{y}}} \mathcal{F}_{\overline{x}}$$
2. *f_* is exact. In particular, $R^p f_* \mathcal{F} = 0$ for $p > 0$.*

Proof. We use the notation introduced in the previous Proposition. In order to compute the stalk of $f_*\mathcal{F}$ at $\overline{y}$ we need to consider the colimit $\varinjlim \mathcal{F}(f^{-1}V)$, where V ranges over the étale neighbourhoods of $\overline{y}$. By Proposition 16.23 we see that this colimit splits as $\bigoplus_{i=1}^n \varinjlim \mathcal{F}(U_i)$, where U_i ranges over the étale neighbourhoods of $\overline{x}_i$, and the claim follows.

For part (2), notice that the claim can be tested on stalks, and that the exactness at the level of stalks follows from (1). □

As another application of lemma 16.21 we prove a more precise version of proposition 16.23 in the case of a finite étale map. This result explains why finite étale maps are a good algebraic substitute for the topological notion of unramified covering: indeed we prove that a finite étale map is locally an isomorphism in the étale topology.

Proposition 16.25. *Let $f : X \longrightarrow Y$ be a finite étale map between locally noetherian schemes and let $y \in Y$. Let $\overline{x}_1, \dots, \overline{x}_n$ be the points in the geometric fibre $X_{\overline{y}}$. Then there exists an étale neighbourhood (V, y') of $\overline{y}$ such that $f^{-1}V$ is the disjoint union of $U_1, \dots, U_n$, where U_i is an étale neighborhood of $\overline{x}_i$ and f induces an isomorphism from U_i to V.*

Proof. The question is local, so we can assume that $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$. We denote by $\mathfrak{p}$ the prime corresponding to y, by $\Bbbk$ the residue field $k(\mathfrak{p})$, by $\overline{\Bbbk}$ its separable closure and by $\overline{B}$ the ring $\overline{\Bbbk} \otimes_A B$ (cf. Lemma 16.21). By Lemma 16.21 c) we can assume that $\overline{B}$ and B have only one prime over $\mathfrak{p}$, and we denote these primes by $\overline{\mathfrak{q}}$ and $\mathfrak{q}$ respectively. We can also assume that $\mathfrak{p}$ is maximal, since if we can find a neighbourhood V with the required properties for closed points, then we can find such a neighbourhood for all points.

The extension $\overline{\Bbbk} \subset \overline{B}$ is étale and $\overline{B}$ has only one prime, so we must have $\overline{B} = \overline{\Bbbk}$ since $\overline{\Bbbk}$ is separably closed. This implies that the extension $\Bbbk = k(\mathfrak{p}) \subset k(\mathfrak{q})$ is trivial. Furthermore,

$A/\mathfrak{p} \subset B/\mathfrak{p}B$ is étale and $A/\mathfrak{p}$ has only one prime ideal, hence $B/\mathfrak{p}B$ is a field. In particular $\mathfrak{q} = \mathfrak{p}B$. Now notice that

$$\overline{B} = \overline{\Bbbk} \otimes_A B = \overline{\Bbbk} \otimes_{\Bbbk} \Bbbk(\mathfrak{q}).$$

Since $\Bbbk \subset \Bbbk(\mathfrak{q})$ is separable, this tensor product splits as the direct product of $[\Bbbk(\mathfrak{q}) : \Bbbk]$ factors. Since by assumption $\overline{B}$ has only one prime we must have $\Bbbk = \Bbbk(\mathfrak{q})$.

We consider the étale neighbourhood of $\mathfrak{p}$ given by $U = \operatorname{Spec} B$. Let $A' = B$ and set $B' = A' \otimes_A B$. Notice that since $\Bbbk = \Bbbk(\mathfrak{q})$ if we consider the coordinate ring of the fibre of B' over $\mathfrak{q} \in \operatorname{Spec} A'$ we obtain

$$B'/\mathfrak{q}B' = \Bbbk \otimes_{A'} B' = \Bbbk \otimes_A B.$$

In particular B' has only one prime over $\mathfrak{q}$, so $f' : A' \longrightarrow B$ is an étale extension with the same properties we have assumed for $f : A \longrightarrow B$. Furthermore there is a section $s : B' \longrightarrow A'$ given by the multiplication $B \otimes_A B \to B$; geometrically, s corresponds to the diagonal morphism. We have already proved that under our assumptions s is an open immersion (cf. Lemma 13.2) and in particular it is étale. Hence $B' \simeq B \times C$ where $\{0\} \times C$ is the kernel of f'.

Observe that we must have $C_{\mathfrak{q}} = 0$, for otherwise B' would have at least two maximal ideals over $\mathfrak{q}$. It follows that there exists $g \in B'$ such that $C_g = 0$, and choosing $V = \operatorname{Spec} B_g$ we have $f^{-1}V \simeq V$ as desired. □

16.6 The stalk of the direct image sheaf. Cohomology commutes with colimits

In this section we compute the stalks of a general direct image.

Theorem 16.26. *Let $f : Z \to Y$ be a separated and quasi-compact morphism of schemes. We will further assume that Y is locally noetherian and that f is locally finitely presented (in particular Z is also locally noetherian). Let $y \in Y$ be a point, and let $\overline{y}$ be the corresponding geometric point. Let $\mathcal{F}$ be an abelian sheaf on Z. We have*

$$(R^p f_* \mathcal{F})_{\overline{y}} = H^p_{\text{ét}}\left(\hat{Z}_{\overline{y}}, \psi^{-1}\mathcal{F}\right),$$

where $\hat{Z}_{\overline{y}} = \operatorname{Spec} \mathcal{O}_{Y,\overline{y}} \times_Y Z$ and $\psi : \hat{Z}_{\overline{y}} \longrightarrow Z$ is the canonical projection.

Remark 16.27. The Theorem is true without the assumption of local noetherianity on X and the assumption of finite presentation of f. The proof of the general case broadly follows the same lines as the special case discussed here, and would be accessible if we had developed the theory of étale morphisms without the assumption of local noetherianity.

Furthermore, having made these (unnecessary) assumptions in our study of étale morphisms, we will need a result on henselisations that we did not prove, namely that if A is a noetherian local ring then its strict henselisation is also noetherian. An indication as to why this is true is given in exercise 13.22.

To compute the stalk of the sheaf $R^p f_* \mathcal{F}$ one takes a (co)limit over the étale neighbourhoods of $\overline{y}$: the idea behind the proof of Theorem 16.26 is that this limit "commutes" with taking étale cohomology. We will prove some results in this direction which are also interesting on their own.

Lemma 16.28. *Let $(A_i)_{i \in I}$ be a filtered injective system of A_0-algebras and let $A = \varinjlim A_i$. Let $X_0, Y_0, Z_0 \to \operatorname{Spec} A_0$ be quasi-compact and separated and write $X_i = X_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} A_i$ (resp. $X = X_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} A$) and define similarly Y_i, Y, Z_i and Z.*

Let $f_0 : Y_0 \longrightarrow X_0$ (resp. $g_0 : Z_0 \longrightarrow X_0$) be an A_0-morphism and let $f_i : Y_i \longrightarrow X_i$ and $f : Y \longrightarrow X$ (resp. $g_i : Z_i \longrightarrow X_i$ and $g : Z \longrightarrow X$) be the induced morphisms over $\operatorname{Spec} A_i$ *(resp.* $\operatorname{Spec} A$*). Assume that f_0 is locally finitely presented.*

- *a) if U is a Zariski open subset of X, then there exists i and a Zariski open subset of U_i of X_i such that $U = U_i \times_{X_i} X$.*

b) $\varinjlim_i \operatorname{Hom}_{X_i}(Z_i, Y_i) = \operatorname{Hom}_X(Z, Y)$. *In particular if both Y_0 and Z_0 are locally finitely presented over X_0 and $h : Z \longrightarrow Y$ is an isomorphism there exists i and an isomorphism $h_i : Z_i \longrightarrow Y_i$ which induces h.*

c) Further assume that X_i and X are noetherian schemes. Let $f : V \longrightarrow X$ be an étale map. Then there exists i and an étale map $f_i : V_i \longrightarrow X_i$ such that f is the pullback of f_i through $\operatorname{Spec} A \longrightarrow \operatorname{Spec} A_i$. *Moreover if $f_j : V_j \longrightarrow X_j$ has the same property there exists $h \geq i, j$ such that $X_h \times_{X_i} V_i$ is isomorphic to $X_h \times_{X_j} V_j$ over X_h.*

Proof. a) By the quasi-compactness of X_0 and the construction of fibre products, X can be covered by a finite number of open affine subsets $U^\alpha = U_0^\alpha \times_{\operatorname{Spec} A_0} \operatorname{Spec} A$, where U_0^α are open affine subsets of X_0. This reduces the proof to the case when X_0 is affine, say $X_0 = \operatorname{Spec} B_0$. Hence $X_i = \operatorname{Spec} B_i$, where $B_i = A_i \otimes_{A_0} B_0$, and $X = \operatorname{Spec} B$ with $B = A \otimes_{A_0} B_0$. Finally, $B = \varinjlim B_i$ since tensor products commute with inductive limits. We can also assume that V is of the form X_f for f in B. By the description of a filtered inductive limit, there exists $i \in I$ and $g \in B_i$ such that g represents f in the colimit. It is then enough to choose $V_i = (X_i)_g$.

b) We have a natural map λ from $\varinjlim_i \operatorname{Hom}_{X_i}(Z_i, Y_i)$ to $\operatorname{Hom}_X(Z, Y)$ given by taking the pullback of maps from Z_i to Y_i. We prove that λ is surjective; the proof that it is injective is similar.

Let $h : Z \longrightarrow Y$ be a morphism over X. We can choose a finite affine Zariski open cover $\mathcal{U} = \{U^\alpha\}$ of X, a finite affine Zariski open cover $\mathcal{V} = \{V^\alpha\}$ of Y and a finite affine Zariski open cover $\mathcal{W} = \{W^\alpha\}$ of Z such that $h(W^\alpha) \subset V^\alpha$ and $f(V^\alpha) \subset U^\alpha$. By part a), possibly after refining the covers we can also assume that they are induced by finite affine Zariski open covers $\mathcal{U}_i = \{U_i^\alpha\}$ of X_i, $\mathcal{V}_i = \{V_i^\alpha\}$ of Y_i and $\mathcal{W}_i = \{W_i^\alpha\}$ of Z_i such that $f(V_i^\alpha) \subset U_i^\alpha$ and $g(W^\alpha) \subset U^\alpha$. We denote by $R^\alpha, S^\alpha, T^\alpha$ the coordinate rings of $U^\alpha, V^\alpha, W^\alpha$ and by $R^{\alpha\beta}, S^{\alpha\beta}, T^{\alpha\beta}$ the coordinate rings of the intersections $U^{\alpha\beta} = U^\alpha \cap U^\beta, V^{\alpha\beta} = V^\alpha \cap V^\beta, W^{\alpha\beta} = W^\alpha \cap W^\beta$ which (by the assumption of quasi-separatedness) are affine. We use similar notations for $U_j^\alpha, \dots$ for $j \geq i$. For every α we then have a commutative diagram of the form

$$\begin{tikzcd} & R^\alpha \arrow[dl, "f^\alpha"'] \arrow[dr, "g^\alpha"] & \\ S^\alpha \arrow[rr, "h^\alpha"] & & T^\alpha \end{tikzcd}$$

Moreover, the maps between the rings $R^{\alpha\beta}, S^{\alpha\beta}, T^{\alpha\beta}$ induced by the maps $f^\alpha, g^\alpha, h^\alpha$ are the same as those determined by $f^\beta, g^\beta, h^\beta$. We want to lift the maps h^α to maps $h_i^\alpha : S_i^\alpha \longrightarrow T_i^\alpha$ with similar properties (we already know by hypothesis that this is possible for f and g). Since S_i^α is finitely presented over R_i^α we can write

$$S_i^\alpha = \frac{R_i^\alpha[t_1, \dots, t_n]}{(P_1^\alpha, \dots, P_m^\alpha)},$$

which induces a similar presentation for S^α and S_j^α for $j \geq i$. The images $h^\alpha(t_1), \dots, h^\alpha(t_n)$ can be represented by elements in T_j^α for j large enough, and similarly the relations given by $h^\alpha(P_j^\alpha) = 0$ will be satisfied for j large enough since $T^\alpha = \varinjlim T_j^\alpha$. Hence we can construct maps $h_j^\alpha : S_j^\alpha \longrightarrow T_j^\alpha$ of R_j^α algebras which induce h^α. We have further relations which say that the induced maps on $S_j^{\alpha\beta}$ by h_j^α and h_j^β agree: again this is true for large enough j since all these rings are finitely presented over $R_j^{\alpha\beta}$. The second claim of part b) follows formally from the first.

c) *First step:* we prove the existence of V_i in the case V and X_0 are affine. Let R_i be the coordinate ring of X_i, R the coordinate ring of X and S the coordinate ring of V. Since S is étale over R, by Exercise 9.9 we can write

$$S \simeq \frac{R[t_1, \dots, t_n]}{(P_1, \dots, P_n)}$$

with the determinant of the Jacobian matrix $\left(\partial f_j/\partial t_i\right)_{i,j=1,\dots,n}$ invertible in S. Since $R = \varinjlim R_i$, there exists i such that $P_1, \dots, P_n$ are represented by polynomials $Q_1, \dots, Q_n \in R_i[t_1, \dots, t_n]$. Choosing a larger index i if necessary, we can assume that the determinant of the Jacobian matrix is invertible in $S_i = R_i[t_1, \dots, t_n]/(Q_1, \dots, Q_n)$. Hence S_i is étale over R_i, and clearly we have $S = S_i \otimes_{R_i} R$, which proves the claim.

Second step: we prove the existence of V_i in the general case. Choose finite affine Zariski open coverings $\mathcal{V} = \{V^\alpha\}$ of V and $\mathcal{U} = \{U^\alpha\}$ of X. By the previous step there exists i and étale morphisms $V_i^\alpha \longrightarrow X_i$ such that $V^\alpha = V_i^\alpha \times_{X_i} X$. We now want to glue the affine subsets V_i^α together.

Consider the intersection $V^{\alpha\beta} = V^\alpha \cap V^\beta \subset V^\alpha$. In particular we have isomorphisms $\varphi^{\alpha\beta} : V^{\alpha\beta} \longrightarrow V^{\beta\alpha}$ which satisfy $\varphi^{\alpha\gamma} = \varphi^{\beta\gamma} \circ \varphi^{\alpha\beta}$ on the intersection of the three open subsets.

Part a), applied to the family of varieties V_j^α for $j \geq i$ and to the Zariski open subset $V^{\alpha\beta}$, implies that we can choose j such that $V^{\alpha\beta} \subset V^\alpha$ is induced from an affine Zariski open subset $V_j^{\alpha\beta} \subset V_j^\alpha$. As $V^\alpha, V^{\alpha\beta}$ are finitely presented over X, by b) we can choose a (possible larger) j so that $\varphi^{\alpha\beta}$ is induced by an isomorphism $\varphi_j^{\alpha\beta} : V_j^{\alpha\beta} \longrightarrow V_j^{\beta\alpha}$. Finally, again by b) we can assume that the relation $\varphi_j^{\alpha\gamma} = \varphi_j^{\beta\gamma} \circ \varphi_j^{\alpha\beta}$ holds. Hence we can glue together the varieties V_j^α and construct a variety V_j, étale over X_j, which induces V.

Third step: we prove the claim about uniqueness. By taking an index larger than i and j we can assume that $i = j$. Hence we have two étale maps $a_i : V_i \longrightarrow X_i$ and $b_i : V_i' \longrightarrow X_i$ such that there exists an isomorphism $c : V \longrightarrow V'$ between the pullbacks $a : V \longrightarrow X$ and $b : V' \longrightarrow X$ of V and V' to X with $a = b \circ c$. As V and V' are finitely presented over X, we can apply b) to obtain the desired uniqueness. □

We now prove that in favourable circumstances colimits commute with étale cohomology.

Theorem 16.29. *Let $(A_i)_{i \in I}$ be a filtered injective system of A_0-algebras and let $A = \varinjlim A_i$. Let $X_0 \to \operatorname{Spec} A_0$ be quasi-compact and separated and write $X_i = X_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} A_i$ (resp. $X = X_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} A$)). Assume that the schemes X_i and X are locally noetherian*[28]. *We further assume that A_i and A are notherian and that X_0 is locally finitely presented over A_0.*

We let $\psi_i : X_i \to X_0$, $\psi : X \to X_0$ be the projections maps. Let $\mathcal{F}_0$ be an abelian sheaf on X_0 and let $\mathcal{F}_i = \psi_i^{-1}\mathcal{F}_0$, $\mathcal{F} = \psi^{-1}\mathcal{F}_0$. Then

a) $\mathcal{F}(X) = \varinjlim \mathcal{F}_i(X_i)$

b) $H^p(X, \mathcal{F}) = \varinjlim H^p(X_i, \mathcal{F}_i)$.

Proof. We construct a presheaf $\mathcal{G}$ on X in the following way. If $V \longrightarrow X$ is an étale open subset of X, choose $V_i \longrightarrow X_i$ such that V is induced from V_i by base change (this is possible by Lemma 16.28 c)) and define

$$\mathcal{G}(V) = \varinjlim_{h \geq i} \mathcal{F}_j(X_j \times_{X_i} V_i).$$

This definition does not depend on the choice of V_i because – as we proved in part c) of the previous Lemma – V_i is essentially unique. If $V' \longrightarrow V$ is a refinement of V we can again apply the previous Lemma, part b), to obtain an essentially unique map $V_i' \longrightarrow V_i$ which induces $V' \longrightarrow V$. We use this map to define restriction morphisms $\mathcal{G}(V) \longrightarrow \mathcal{G}(V')$.

We now prove that $\mathcal{G}$ is a sheaf. Any covering of an étale map $U \longrightarrow X$ can be refined to a finite covering by quasi compactness, hence it is enough to prove the sheaf property for finite coverings. Let $\mathcal{U} = \{U^\alpha \longrightarrow U\}$ be a finite étale covering of an étale map $U \longrightarrow X$. By the previous lemma there exist an index i, an open étale map $U_i \longrightarrow X$, and a finite covering $\mathcal{U}_i = \{U_i^\alpha\}$ of U_i such that U_i induces U by pullback, and the covering $\mathcal{U}_i$ induces $\mathcal{U}$. For $j \geq i$ we denote by U_j and by

[28] again these hypotheses are not necessary, but we need to assume them since our treatment of étale morphisms requires them. We are not going to use these hypotheses directly in the proof of the theorem, but only to apply Lemma 16.28 c)

$\mathcal{U}_j = \{U_j^\alpha\}$ the étale map and the étale covering obtained by pullback from $\mathcal{U}_i$ and U_i. Since $\mathcal{F}_j$ is a sheaf, we obtain that for all $j \geq i$ the diagram characterising the sheaf property,

$$\mathcal{F}_j(U_j) \longrightarrow \prod_{\alpha_0} \mathcal{F}_j(U_j^{\alpha_0}) \rightrightarrows \prod_{\alpha_0,\alpha_1} \mathcal{F}_j(U_j^{\alpha_0} \times_{U_j} U_j^{\alpha_1}),$$

is an equalizer. Since filtered inductive limits of abelian groups are exact the analogous diagram for $\mathcal{G}$ is also an equalizer, so $\mathcal{G}$ is a sheaf.

In order to prove a) it now suffices to show that $\mathcal{G} \simeq \mathcal{F}$. We have a natural map $\mu : \mathcal{G} \longrightarrow \mathcal{F}$ induced by the natural maps $\mathcal{F}_i(U_i) \longrightarrow \mathcal{F}(X \times_{X_i} U_i)$. It is enough to check that it is an isomorphism on stalks. Let $x \in X$ and let x_i be its image in X_i. Notice that for all i we have $\mathcal{F}_{\overline{y}} \simeq \mathcal{F}_{i,\overline{y}_i} \simeq \mathcal{F}_{0,\overline{x}_0}$. Now let $\xi \in \mathcal{G}_{\overline{x}}$ and suppose that $\mu(x) = 0$. By definition it is represented by some element in $\mathcal{F}_i(U_i)$ for some étale neighbourhood of $\overline{x}$. Given the isomorphism $\mathcal{F}_{\overline{y}} \simeq \mathcal{F}_{i,\overline{y}_i} \simeq \mathcal{F}_{0,\overline{x}_0}$, if $\mu(x) = 0$ there exists a neighbourhood U_i' refining U_i such that $x'_{U_i'} = 0$, hence $x = 0$ also in $\mathcal{G}_{\overline{x}}$. Surjectivity can be checked similarly.

We now prove b) by induction on p, the base case $p = 0$ being the content of part a). Let now $p \geq 1$; the reader may find it useful, on a first reading, to set $p = 1$ and understand the proof in this special case. Adjunction yields a compatible system of morphisms $H^p(X_i, \mathcal{F}_i) \longrightarrow H^p(X_i, \mathcal{F}_i)$ for all i, hence for every $p \geq 0$ we have a map

$$\lambda : \varinjlim H^p(X_i, \mathcal{F}_i) \longrightarrow H^p(X_i, \mathcal{F}_i).$$

We want to prove that this map is an isomorphism, and we will do so by using Čech cohomology. Let $\mathcal{U}_i = \{U_i^\alpha\}$ be a finite covering of X_i. Then for all $j \geq i$ we obtain an induced covering $\mathcal{U}_j$ of X_j. By induction we have

$$\varinjlim_{j \geq i} H^\ell(U_j^\alpha, \mathcal{F}_j) = H^\ell(U_j^\alpha, \mathcal{F}_j)$$

for all $\ell < p$. This implies that for all m and for all $\ell < p$ we have

$$\varinjlim_{j \geq i} \check{H}^m(\mathcal{U}_j, \underline{H}^\ell(\mathcal{F}_j)) = \check{H}^m(\mathcal{U}, \underline{H}^\ell(\mathcal{F})).$$

For all $j \geq i$ we now consider the spectral sequence $E_2^{m,\ell}(j) = \check{H}^m(\mathcal{U}_j, \underline{H}^\ell(\mathcal{F}_j))$, which converges to $H^{m+\ell}(X_j, \mathcal{F}_j)$, and similarly $E_2^{m,\ell} = \check{H}^m(\mathcal{U}, \underline{H}^\ell(\mathcal{F}))$, which converges to $H^{m+\ell}(X, \mathcal{F})$. All the relevant maps between these spectral sequence are natural and it is possible to check that for all r and for $j' \geq j$ there is a compatible system of morphisms

$$E_r^{m,\ell}(j) \longrightarrow E_r^{m,\ell}(j') \longrightarrow E_r^{m,\ell}$$

which commutes with differentials. In particular we have a map from $\varinjlim E_r^{m,\ell}(j)$ to $E_r^{m,\ell}$. By what we have already proven, we have

$$\varinjlim_{j \geq i} E_2^{m,\ell}(j) = E_2^{m,\ell} \qquad \text{for all } m \text{ and for all } \ell < p. \tag{19}$$

Now the only groups involved in the computation of the terms $E_r^{m,\ell}(j)$ for $\ell + m \leq p$ and $\ell < p$ correspond to indices lying in the range of formula 19, hence we obtain

$$\varinjlim_{j \geq i} E_r^{m,\ell}(j) = E_r^{m,\ell} \qquad \text{for all } m, \ell \text{ such that } m + \ell \leq p \text{ and for all } \ell < p.$$

In particular, this implies a similar property for the filtration of $H^p(X_j, \mathcal{F}_j)$:

$$\varinjlim_{j \geq i} F^m H^p(X_j, \mathcal{F}_j) = F^m H^p(X, \mathcal{F}) \qquad \text{for } m \geq 1; \tag{20}$$

this can be proven by decreasing induction on m, starting with $m = p + 1$ (for which the groups $F^m H^p(X_j, \mathcal{F}_j)$ are zero).

We now prove that the map λ is injective. Let $\lambda(\xi) = 0$ with $\xi \in H^p(X_i, \mathcal{F}_i)$. We can choose an étale covering $\mathcal{U}_i = \{U_i^\alpha\}$ of X_i as above such that $\xi|_{U_i^\alpha} = 0$ for all α. Hence $\xi \in F^1 H^p(X_i, \mathcal{F}_i)$, and by equation 20 we have that ξ is equivalent to zero in the limit.

To prove that it is surjective let $\xi \in H^p(X, \mathcal{F})$. We can choose a finite covering $\mathcal{U} = \{U^\alpha\}$ of X such that $\xi|_{U^\alpha}$ for all α hence $\xi \in F^1 H^p(X, \mathcal{F})$. We can assume that $\mathcal{U}$ is induced from an étale covering $\mathcal{U}_i$ of X_i and we conclude again by equation 20. □

We now prove Theorem 16.26 by applying the previous Theorem in the special situation when the schemes $\operatorname{Spec} A_i$ are the affine étale neighbourhoods of $y \in Y$.

Proof of theorem 16.26. We can assume that $Y = \operatorname{Spec} A_0$ is affine with A_0 noetherian. We are interested in $(R^p f_* \mathcal{F})_{\overline{y}}$. Notice that $R^p f_* \mathcal{F}$ is the sheaf corresponding to the presheaf

$$V \mapsto H^p\left(f^{-1}(V), \mathcal{F}\right)$$

To check this, take an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$, apply f_*, and – denoting by $\mathcal{G}$ the presheaf given by $H^p(f_* \mathcal{I}^\bullet)$ – observe that we have

$$\begin{aligned} \mathcal{G}(U) &= H^p\left(f_* I^0(U) \to f_* I^1(U) \to \cdots\right) \\ &= H^p\left(I^0(f^{-1}U) \to I^1(f^{-1}(U)) \to \cdots\right) \\ &= H^p(f^{-1}U, \mathcal{F}). \end{aligned}$$

We are interested in the stalks, which are the same for a presheaf and for the associated sheaf. Thus we have

$$(R^p f_* \mathcal{F})_{\overline{y}} = \varinjlim_{\overline{y} \to V \to Y} H^p(f^{-1}V, \mathcal{F}),$$

where the colimit can be taken on the affine neighbourhoods only. We apply the previous Theorem in the case where A_i are the coordinate rings of the affine étale neighbourhoods of $\overline{y}$ and $X_0 = Z$. To apply the theorem we need to know that $X_0 = Z$ and $\mathcal{O}_{Y,\overline{y}} \times_Y Z$ are locally noetherian. This is true by assumption for the former; for the latter, since it is locally finitely presented over $\mathcal{O}_{Y,\overline{y}} \times_Y Z$, this follows from the fact that the strict henselisation of a local noetherian ring is noetherian. Hence we have that the colimit is equal to

$$H^p\left((\varinjlim A_i) \times_A Z, \psi^{-1}, \mathcal{F}\right),$$

proving the claim. □

16.7 Exercises

Exercise 16.30. Let $f : X \longrightarrow Y$ be a map of schemes.

a) Let $\mathcal{G}$ be a sheaf on Y. Prove that $f_P^{-1}\mathcal{G}$ is a separated presheaf.

b) Let $\mathcal{G}$ be a presheaf on Y. Prove that $f^{-1}(\mathcal{G}^\sharp) = (f_P^{-1}(\mathcal{G}))^\sharp$

c) For an abelian group A and a scheme Z denote by A_Z the sheaf associated with the constant presheaf $U \mapsto A$ for all non empty U. Prove that $f^{-1}A_Y = A_X$.

d) construct a natural map $H^*(f) : H^*_{\text{ét}}(Y, A_Y) \longrightarrow H^*_{\text{ét}}(X, A_X)$.

Exercise 16.31. Prove Proposition 16.10.

17 Cohomology of points and curves

In this section we describe the étale cohomology of points (spectra of fields) and of smooth curves over algebraically closed fields. The results for curves will rely heavily on the results for points: indeed, our general strategy will be that of comparing the cohomology of the curve with the cohomology of its generic point, for which the calculation will be reduced to one in Galois cohomology (which, while not easy, is possibly a little more familiar to the reader than the étale side of the story).

17.1 Cohomology of abelian sheaves on $(\operatorname{Spec} K)_{\text{ét}}$

We begin our discussion by relating the étale cohomology of (spectra of) fields with their Galois cohomology. As already hinted at in exercise 11.44, the small étale site of a scheme of the form $\operatorname{Spec} K$ with K a field is relatively easy to describe:

Theorem 17.1 (Étale sheaves on $\operatorname{Spec} K$)**.** *Let L/K be a finite Galois (in particular, separable) extension of fields, let $S = \operatorname{Spec} K$, and let $\mathcal{F}$ be a sheaf on the small étale site of S.*

1. *There is a natural action of $\operatorname{Gal}(L/K)$ on $\mathcal{F}(\operatorname{Spec} L)$.*
2. *There is a natural identification $\mathcal{F}(\operatorname{Spec} L)^{\operatorname{Gal}(L/K)} = \mathcal{F}(K)$.*
3. *The category of étale abelian sheaves on the small étale site of $\operatorname{Spec} K$ is equivalent to the category of $\operatorname{Gal}(K^{\text{sep}}/K)$-modules, that is, discrete abelian groups sets equipped with a continuous action of $\operatorname{Gal}(K^{\text{sep}}/K)$, where K^{sep} denotes a separable closure of K.*
4. *Suppose now that $\mathcal{F}$ is an abelian étale sheaf on $(\operatorname{Spec} K)_{\text{ét}}$. If K^{sep} is not a finite extension of K, then $\operatorname{Spec} K^{\text{sep}}$ is not an element of $(\operatorname{Spec} K)_{\text{ét}}$, and we can define*

$$\mathcal{F}(\operatorname{Spec} K^{\text{sep}}) = \varinjlim_{L/K \text{ finite separable}} \mathcal{F}(L);$$

notice that if K^{sep}/K is a finite extension, then the previous definition gives back $\mathcal{F}(\operatorname{Spec} K^{\text{sep}})$ (because this is one of the objects on which we take the colimit). Also notice that (independently of whether K^{sep}/K is a finite or infinite extension) there is a natural action of $\operatorname{Gal}(K^{\text{sep}}/K)$ on $\mathcal{F}(\operatorname{Spec} K^{\text{sep}})$, obtained by passage to the colimit when K^{sep}/K is not finite. For this Galois action we have

$$H^i_{\text{ét}}(\operatorname{Spec} K, \mathcal{F}) = H^i(\operatorname{Gal}(K^{\text{sep}}/K), \mathcal{F}(K^{\text{sep}})).$$

Proof. For simplicity of notation let $G = \operatorname{Gal}(L/K)$.

1. This is true by functoriality: G acts on $\operatorname{Spec} L$, hence it acts on $\mathcal{F}(\operatorname{Spec} L)$.
2. Consider the étale covering $\operatorname{Spec} L \to \operatorname{Spec} K$. The sheaf condition yields an exact sequence

$$0 \to \mathcal{F}(\operatorname{Spec} K) \to \mathcal{F}(\operatorname{Spec} L) \rightrightarrows \mathcal{F}(\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L). \tag{21}$$

In order to better understand this sequence, recall from standard Galois theory that $L \otimes_K L$ is isomorphic to L^G (the direct product of $|G|$ copies of L, indexed by the elements of G), with the isomorphism being given by

$$\begin{array}{rccc} \varphi: & L \otimes_K L & \to & L^G \\ & x \otimes y & \mapsto & (xg(y))_{g \in G}. \end{array}$$

Let us recall why this is an isomorphism. It is certainly a map of rings, and both sides are $[L : K]$-dimensional vector spaces over L. Furthermore, by construction the image is an L-vector subspace of L^G, so it suffices to show that the image contains an L-basis of L^G.

Let $m := [L : K]$ and let $\ell_1, \dots, \ell_m$ be a basis of L as a K-vector space. The image of the morphism above contains the m vectors $\varphi(1 \otimes \ell_1), \dots, \varphi(1 \otimes \ell_m)$, which we claim are linearly independent over L. To see this, suppose by contradiction that they are not. Denote by $\sigma_1, \dots, \sigma_m$ the elements of G, considered as homomorphisms $L \to L$. The matrix

$$\begin{pmatrix} \sigma_1(x_1) & \sigma_2(x_1) & \cdots & \sigma_n(x_1) \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_1(x_n) & \sigma_2(x_n) & \cdots & \sigma_n(x_n) \end{pmatrix},$$

whose rows are precisely $\varphi(1 \otimes x_1), \dots, \varphi(1 \otimes x_n)$, would then have zero determinant, which implies that there would exist scalars $\lambda_1, \dots, \lambda_n$ such that the corresponding linear combination of the *rows* would be zero. But this means

$$\sum_{i=1}^{m} \lambda_i \sigma_i(x_j) = 0 \quad \forall j,$$

hence by K-linearity $\sum_{i=1}^{m} \lambda_i \sigma_i(x) = 0$ for every $x \in L$. This contradicts Artin's theorem on the independence of characters.

In particular, we obtain $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong \coprod_{g \in G} \operatorname{Spec} L$, so $\mathcal{F}(\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L) = \coprod_{g \in G} \mathcal{F}(\operatorname{Spec} L)$. We now describe the two arrows $\mathcal{F}(\operatorname{Spec} L) \to \coprod_{g \in G} \mathcal{F}(\operatorname{Spec} L)$. At the level of rings, the two maps from $\mathcal{F}(\operatorname{Spec} L)$ to the copy of $\mathcal{F}(\operatorname{Spec} L)$ indexed by $g \in G$ are induced respectively by

$$\begin{array}{ccccc} L & \xrightarrow{\pi_1^\#} & L \otimes_K L & \xrightarrow{\varphi} & L \\ \ell & \mapsto & \ell \otimes 1 & \mapsto & \ell \end{array}$$

and

$$\begin{array}{ccccc} L & \xrightarrow{\pi_2^\#} & L \otimes_K L & \xrightarrow{\varphi} & L \\ \ell & \mapsto & 1 \otimes \ell & \mapsto & g(\ell), \end{array}$$

so we obtain the two maps

$$\begin{array}{cccc} \varphi_1, \varphi_2 : & \mathcal{F}(\operatorname{Spec} L) & \to & \coprod_{g \in G} \mathcal{F}(\operatorname{Spec} L) \\ & s & \mapsto & (s, s, \dots, s) \\ & s & \mapsto & (g(s))_{g \in G} \end{array}$$

The exactness of (21) then describes $\mathcal{F}(\operatorname{Spec} K)$ as the set of $s \in \mathcal{F}(\operatorname{Spec} L)$ such that $\varphi_1(s) = \varphi_2(s)$, that is,

$$\mathcal{F}(\operatorname{Spec} K) = \{s \in \mathcal{F}(\operatorname{Spec} L) : (s, \cdots, s) = (g(s))_{g \in G}\} = \mathcal{F}(\operatorname{Spec} L)^{\operatorname{Gal}(L/K)}.$$

3. We describe the functors that give the equivalence. Given an abelian sheaf $\mathcal{F}$, the corresponding $\operatorname{Gal}(K^{\text{sep}}/K)$-module is $\varinjlim_{L/K \text{ Galois}} \mathcal{F}(L)$ with its natural Galois action (see part (1)). Conversely, given a $\operatorname{Gal}(K^{\text{sep}}/K)$-module A, we may define a sheaf on $(\operatorname{Spec} K)_{\text{ét}}$ by the formula
$$\mathcal{F}(\operatorname{Spec} L) = A^{\operatorname{Gal}(K^s/L)}$$
on the connected objects, and then by $\mathcal{F}\left(\coprod_i \operatorname{Spec} L_i\right) = \prod_i \mathcal{F}(\operatorname{Spec} L_i)$ in the general case (recall that the étale site of $\operatorname{Spec} K$ consists of finite disjoint unions of objects of the form $\operatorname{Spec} L_i$ with L_i/K finite and separable, see proposition 7.2). It remains to see that these two functors give an equivalence; let's denote them by

$$\begin{array}{ccc} \{\text{continuous } \operatorname{Gal}(K^{\text{sep}}/K)\text{-modules}\} & \to & \{\text{abelian sheaves on } (\operatorname{Spec} K)_{\text{ét}}\} \\ M & \mapsto & \mathcal{F}_M \end{array}$$

and

$$\begin{array}{ccc} \{\text{abelian sheaves on } (\operatorname{Spec} K)_{\text{ét}}\} & \to & \{\text{continuous } \operatorname{Gal}(K^{\text{sep}}/K)\text{-modules}\} \\ \mathcal{F} & \mapsto & \mathcal{F}\left(\operatorname{Spec} K^{\text{sep}}\right). \end{array}$$

One should check that $\mathcal{F}_M$ is indeed a sheaf, but this is very similar to what we did in part (2) and we omit the details. Notice that (thanks to our explicit description of the étale site of $\operatorname{Spec} K$) it suffices to check the sheaf condition on morphisms of the form $\operatorname{Spec} L \to \operatorname{Spec} K$. Using that by construction both $\mathcal{F}_M(\operatorname{Spec} L)$ and $\mathcal{F}_M(\operatorname{Spec} K)$ embed into $\mathcal{F}_M(\operatorname{Spec} F)$, where F/K is Galois and contains L, the verification of the sheaf condition boils down to precisely what we did in (2).

Let us check that the two functors we described are quasi-inverse to each other. It is clear by construction that $M \mapsto \mathcal{F}_M \to \mathcal{F}_M(\operatorname{Spec} K^{\text{sep}})$ is (isomorphic to) the identity, for by definition

$$\mathcal{F}_M(\operatorname{Spec} K^{\text{sep}}) = \varinjlim_{L/K \text{ finite separable}} \mathcal{F}_M(L) = \varinjlim_{L/K \text{ finite separable}} M^{\operatorname{Gal}(K^{\text{sep}}/L)} = M,$$

where the last equality depends on the fact that M is a *continuous* G-module (the stabiliser of every $m \in M$ is open in $\operatorname{Gal}(K^{\text{sep}}/K)$, so it corresponds to some *finite* extension L/K).

Conversely, let us start with a sheaf $\mathcal{F}$ and consider the associated module $M := \mathcal{F}(\operatorname{Spec} K^{\text{sep}})$. That the action of G on M is continuous is due to the fact that every element m of M is an element of $\mathcal{F}(\operatorname{Spec} L)$ for some *finite* separable extension L/K, so the open subgroup $\operatorname{Gal}(K^{\text{sep}}/L)$ of G stabilises m. Finally, for any finite separable extension L/K we have

$$\mathcal{F}_M(\operatorname{Spec} L) = M^{\operatorname{Gal}(K^{\text{sep}}/L)} = \left(\varinjlim_{F/K \text{ finite separable}} \mathcal{F}(\operatorname{Spec} F) \right)^{\operatorname{Gal}(K^{\text{sep}}/L)};$$

since the finite, Galois, separable extension of K containing L are cofinal in the system of all finite separable extensions of K, we may take the colimit only over these, and we get

$$\begin{aligned}
\left(\varinjlim_{\substack{F/L/K \text{ finite} \\ \text{separable Galois}}} \mathcal{F}(\operatorname{Spec} F) \right)^{\operatorname{Gal}(K^{\text{sep}}/L)} &= \varinjlim_{\substack{F/L/K \text{ finite} \\ \text{separable Galois}}} \mathcal{F}(\operatorname{Spec} F)^{\operatorname{Gal}(K^{\text{sep}}/L)} \\
&= \varinjlim_{\substack{F/L/K \text{ finite} \\ \text{separable Galois}}} \mathcal{F}(\operatorname{Spec} F)^{\operatorname{Gal}(F/L)} \\
&= \varinjlim_{\substack{F/L/K \text{ finite} \\ \text{separable Galois}}} \mathcal{F}(\operatorname{Spec} L) \\
&= \mathcal{F}(\operatorname{Spec} L)
\end{aligned}$$

where the first equality is obvious, the second follows from the fact that $\operatorname{Gal}(K^{\text{sep}}/F)$ acts trivially on $\mathcal{F}(\operatorname{Spec} F)$ by functoriality, and the third follows from part (2).

4. Under the equivalence described above, the global sections functor $\mathcal{F} \mapsto \mathcal{F}(\operatorname{Spec} K)$ becomes $A \mapsto A^{\operatorname{Gal}(K^s/K)}$. By definition, étale cohomology is given by the derived functors of the former and Galois cohomology is given by the derived functors of the latter. As the categories are equivalent, the claim follows.

□

17.2 Tools from the previous lectures

For the reader's convenience, we collect and re-state here some results proven in the previous lectures:

Theorem 17.2 (Lemma 16.29, see also [Sta19, Tag 09YQ])**.** *Let $X = \lim_{i\in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of abelian sheaves on $(X_i, f_{i'i})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \varinjlim f_i^{-1}\mathcal{F}_i$. Then*

$$\varinjlim_{i\in I} H^p_{\text{ét}}(X_i, \mathcal{F}_i) = H^p_{\text{ét}}(X, \mathcal{F}).$$

for all $p \geq 0$.

Proposition 17.3 (Theorem 16.24)**.** *Let $f : X \to Y$ be a finite morphism and $\mathcal{F} \in \mathbf{Ab}(X_{\text{ét}})$. Then $R^q f_*\mathcal{F} = 0$ for $q > 0$.*

Proposition 17.4 (Theorem 16.26)**.** *Let $f : X \to Y$ be a quasi-compact quasi-separated morphism. Then for every abelian sheaf $\mathcal{F}$ on X and any geometric point $\overline{y}$ of Y we have*

$$(R^p f_*\mathcal{F})_{\overline{y}} = H^p\left((X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\overline{y}}), \pi_1^{-1}\mathcal{F}\right),$$

where $\pi_1 : X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\overline{y}} \to X$ is the canonical projection.

We shall also need the following fact, known as the Leray spectral sequence, which is in fact a special case of the Grothendieck spectral sequence (corollary 14.23):

Theorem 17.5 (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism and let $\mathcal{F}$ be an abelian sheaf on X. There is a second-page spectral sequence*

$$E_2^{p,q} = H^p(Y_{\text{ét}}, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X_{\text{ét}}, \mathcal{F}).$$

Proof. It suffices to apply corollary 14.23 to the categories $\mathcal{A} = \mathbf{Ab}(X_{\text{ét}})$, $\mathcal{B} = \mathbf{Ab}(Y_{\text{ét}})$ and $\mathcal{C} = \mathbf{Ab}$, and to the functors $F = f_*$ and $G = \Gamma(Y_{\text{ét}}, -)$. We just need to check:

- that F carries injective objects of $\mathcal{A}$ to G-acyclic objects of $\mathcal{B}$: this follows from lemma 16.3;
- that $G \circ F = \Gamma(X_{\text{ét}}, -)$. This follows from the definitions, because for any $\mathcal{F} \in \mathbf{Ab}(X_{\text{ét}})$ we have
$$(G \circ F)(\mathcal{F}) = \Gamma(Y_{\text{ét}}, f_*\mathcal{F}) = (f_*\mathcal{F})(Y) = \mathcal{F}(f^{-1}Y) = \mathcal{F}(X) = \Gamma(X_{\text{ét}}, \mathcal{F}).$$

□

17.3 The fundamental exact sequence

Theorem 17.6. *Let X be a connected, locally Noetherian normal scheme. The following sequence in $\mathbf{Ab}(X_{\text{ét}})$ is exact (surjective on the right if X is assumed to be regular):*

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_m \to \bigoplus_{x\in X^{(1)}} (i_x)_*\mathbb{Z}_x \dashrightarrow 0. \tag{22}$$

Proof. We claim that it suffices to show that (22) is exact *in the Zariski topology* upon pullback to every $U \in |X_{\text{ét}}|$. Indeed:

- injectivity: $\mathbb{G}_m \to j_*\mathbb{G}_m$ is injective if and only if $\mathbb{G}_m(U) \to (j_*\mathbb{G}_m)(U)$ is injective for every $U \in X_{\text{ét}}$. In particular, if injectivity fails for some $U \in |X_{\text{ét}}|$, then it also fails (in the Zariski topology) for the object U in U_{Zar}. Thus proving injectivity upon restriction to every U_{Zar} is enough to show injectivity on $X_{\text{ét}}$.
- surjectivity (in the regular case): as we know from theorem 11.30, surjectivity means that for every $U \in |X_{\text{ét}}|$ and every $s \in \left(\bigoplus_{x\in X_{(0)}} (i_x)_*\mathbb{Z}\right)(U)$ there is an étale covering $\mathcal{U} = (U_i \to U)$ of U such that for every i the restriction $s|_{U_i}$ is in the image of $j_*\mathbb{G}_{m,\eta}(U_i) \to \left(\bigoplus_{x\in X_{(0)}} (i_x)_*\mathbb{Z}\right)(U_i)$. Suppose that (22) is exact on U_{Zar}: then there exists a *Zariski* open cover with the same property. Since a Zariski covering is in particular an étale covering, and this holds for every U, we get the desired surjectivity.

- exactness at $j_*\mathbb{G}_m$: this is proven exactly as above; given a section s over U that maps to 0, we need to find an étale covering such that the restrictions of s to the objects in the covering come from $\mathbb{G}_m$. However, if the sequence is exact in U_{Zar}, we can a Zariski covering with the same property, and this is in particular an étale covering.

Since étale maps preserve local Noetherianity[29], normality[30] and regularity[31] and X has these properties, then so does every U we consider. In other words, we are reduced to showing that (22) is exact *in the Zariski topology* for every connected, Noetherian normal scheme X.

Remark 17.7. More abstractly: exactness can be checked on (étale) stalks, which are obtained as a filtered colimit over sequences in the Zariski topology. If each Zariski sequence is exact, given that filtered colimits are exact then also the limit sequence is exact.

Since the question of exactness can be tested on stalks, it can also be tested on a (Zariski) open covering; let $\operatorname{Spec} A \hookrightarrow X$ be an open affine subscheme (with A Noetherian, local, and integrally closed, and A regular – hence an UFD – if X is regular). Over $\operatorname{Spec} A$, the sections of (22) are given by

$$0 \to A^\times \to \operatorname{Frac}(A)^\times \xrightarrow{\operatorname{div}} \bigoplus_{\mathfrak{p}:\operatorname{ht}\mathfrak{p}=1} \mathbb{Z} \dashrightarrow 0, \tag{23}$$

where as usual the divisor map takes $a \in \operatorname{Frac}(A)^\times$ to the collection $(v_\mathfrak{p}(a))_\mathfrak{p}$. Clearly $A^\times \to \operatorname{Frac}(A)^\times$ is injective (since A is a domain). Notice that $v_\mathfrak{p}(a)$ is 0 if and only if $a \in A_\mathfrak{p}^\times$; in particular, a belongs to the localisation $A_\mathfrak{p}$. We now recall the following standard lemma from commutative algebra:

Lemma 17.8. *Let A be a Noetherian, integrally closed domain. Then*

$$\bigcap_{\mathfrak{p}:\operatorname{ht}\mathfrak{p}=1} A_\mathfrak{p} = A.$$

The lemma implies immediately that the kernel of div is contained in A; furthermore, every non-unit of A is contained in some prime of height 1 (this is essentially Krull's principal ideal theorem), so the kernel of div consists precisely of the units of A. In other words, we have shown that the solid part of (23) is an exact sequence, hence the same statement holds for the solid part of (22) (as a sequence of sheaves). Finally, suppose that our scheme X is regular, in which case we can assume that A is a Noetherian UFD (exactness of a sequence of sheaves is a local problem, so we can assume that A is local, and a regular local ring is a UFD). In this case every prime ideal $\mathfrak{q}$ of A of height one is generated by a single element $f_\mathfrak{q} \in A \subset \operatorname{Frac}(A)^\times$. Such an element is clearly prime in A, hence not contained in any other prime ideal, so $\operatorname{div}(f_\mathfrak{q}) = (v_\mathfrak{p}(f_\mathfrak{q}))_\mathfrak{p}$ is 0 for $\mathfrak{p} \neq \mathfrak{p}$ and is 1 for $\mathfrak{p} = \mathfrak{q}$. This proves the desired surjectivity.

□

17.4 Tools from Galois cohomology

Definition 17.9. A field K is said to be C_1 if the following holds: for every polynomial $f \in K[t_0, \dots, t_{n-1}]$ homogeneous of degree d, $1 \leq d < n$, there exists $(a_0, \dots, a_{n-1}) \in K^n \setminus \{(0, \dots, 0)\}$ such that $f(a_0, \dots, a_{n-1}) = 0$. In other words, every projective variety in $\mathbb{P}_{n-1,K}$ defined by an equation of degree at most $n-1$ has a K-rational point.

Definition 17.10. Let K be a field. The **Brauer group** of K is

$$\operatorname{Br}(K) := H^2\left(\operatorname{Gal}\left(\overline{K}/K\right), \overline{K}^\times\right).$$

[29]By definition, an étale map is locally of finite presentation, and a finitely presented ring over a Noetherian ring is Noetherian by Hilbert's basis theorem.

[30]this is not obvious, and depends on Serre's characterisation of *normal* as (S_2)+(R_1). For our applications, however, we won't need any deep results: we shall only be interested in the case of curves, and for a Noetherian local ring of dimension 1 being regular and being integrally closed are equivalent.

[31]Corollary 8.5

Theorem 17.11. *Let K be a C_1 field: then the Brauer group of K is trivial.*

Theorem 17.12 (Tsen)**.** *Let k be an algebraically closed field and let X be a curve over k. Let K be the field of rational functions on X: then K is a C_1 field.*

Theorem 17.13. *Let K be a field. Suppose that for every finite separable extension K'/K the Brauer group of K' vanishes: then $H^q\left(\mathrm{Gal}(K^{\mathrm{sep}}/K), (K^{\mathrm{sep}})^\times\right) = 0$ for all $q \geq 1$.*

Corollary 17.14. *Let k be an algebraically closed field and let K be an extension of k of transcendence degree 1. Then $H^q\left(\mathrm{Gal}\left(K^{\mathrm{sep}}/K\right), (K^{\mathrm{sep}})^\times\right) = 0$ for every $q > 0$.*

Proof. By theorem 17.13 it suffices to show that if K'/K is any finite extension we have

$$H^2\left(\mathrm{Gal}\left(\overline{K}/K'\right), \overline{K}^\times\right) = 0.$$

By theorem 17.11 it suffices to show that K' is a C_1-field, and this follows from theorem 17.12. More precisely, since K'/K is finite we have $\mathrm{trdeg}_k(K') = \mathrm{trdeg}_k(K) = 1$, and we can write K' as the union (colimit) of all the *finitely generated* extensions K'' of k contained in K' and of transcendence degree 1. Any such extension is the function field of a curve over k, so theorem 17.12 implies $\mathrm{Br}(K'') = 0$. We then obtain

$$\mathrm{Br}(K') = \varinjlim_{K''} \mathrm{Br}(K'') = (0)$$

as desired. □

17.5 Cohomology of curves over an algebraically closed field

We set the notation for the whole section. Let k be an algebraically closed field, X/k be a smooth projective connected curve of genus g, and let $\eta = \mathrm{Spec}\, k(X)$ be its generic point. Denote by $j : \eta \to X$ the inclusion of this generic point and by $i_x : x \to X$ the inclusion of a closed point.

Lemma 17.15. *Let K be a field of transcendence degree 1 over an algebraically closed field k. We have $H^q\left((\mathrm{Spec}\, K)_{\text{ét}}, \mathbb{G}_m\right) = 0$ for all $q > 0$. In particular, $H^q(\eta_{\text{ét}}, \mathbb{G}_{m,\eta})$ is trivial for all $q > 0$.*

Proof. Let $G = \mathrm{Gal}\left(K^{\mathrm{sep}}/K\right)$. For any finite separable extension L of K we have $\mathbb{G}_{m,\eta}(L) = L^\times$, so the G-module corresponding to $\mathbb{G}_{m,\eta}$ under the equivalence of theorem 17.1 is

$$\varinjlim_{L \text{ finite separable}} L^\times = (K^{\mathrm{sep}})^\times.$$

By theorem 17.1 again we then obtain

$$H^q\left(\eta_{\text{ét}}, \mathbb{G}_{m,\eta}\right) = H^q\left(\mathrm{Gal}(K^{\mathrm{sep}}/K), (K^{\mathrm{sep}})^\times\right),$$

and the latter vanishes for every $q \geq 1$ by corollary 17.14. □

Lemma 17.16. *The higher direct images of $\mathbb{G}_{m,\eta}$ via j vanish; in symbols, we have $R^p j_* \mathbb{G}_{m,\eta} = 0$ for all $p > 0$.*

Proof. It suffices to check that the stalks of these sheaves are trivial at every geometric point. Let $\overline{s}$ be a geometric point of X; by proposition 17.4 we have

$$\left(R^p j_* \mathbb{G}_{m,\eta}\right)_{\overline{s}} = H^p\left((\eta \times_X \mathrm{Spec}\, \mathcal{O}_{X,\overline{s}})_{\text{ét}}, \mathbb{G}_m\right),$$

where we omit the pullback map from η to $\eta \times_X \mathcal{O}_{X,\overline{x}}$ (notice however that the pullback of $\mathbb{G}_{m,\eta}$ to $\eta \times_X \mathcal{O}_{X,\overline{s}}$ is isomorphic to the multiplicative group of this latter scheme). From the construction of the fibred product of schemes we know that $\eta \times_X \mathcal{O}_{X,\overline{s}}$ is the same as $\eta \times_{\mathrm{Spec}\, A} \mathcal{O}_{X,\overline{s}}$, where $\mathrm{Spec}\, A$

is an open affine neighbourhood of s in X. In this case η is simply $\operatorname{Spec} K$ with $K = \operatorname{Frac}(A)$ and everything is sight is affine, so we are reduced to considering

$$\begin{aligned}(R^p j_* \mathbb{G}_{m,\eta})_{\overline{s}} &= H^p\left((\eta \times_X \operatorname{Spec} \mathcal{O}_{X,\overline{s}})_{\text{ét}}, \mathbb{G}_m\right) \\ &= H^p\left((\operatorname{Spec} K \times_{\operatorname{Spec} A} \operatorname{Spec} \mathcal{O}_{X,\overline{s}})_{\text{ét}}, \mathbb{G}_m\right) \\ &= H^p\left((\operatorname{Spec} K \otimes_A \mathcal{O}_{X,\overline{s}})_{\text{ét}}, \mathbb{G}_m\right)\end{aligned}$$

Now recall that $\mathcal{O}_{X,\overline{s}} = \mathcal{O}^{sh}_{X,s}$ is strictly Henselian (proposition 13.18), and notice that taking the tensor product over A with $\operatorname{Frac}(A)$ amounts to taking a localisation of $\mathcal{O}_{X,\overline{s}}$. Now the maximal ideal of this ring is principal (see exercise 13.23 for more details on the henselisation of a DVR), and the uniformiser gets inverted in K (this is because the maximal ideal is generated by a uniformiser of $\mathcal{O}_{X,x}$, which is certainly inverted in K). It follows that the localisation of $\mathcal{O}_{X,\overline{s}}$ that we are taking is its fraction field. Notice furthermore that since every element of $\mathcal{O}_{X,\overline{s}}$ is algebraic over $\mathcal{O}_{X,x}$ we have

$$\operatorname{trdeg}_k L = \operatorname{trdeg}_k \operatorname{Frac} \mathcal{O}_{X,x} = 1,$$

so the stalk we are trying to study is $H^p\left((\operatorname{Spec} L)_{\text{ét}}, \mathbb{G}_m\right)$ with L of transcendence degree 1 over the algebraically closed field k. It then follows from lemma 17.15 that this cohomology group vanishes, so all the stalks of $R^p j_* \mathbb{G}_{m,\eta}$ at closed points are 0. As for the generic point, the stalk is $H^p\left((\operatorname{Spec} K \otimes_K \mathcal{O}_{X,\overline{\eta}})_{\text{ét}}, \mathbb{G}_m\right)$, which is zero because $\mathcal{O}_{X,\overline{\eta}}$ is the strict henselisation of a field, hence a separably closed field. It follows that $R^p j_* \mathbb{G}_{m,\eta} = 0$ as desired. $\square$

Lemma 17.17. *We have*

$$H^q(X, j_* \mathbb{G}_{m,\eta}) = 0$$

for all $q \geq 1$.

Proof. The Leray spectral sequence for $j : \eta \to X$ yields $H^q(X_{\text{ét}}, R^p j_* \mathbb{G}_{m,\eta}) \Rightarrow H^{p+q}(\eta_{\text{ét}}, \mathbb{G}_{m,\eta})$. By lemma 17.16 we have $R^p j_* \mathbb{G}_{m,\eta} = 0$ for all $p > 0$, so the spectral sequence degenerates on page 2 and we get

$$H^q(X_{\text{ét}}, j_* \mathbb{G}_{m,\eta}) = H^q(\eta_{\text{ét}}, \mathbb{G}_{m,\eta});$$

by lemma 17.15 we are done. $\square$

Lemma 17.18. *Let X be a smooth projective connected curve over an algebraically closed field k. Then for every $q \geq 1$ we have*

$$H^q\left(X_{\text{ét}}, \bigoplus_{x \in X^{(1)}} (i_x)_* \mathbb{Z}_x\right) = 0$$

Proof. The cohomology group we want to compute is given by

$$H^q\left(X_{\text{ét}}, \bigoplus_{x \in X^{(1)}} (i_x)_* \mathbb{Z}_x\right) = H^q\left(X_{\text{ét}}, \bigoplus_{x \in X_{(0)}} (i_x)_* \mathbb{Z}_x\right) = \bigoplus_{x \in X_{(0)}} H^q\left(X_{\text{ét}}, (i_x)_* \mathbb{Z}_x\right)$$

where the first equality follows from the fact that a closed irreducible subscheme of a connected curve is (topologically) either a point or the whole curve (so points of codimension 1 are simply points of dimension 0), while the second equality comes from theorem 17.2 (used in the special case where $f_i : X \to X_i$ is the identity for every i). Finally, $H^q\left(X_{\text{ét}}, (i_x)_* \mathbb{Z}_x\right)$ vanishes for all x and all $q \geq 1$. To see this, apply proposition 17.3 (notice that i_x is finite because it is a closed immersion) to get that $R^p(i_x)_* \mathbb{Z}$ is zero for $p > 0$, and then apply the Leray spectral sequence to obtain $H^p\left(X_{\text{ét}}, (i_x)_* \mathbb{Z}\right) = H^p(x_{\text{ét}}, \mathbb{Z})$. This latter cohomology group vanishes because $x = \operatorname{Spec} k$ and k is algebraically closed. $\square$

Theorem 17.19. *Let X be a smooth, projective, connected curve of genus g over the algebraically closed field k. The following hold:*

1. $H^0(X_{\text{ét}}, \mathbb{G}_m) = k^\times$;

2. $H^1(X_{\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}(X)(k)$;

3. $H^q(X_{\text{ét}}, \mathbb{G}_m) = (0)$ *for every* $q \geq 2$.

Proof. Consider the fundamental exact sequence (22), which is also surjective in our case because a smooth scheme is regular by theorem 10.13. Taking étale cohomology of this sequence we get

$$\begin{aligned} 0 &\to H^0(X_{\text{ét}}, \mathcal{O}_X)^\times \to H^0(X_{\text{ét}}, j_*\mathbb{G}_{m,\eta}) \to H^0\left(X_{\text{ét}}, \bigoplus_{x \in X^{(1)}} (i_x)_*\mathbb{Z}_x\right) \\ &\to H^1(X_{\text{ét}}, \mathbb{G}_m) \to H^1(X_{\text{ét}}, j_*\mathbb{G}_m) \end{aligned}$$

and for every $q \geq 1$

$$H^q\left(X_{\text{ét}}, \bigoplus_{x \in X^{(1)}} (i_x)_*\mathbb{Z}_x\right) \to H^{q+1}(X_{\text{ét}}, \mathbb{G}_m) \to H^{q+1}(X_{\text{ét}}, j_*\mathbb{G}_m).$$

Lemma 17.18 shows that $H^q\left(X_{\text{ét}}, \bigoplus_{x \in X^{(1)}}(i_x)_*\mathbb{Z}_x\right)$ vanishes for all $q \geq 1$, and lemma 17.17 tells us that all the higher $H^i(X_{\text{ét}}, j_*\mathbb{G}_m)$ vanish. This leads us to the sequences

$$0 \to H^0(X_{\text{ét}}, \mathcal{O}_X)^\times \to H^0(X_{\text{ét}}, j_*\mathbb{G}_{m,\eta}) \to \bigoplus_{x \in X^{(1)}} H^0\left(X_{\text{ét}}, (i_x)_*\mathbb{Z}_x\right) \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0$$

and for every $q \geq 1$

$$0 \to H^{q+1}(X_{\text{ét}}, \mathbb{G}_m) \to 0.$$

This already shows that $H^q(X_{\text{ét}}, \mathbb{G}_m) = 0$ for $q \geq 2$, and clearly $H^0(X_{\text{ét}}, \mathbb{G}_m) = \Gamma(X, \mathcal{O}_X)^\times = k^\times$ since X is projective. It remains to understand $H^1(X_{\text{ét}}, \mathbb{G}_m)$; for this, consider again

$$0 \to H^0(X_{\text{ét}}, \mathcal{O}_X)^\times \to H^0(X_{\text{ét}}, j_*\mathbb{G}_{m,\eta}) \to \bigoplus_{x \in X_{(0)}} H^0\left(X_{\text{ét}}, (i_x)_*\mathbb{Z}_x\right) \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0$$

where (as already observed in the proof of lemma 17.18) we may replace the sum over codimension 1 points with the sum over the closed points. We have $H^0(X_{\text{ét}}, \mathcal{O}_X)^\times = \Gamma(X, \mathcal{O}_X)^\times = k^\times$, $H^0(X_{\text{ét}}, j_*\mathbb{G}_{m,\eta}) = H^0(\eta_{\text{ét}}, \mathbb{G}_{m,\eta}) = k(X)^\times$, and by definition

$$H^0\left(X_{\text{ét}}, \bigoplus_{x \in X_{(0)}} (i_x)_*\mathbb{Z}_x\right) = \bigoplus_{x \in X_{(0)}} \mathbb{Z}$$

is the group $\mathrm{Div}(X)$ of (Weil) divisors on X. Thus we may further rewrite the previous exact sequence as

$$0 \to k^\times \to k(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0;$$

this shows that $H^1(X_{\text{ét}}, \mathbb{G}_m)$ is the cokernel of $\mathrm{div} : k(X)^\times \to \mathrm{Div}(X)$, which (more or less by definition) is the Picard group of X. This finishes the proof. $\square$

Corollary 17.20. *Let X be a smooth, projective, connected curve of genus g over the algebraically closed field k. For every n not divisible by the characteristic of k we have*

- $H^0(X, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$;
- $H^1_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Pic}(X)(k)[n]$
- $H^2_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$;

- $H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}) = (0)$ *for* $i > 2$.

Proof. To ease the notation we drop the subscript ét from our cohomology groups. Observe that by lemma 12.17 the étale sheaves μ_n and $\mathbb{Z}/n\mathbb{Z}$ on $X_{\text{ét}}$ are isomorphic, so we may equally well compute the cohomology of μ_n instead. Since n is invertible on X, we can consider the Kummer sequence (exact by proposition 12.16)

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0; \tag{24}$$

taking étale cohomology, we obtain the long exact sequence

$$\begin{aligned} 0 &\to H^0(X, \mu_n) \to H^0(X, \mathbb{G}_m) \to H^0(X, \mathbb{G}_m) \to H^1(X, \mu_n) \\ &\to H^1(X, \mathbb{G}_m) \xrightarrow{n} H^1(X, \mathbb{G}_m) \to H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m). \end{aligned}$$

We know several of these terms explicitly. Indeed, by theorem 17.19 we have $H^1(X, \mathbb{G}_m) = \operatorname{Pic}(X)(k)$ and $H^2(X, \mathbb{G}_m) = 0$; moreover, as X is connected and proper, we have $H^0(X, \mathbb{G}_m) = \Gamma(X, \mathcal{O}_X^\times) = \Gamma(X, \mathcal{O}_X)^\times = k^\times$. The previous sequence then becomes

$$0 \to \mu_n(k) \to k^\times \xrightarrow{\cdot n} k^\times \to H^1(X, \mu_n) \to \operatorname{Pic}(X)(k) \xrightarrow{[n]} \operatorname{Pic}(X)(k) \to H^2(X, \mu_n) \to 0,$$

and since k is algebraically closed the map $k^\times \to k^\times$ is surjective, so we get

$$0 \to H^1(X, \mu_n) \to \operatorname{Pic}(X)(k) \xrightarrow{[n]} \operatorname{Pic}(X)(k) \to H^2(X, \mu_n) \to 0.$$

This prove that $H^1(X, \mu_n) = \operatorname{Pic}(X)(k)[n]$ and $H^2(X, \mu_n) = \frac{\operatorname{Pic}(X)(k)}{n \operatorname{Pic}(X)(k)}$. Now since $\operatorname{Pic}(X)(k)$ sits in an exact sequence

$$0 \to \operatorname{Pic}^0(X)(k) \to \operatorname{Pic}(X)(k) \xrightarrow{\deg} \mathbb{Z} \to 0$$

and $\operatorname{Pic}^0(X)(k)$ is the group of k-rational points of an abelian variety (hence in particular an n-divisible group) we see that

$$H^2(X, \mu_n) = \frac{\operatorname{Pic}(X)(k)}{n \operatorname{Pic}(X)(k)} \cong \frac{\mathbb{Z}}{n\mathbb{Z}}.$$

Moreover, for every $q \geq 2$ the long exact sequence in cohomology attached to (24) contains the segment

$$H^q(X, \mathbb{G}_m) \to H^{q+1}(X, \mu_n) \to H^{q+1}(X, \mathbb{G}_m),$$

and since we have $H^q(X, \mathbb{G}_m) = H^{q+1}(X, \mathbb{G}_m) = 0$ we obtain $H^{q+1}(X, \mu_n) = (0)$ for every $q \geq 2$. $\square$

17.5.1 Action of Frobenius on $H^2_{\text{ét}}$ of a curve

Let X_0 be a smooth projective curve defined over a finite field $\mathbb{F}_q$ of characteristic p, and let $k = \overline{\mathbb{F}_q}$. Denote by X the basechange of X_0 to k. As we have seen in the proof of corollary 17.20, for every prime $\ell \neq p$ we can describe $H^2(X, \mathbb{Z}/\ell^n\mathbb{Z})$ as the cokernel of $[\ell^n] : \operatorname{Pic}(X)(k) \to \operatorname{Pic}(X)(k)$. Furthermore, we have an exact sequence

$$0 \to \operatorname{Pic}^0(X)(k) \to \operatorname{Pic}(X)(k) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

where the last arrow sends the equivalence class of a divisor to its degree. Since it is well-known that $\operatorname{Pic}^0(X)(k)$ is an abelian variety (often denote by $\operatorname{Jac}(X)$, the Jacobian of X), hence in particular that $[\ell^n] : \operatorname{Pic}^0(X)(k) \to \operatorname{Pic}^0(X)(k)$ is surjective, we have concluded that $H^2(X, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathbb{Z}/\ell^n\mathbb{Z}$. But we can be slightly more precise: indeed, all the above sequences are functorial, hence compatible with the action of the relative Frobenius of X! Thus in order to understand the action of $F := \mathrm{Fr}_{X,q}$ on $H^2(X, \mathbb{Z}/n\mathbb{Z})$ it suffices to understand its action on $\mathbb{Z} \cong \frac{\mathrm{Pic}(X)(k)}{\mathrm{Pic}^0(X)(k)}$. In turn, $\mathrm{Pic}(X)$ sits in a (Frobenius-equivariant) sequence

$$K^\times \to \mathrm{Div}(X) \to \mathrm{Pic}(X)(k),$$

and the composition

$$\mathrm{Div}(X) \to \mathrm{Pic}(X)(k) \to \frac{\mathrm{Pic}(X)(k)}{\mathrm{Pic}^0(X)(k)} \cong \mathbb{Z}$$

simply sends a divisor to its degree. It follows that in order to understand the action of F on $\mathbb{Z}$ it suffices to understand its action on any divisor of degree 1, for example a single point $P \in X(k)$. Since it is clear that

$$F^*[P] = \sum_{Q : F(Q)=P} [Q],$$

we have $\deg F^*[P] = \#\{Q : F(Q) = P\} = \deg F = q$, so F acts on $\mathbb{Z}$ as multiplication by q, and so the same is true for its action of $H^2(X, \mathbb{Z}/\ell^n\mathbb{Z})$. By passing to the limit in n and tensoring by $\mathbb{Q}_\ell$, we see that $H^2(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$, where the action of Frobenius on $\mathbb{Q}_\ell$ is multiplication by q. This is precisely what we would expect from Poincaré duality: on the one hand $\dim H^2(X, \mathbb{Q}_\ell)$ must be equal to $\dim H^{2-2}(X, \mathbb{Q}_\ell) = 1$, and on the other the action of Frobenius on $H^0(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ is trivial, so the action on $H^2(X, \mathbb{Q}_\ell)$ should be multiplication by q by the functional equation for the ζ-function of X (see theorem 4.9 and its proof).

17.6 Exercises

Exercise 17.21. Let $k = \overline{\mathbb{F}_p}$ and let X be the nodal cubic $y^2 = x^2(x-1) \subset \mathbb{A}^2_k$. Show that X is not regular and that $H^1(X_{\text{ét}}, \mathbb{Z}_X)$ is nonzero.

Hint. Consider the normalisation morphism $\mathbb{A}^1 \to X$. Alternative approach: use the interpretation of the first étale cohomology group in terms of torsors, theorem 18.16 below.

Exercise 17.22 (For those who have never seen a proof of Tsen's theorem)**.** Let k be an algebraically closed field and let $K = k(x)$. Prove that K is a C_1 field.

Hint. We are looking for a solution to an equation in rational functions (or – morally – in polynomials). Consider this as a system of equations in the coefficients and let the degree of the unknowns grow...

Exercise 17.23. Let p be a prime number and let $E/\mathbb{F}_p$ be the elliptic curve of equation $y^2 = x(x^2+1)$ (as usual, we mean that E is the projective closure of the affine curve given by this equation).

1. Describe three nontrivial $(\mathbb{Z}/2\mathbb{Z})$-torsors $X \to E$ (these may be defined over an extension of $\mathbb{F}_p$).

2. Describe the action of Frobenius on these torsors.

3. Describe the 2-torsion points of E and the action of Frobenius on them.

For the next exercise we need a definition:

Definition 17.24. Let $Y \to X$ be a faithfully flat map, and let G be a finite group acting on Y over[32] X on the right. Then $Y \to X$ is called a Galois covering of X with group G if the morphism

$$\begin{array}{ccc} G \times Y & \to & Y \times_X Y \\ (g, y) & \mapsto & (yg, y) \end{array}$$

is an isomorphism.

[32] that is, for every g in G the maps $Y \to X$ and $Y \xrightarrow{g} Y \to X$ coincide

Exercise 17.25 (Hochschild-Serre spectral sequence)**.** Let $f : Y \to X$ be a Galois covering with group G and let $\mathcal{F}$ be an étale sheaf of abelian groups on X.

1. Prove that f is étale (*hint*: being étale is an fpqc-local property, so it can be tested on a suitable fpqc-covering).

2. Prove that $\mathcal{F}(X) = \mathcal{F}(Y)^G$.

3. Deduce that there exists a spectral sequence

$$E_2^{p,q} = H^p\left(G, H^q(Y_{\text{ét}}, \mathcal{F}|_Y)\right) \Rightarrow H^{p+q}(X_{\text{ét}}, \mathcal{F}).$$

   Hint. For this you'll need to check that if $\mathcal{I}$ is an injective sheaf on X, then $\mathcal{I}(Y)$ is acyclic for the functor $\Gamma(G, -)$. This can be done by noticing that the complex

$$\mathcal{I}(X) \to \mathcal{I}(Y) \to \mathcal{I}(Y \times_X Y) \to \mathcal{I}(Y \times_X Y \times_X Y) \to \cdots$$

   is isomorphic to the complex of inhomogeneous chains for G acting on $\mathcal{I}(Y)$. And now it suffices to notice that this complex computes the Čech cohomology of $\mathcal{I}$ for the covering $Y \to X$...

4. Let now X be a scheme over the field k. For every extension L of k, let $X_L = X \times_{\operatorname{Spec} k} \operatorname{Spec} L$. Passing to the limit in the previous spectral sequences, deduce that there exists a spectral sequence

$$E_2^{p,q} = H^p\left(\operatorname{Gal}(K^{\text{sep}}/K), H^q(X_{K^{\text{sep}}}, \mathcal{F}|_{X_{K^{\text{sep}}}})\right) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Exercise 17.26. Let $k = \mathbb{F}_p$ and $X = \mathbb{P}_{1,k}$.

1. Compute $H^q(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ for every $q \geq 0$ and every n with $(n, p) = 1$.

2. Compute $H^q(X_{\text{ét}}, \mu_n)$ for every $q \geq 0$ and every n with $(n, p) = 1$.

18 Final comments

18.1 Étale cohomology as a Weil theory; the necessity of torsion coefficients

Now that we have constructed étale cohomology, we might be tempted to think that the 'correct' way to recover an algebraic analogue of the usual (topological) cohomology with constant coefficients $\mathbb{Z}$ is to simply take étale cohomology of the constant sheaf $\mathbb{Z}$. However, this is not the case, as the following lemma shows:

Lemma 18.1. *Let X be a regular scheme. Then $H^1(X_{\text{ét}}, \mathbb{Z}_X) = 0$.*

To prove this we need two intermediate lemmas:

Lemma 18.2. *Let X be a scheme and $i_x : x \to X$ a (not necessarily closed) point. Then $H^1(X_{\text{ét}}, (i_x)_*\mathbb{Z}) = 0$.*

Proof. The Leray spectral sequence for i_x

$$E_2^{p,q} = H^p(X_{\text{ét}},\ R^q(i_x)_*\mathbb{Z}) \Rightarrow H^{p+q}(x_{\text{ét}},\ \mathbb{Z})$$

implies $H^1(X_{\text{ét}}, (i_x)_*\mathbb{Z}) \subset H^1(x_{\text{ét}}, \mathbb{Z})$. But

$$H^1(x_{\text{ét}}, \mathbb{Z}) = H^1(\mathrm{Gal}(\overline{k(x)}/k(x)), \mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}\left(\mathrm{Gal}(\overline{k(x)}/k(x)), \mathbb{Z}\right) = 0$$

where the first equality comes from our identification of the étale cohomology of points with Galois cohomology of their residue fields, and the vanishing of Galois cohomology follows from the fact that $\mathrm{Gal}(\overline{k(x)}/k(x))$ is a profinite group while $\mathbb{Z}$ has no torsion. $\square$

Lemma 18.3. *Let X be a regular irreducible scheme and let $j : \eta \to X$ be the inclusion of the generic point of X. The adjunction map $\mathbb{Z}_X \to j_*\mathbb{Z}_\eta$ is an isomorphism.*

Proof. We show that for every geometric point $\overline{x} \to X$ the map of stalks $\mathbb{Z}_{X,\overline{x}} \to (j_*\mathbb{Z}_\eta)_{\overline{x}}$ is an isomorphism. On the one hand $\mathbb{Z}_{X,\overline{x}} = \varinjlim_{(V,\overline{v})} \mathbb{Z}_X(V) = \mathbb{Z}$, where the colimit can be taken over the *connected* étale neighbourhoods $(V,\ \overline{v})$ of $(X, \overline{x})$ as X is irreducible. On the other hand $(j_*\mathbb{Z}_\eta)_{\overline{x}} = \varinjlim_{(V,\overline{v})} \mathbb{Z}_\eta(\eta) \times_X V$, where again the colimit can be taken over the connected étale neighbourhoods $(V, \overline{v})$ of $(X, \overline{x})$. As $V \to X$ is étale and η is a point, the scheme $\eta \times_X V$ is the disjoint union of the generic points of V. As X is regular, V is regular too (corollary 8.5); furthermore V is connected, and therefore irreducible. Hence $\eta \times_X V$ is one point, $\mathbb{Z}_\eta(\eta \times_X V) = \mathbb{Z}$ and $(j_*\mathbb{Z}_\eta)_{\overline{x}} = \mathbb{Z}$. One checks easily that the map $\mathbb{Z}_{X,\overline{x}} = \mathbb{Z} \to (j_*\mathbb{Z}_\eta)_{\overline{x}} = \mathbb{Z}$ is the identity, hence the result. $\square$

Proof of lemma 18.1. Since X is regular we can immediately reduce to the case of X being irreducible. By the previous lemma, $\mathbb{Z}_X$ is isomorphic to $j_*\mathbb{Z}_\eta$, so by lemma 18.2 we obtain

$$H^1(X_{\text{ét}}, \mathbb{Z}_X) = H^1(X_{\text{ét}}, j_*\mathbb{Z}_\eta) = (0)$$

as claimed. $\square$

Lemma 18.1 is the reason why, in order to recover a sensible cohomological theory from étale cohomology, one only considers *torsion* sheaves, that is, sheaves $\mathcal{F}$ for which $\mathcal{F} = \varinjlim_n \ker(n : \mathcal{F} \to \mathcal{F})$. Just like in the theory of the étale fundamental group one cannot expect to get an algebraic version of the universal cover and *approximates* it by a limit over finite covers (which ultimately leads to replacing the topological fundamental group by some profinite completion of it), also in étale cohomology the sheaf $\mathbb{Z}$ is badly behaved and should be approximated by some sheaf of profinite groups. The natural choice to make is to replace $\mathbb{Z}$ by $\widehat{\mathbb{Z}}$, hence to consider $H(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ for every n. An alternative (and very effective) choice is to only consider one prime at a time, which leads to the definition of ℓ-adic cohomology:

Definition 18.4. Let X be a scheme over a field k. Let ℓ be a prime different from the characteristic of k. We (re)define

$$H^i(X_{\text{ét}}, \mathbb{Z}_\ell) := \varprojlim_n H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

It is by definition a $\mathbb{Z}_\ell$-module, and we can therefore (re)define

$$H^i(X_{\text{ét}}, \mathbb{Q}_\ell) := \varprojlim_n H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Remark 18.5. Notice that the symbol $H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$ would already have a natural meaning, namely the étale cohomology of the constant sheaf $\mathbb{Z}_\ell$. However, for reasons similar to those of lemma 18.1 these cohomology groups are quite different from their topological analogues (in particular, $H^1(X_{\text{ét}}, \mathbb{Z}_\ell)$ vanishes for all X normal by a proof similar to that of lemma 18.1), so this notion of $\mathbb{Z}_\ell$-cohomology is not considered interesting. This is the reason why the symbol $H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$ is almost universally taken to mean the projective limit above; a similar comment applies to $H^i(X_{\text{ét}}, \mathbb{Q}_\ell)$.

As is clear from the proof of lemma 18.1, the crux of the matter is the fact that $\mathbb{Z}$ (and also $\mathbb{Z}_\ell, \mathbb{Q}_\ell, \ldots$) are torsion-free, and since the étale cohomology of points is in a sense built out of profinite groups one cannot expect any good behaviour for these sheaves. This explains the importance of torsion sheaves in the étale theory.

With these definitions at hand, we can finally give a meaning to theorem 3.6:

Theorem 18.6. *Let k be the algebraic closure of the finite field $\mathbb{F}_p$. Let ℓ be a prime different from p. Étale cohomology $H^\bullet_{\acute{e}t}(-, \mathbb{Q}_\ell)$ is a Weil cohomology on* $\mathrm{SmProj}(k)$.

Remark 18.7. Notice that our computation of the cohomology of curves yields

$$\dim_{\mathbb{Q}_\ell} H^0(X_{\text{ét}}, \mathbb{Q}_\ell) = 1, \quad \dim_{\mathbb{Q}_\ell} H^1(X_{\text{ét}}, \mathbb{Q}_\ell) = 2g(X), \quad \dim_{\mathbb{Q}_\ell} H^2(X_{\text{ét}}, \mathbb{Q}_\ell) = 1,$$

which are the 'correct' Betti numbers for a curve. Furthermore, it is clear by definition that (if X comes from base-change from a certain X_0 defined over the finite field $\mathbb{F}_q$) the relative Frobenius $\mathrm{Fr}_{\overline{X},q}$ acts trivially on $H^0(X_{\text{ét}}, \mathbb{Q}_\ell)$; we shall show in the next section that it also has the 'correct' action on $H^2(X_{\text{ét}}, \mathbb{Q}_\ell)$. This only leaves open the question of the action of Frobenius on the étale H^1, which we shall investigate in section 18.3 below by describing a nondegenerate pairing $H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \times H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \to H^2(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ (namely Poincaré duality; recall that the duality is used in the proof of theorem 5.1 to show that all the eigenvalues of Frobenius acting on the étale H^1 have absolute value $q^{1/2}$).

18.2 $H^1_{\text{ét}}$ and torsors

In this section we will prove that for any abelian sheaf $\mathcal{G} \in \mathbf{Ab}(X_{\text{ét}})$, the (pointed) set $H^1(X_{\text{ét}}, \mathcal{G})$ is canonically isomorphic to the (pointed) set of isomorphism classes of $\mathcal{G}$-torsors.

Definition 18.8 (Torsor)**.** Let C be a site and $\mathcal{G}$ be a sheaf of groups on C. Then a **$\mathcal{G}$-torsor** is a sheaf of sets $\mathcal{F}$ on C with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that for all $U \in C$ the following holds: the action of $\mathcal{G}(U)$ on $\mathcal{F}(U)$ is free (i.e. all point stabilizers are trivial) and transitive and there exists a cover $\{U_i \to U\}$ of U such that $\forall i : \mathcal{F}(U_i) \neq \emptyset$. A **morphism of $\mathcal{G}$-torsors** is a morphism of sheaves of sets that is compatible with the action of $\mathcal{G}$. A $\mathcal{G}$-torsor is called **trivial** if it is isomorphic to $\mathcal{G}$ as a $\mathcal{G}$-torsor.

Exercise 18.9. Prove that $\mathcal{F}$ is trivial if and only if $\mathcal{F}(U) \neq \emptyset$. Deduce that (if C is the étale site of a scheme) the last condition in the definition of a torsor is equivalent to the fact that $\mathcal{F}$ is locally isomorphic to $\mathcal{G}$ for the étale topology.

Lemma 18.10. *All morphisms of $\mathcal{G}$-torsors are isomorphisms (hence the category of $\mathcal{G}$-torsors is a groupoid).*

Proof. As the action of $\mathcal{G}(U)$ on $\mathcal{F}(U)$ is free and transitive when the latter is nonempty, any $\mathcal{G}$-equivariant morphism

$$\mathcal{F}_1(U) \to \mathcal{F}_2(U)$$

must be an isomorphism whenever $\mathcal{F}_1(U)$ is nonempty (notice that in this case $\mathcal{F}_2(U)$ is also nonempty). As the property of being an isomorphism is C-local (that is, it can be checked on a covering: exercise) and by assumption we have an étale covering $U_i \to U$ for which $\mathcal{F}_1(U_i) \neq \emptyset$ the claim follows. □

Example 18.11. Let S be a scheme over which n is invertible (that is, $n \in \mathcal{O}_S(S)^\times$). Consider the sheaf of n-th roots of unity,

$$\mu_{n,S} = \ker(\cdot^n : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S})$$

in the big étale site of S. Then for every $a \in \mathcal{O}_S(S)^\times$, the fibre sheaf

$$\mathcal{F}_a(U) = (\cdot^n)^{-1}\{a|_U\}$$

is a $\mu_{n,S}$-torsor.

Let's see this in more detail in the affine case. For any A-algebra $f : A \to B$ the fibre is

$$\mathcal{F}_a(B) = \{b \in B^\times : b^n = f(a)\}.$$

The group $\mu_{n,A}(B) = \{\zeta \in B^\times \ : \ \zeta^n = 1\}$ acts on it by multiplication. This action is clearly free (every $b \in \mathcal{F}_a(B)$ is a unit), and it's transitive since the ratio $\dfrac{b}{b'}$ is in $\mu_{n,A}(B)$ for any choice of $b, b' \in \mathcal{F}_a(b)$. Finally, we know from our study of étale morphisms that $B \to C := B[x]/(x^n - f(a))$ is standard étale since we have

$$\left(x^n - f(a), n \cdot \overline{x}^{n-1}\right) = \left(x^n - f(a), \overline{x}^{n-1}\right) = (f(a), x^{n-1}) = (1)$$

as n is invertible in A (hence also in B and in C) and $f(a) \in B^\times$. By construction we have that $\mathcal{F}_a(C)$ is nonempty, so $\mathcal{F}_a$ is trivialised by the étale cover C of B. By gluing together these local descriptions, one checks easily that $\mathcal{F}_a$ is indeed a $\mu_{n,S}$ torsor.

Definition 18.12. Let C be a site and $\mathcal{G}$ be a sheaf of groups on C. Suppose that we have sheaves $\mathcal{X}$ and $\mathcal{Y}$ on C with a right- and left- action of $\mathcal{G}$, respectively. Then we can define the sheaf

$$\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y} := \left(T \mapsto \frac{\mathcal{X}(T) \times \mathcal{Y}(T)}{\mathcal{G}(T)}\right)^{\#},$$

where $\mathcal{G}$ acts on the right on $\mathcal{X} \times \mathcal{Y}$ by $(x, y)g = (xg, g^{-1}y)$. The sheaf $\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}$ is called the **contracted product** of X and Y and often denoted by $\mathcal{X} \wedge^{\mathcal{G}} \mathcal{Y}$.

Definition 18.13. Let $\mathcal{X}, \mathcal{Y}$ be as in the previous definition and let $\mathcal{Z}$ be a further sheaf of sets. A **$\mathcal{G}$-bilinear morphism** (or $\mathcal{G}$-equivariant morphism) $\mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a morphism of sheaves such that $\forall U$ the equality

$$\psi(xg, y) = \psi(x, gy)$$

holds for every $g \in \mathcal{G}(U), x \in \mathcal{X}(U), y \in \mathcal{Y}(U)$.

Lemma 18.14. *The contracted product $\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}$ satisfies the following universal property: any $\mathcal{G}$-bilinear morphism of sheaves $\psi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ factors uniquely through the map $\mathcal{X} \times \mathcal{Y} \to \mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}$.*

Proof. By adjunction $\mathrm{Hom}_{\mathbf{Sh}}(\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}, \mathcal{Z}) = \mathrm{Hom}_{\mathbf{PSh}}(\mathcal{W}, \iota\mathcal{Z})$, where $\iota : \mathbf{Sh} \to \mathbf{PSh}$ is the forgetful functor and $\mathcal{W}$ is the presheaf $T \mapsto \frac{\mathcal{X}(T) \times \mathcal{Y}(T)}{\mathcal{G}(T)}$. The claim now follows because on every T the map

$$\mathcal{X}(T) \times \mathcal{Y}(T) \to \mathcal{Z}(T)$$

factors via $\frac{\mathcal{X}(T) \times \mathcal{Y}(T)}{\mathcal{G}(T)}$ by definition of the quotient set. □

Remark 18.15. Let C be a site. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are locally isomorphic sheaves of sets on C (that is, for every U there is a covering $(U_i \to U)_{i \in I}$ such that $\mathcal{X}|_{U_i} \cong \mathcal{Y}|_{U_i}$). We define the sheaf of sets $\mathcal{I} := \mathcal{I}som(\mathcal{X}, \mathcal{Y})$ (whose values on U are the set of isomorphisms between $\mathcal{X}|_U$ and $\mathcal{Y}|_U$) and the sheaf of groups $\mathcal{G} := \mathcal{A}ut(\mathcal{X})$ (which is the sheaf[33] that on U takes the value $\mathrm{Aut}(\mathcal{X}(U))$). There are an obvious actions of $\mathcal{G}$ on $\mathcal{I}$ (on the right) and on $\mathcal{X}$ (on the left). The map

$$\begin{array}{ccc} \mathcal{I}(U) \times \mathcal{X}(U) & \to & \mathcal{Y}(U) \\ (i, x) & \mapsto & i(x) \end{array}$$

is $\mathcal{G}$-bilinear and thus it induces a morphism $\mathcal{I} \otimes_{\mathcal{G}} \mathcal{X} \to \mathcal{Y}$. We show that this is an isomorphism by describing the inverse. For any $U \in C$, fix[34] $i \in \mathcal{I}(U)$ and consider

$$\begin{array}{cccc} \varphi_{U,i}: & \mathcal{Y}(U) & \to & \frac{\mathcal{I}(U) \times \mathcal{X}(U)}{\mathcal{G}(U)} \\ & y & \mapsto & [(i, i^{-1}(y))]. \end{array}$$

This is easily checked to be a two-sided inverse to the map above. If we choose a different isomorphism j, then $j^{-1}i$ is an element of $\mathrm{Aut}(\mathcal{X}(U)) = \mathcal{G}(U)$, and we get

$$\varphi_{U,j}(y) = [(j, j^{-1}(y))] = [(j(j^{-1}i), (j^{-1}i)^{-1}j^{-1}(y))] = [(i, i^{-1}(y))] = \varphi_{U,i}(y),$$

so $\varphi_{U,i}$ does not depend on the choice of i (so we denote it by φ_U). Independence of i implies that the various φ_U glue to give an inverse morphism $\mathcal{Y} \to \mathcal{I} \otimes_{\mathcal{G}} \mathcal{X}$.

The key point to take home is that given two locally isomorphic sheaves $\mathcal{X}, \mathcal{Y}$ one can get $\mathcal{Y}$ from $\mathcal{X}$ by 'twisting' it by the $\mathcal{I} = \mathcal{I}som(\mathcal{X}, \mathcal{Y})$.

Theorem 18.16. *Let $\mathcal{C}$ be the (small or big) étale site of a scheme S and let $\mathcal{G}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a canonical bijection*

$$\{\mathcal{G} - torsors\}/\cong \; \leftrightarrow H^1_{\text{ét}}(S, \mathcal{G}) \; .$$

Proof. We begin by noticing that by lemma 12.17 for every sheaf $\mathcal{F}$ there is a canonical isomorphism

$$H^0_{\text{ét}}(S, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, \mathcal{F}(S)) = \mathrm{Hom}(\mathbb{Z}_S, \mathcal{F}) \, .$$

Thus the functors $\Gamma(S, -)$ and $\mathrm{Hom}(\mathbb{Z}_S, -)$ are isomorphic, and therefore so are their derived functors: it follows that

$$H^1_{\text{ét}}(S, \mathcal{F}) = R^1\Gamma(S, \mathcal{F}) = R^1 \mathrm{Hom}(\mathbb{Z}_S, \mathcal{F}) = \mathrm{Ext}^1(\mathbb{Z}_S, \mathcal{F}).$$

We now recall the following fact, whose proof in the general case is essentially identical to the case of abelian groups:

Lemma 18.17. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$ is in (natural) bijection with the set of isomorphism classes of exact sequences $0 \to B \to E \to A \to 0$, where two sequences $0 \to B \to E \to A \to 0$ and $0 \to B \to E' \to A \to 0$ are isomorphic if there exists a commutative diagram as follows:*

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Remark 18.18. In such a diagram the map f is necessarily an isomorphism because of the 5-lemma.

[33] notice that local automorphisms glue, hence there is no need to sheafify: $\mathcal{A}ut(\mathcal{X})$ is already a sheaf

[34] for now we describe the map on a suitably fine covering of U, so that we may assume that $\mathcal{I}(U)$ is nonempty. Then we check that the maps thus constructed are independent of any choice and therefore glue to give a globally defined morphism.

We will now show that $\mathrm{Ext}^1(\mathbb{Z}_S, \mathcal{G})$ is canonically isomorphic to the set of isomorphism classes of $\mathcal{G}$-torsors. Suppose that $\mathcal{G} \to \mathcal{E} \to \mathbb{Z}_S$ is an extension of étale sheaves of abelian groups. It is clear that étale-locally the sequence splits: surjectivity of $\mathcal{E} \to \mathbb{Z}_S$ on the stalks shows that étale-locally around every point $\mathcal{E}(U) \to \mathbb{Z}_S(U)$ is surjective, and since $\mathbb{Z}_S(U) \cong \mathbb{Z}^{\pi_0(U)}$ is free the map has a section. It follows that $\mathcal{E}$ is étale-locally isomorphic to $\mathcal{G} \oplus \mathbb{Z}_S$. We then know from remark 18.15 that the important objects to look at are $\mathcal{A}ut(\mathcal{G} \oplus \mathbb{Z}_S)$ and $\mathcal{I}som(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})$ (here the isomorphisms in question are isomorphisms *of extensions*).

Consider first the automorphism sheaf. Any automorphism of extensions is of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "="'] & \mathcal{G} \oplus \mathbb{Z}_S \arrow[r] \arrow[d, "\varphi"] & \mathbb{Z}_S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G} \oplus \mathbb{Z}_S \arrow[r] & \mathbb{Z}_S \arrow[r] & 0,
\end{tikzcd}$$

which means that it sends (g, n) to $(g+\psi(n), n)$. In particular, any such φ is completely determined by $\psi(1)$, and conversely any such choice gives an automorphism, so $\mathcal{G} \cong \mathcal{A}ut(\mathcal{G} \oplus \mathbb{Z}_S)$; in an imprecise but suggestive way, g acts on $\mathcal{G} \oplus \mathbb{Z}_S$ via the matrix $\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$. As a consequence of the fact that clearly $\mathcal{A}ut(\mathcal{G} \oplus \mathbb{Z}_S) \cong \mathcal{G}$ acts on $\mathcal{I}som(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})$ on the right, this gives $\mathcal{I}som(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})$ the structure of a $\mathcal{G}$-torsor. We have thus constructed a map that given an extension $\mathcal{E}$ yields a $\mathcal{G}$-torsor $\mathcal{I}som(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})$, that is, we have constructed a map

$$\mathrm{Ext}(\mathbb{Z}_S, \mathcal{G}) \to H^1_{\text{ét}}(S, \mathcal{G}).$$

We now construct the inverse. Suppose we have a $\mathcal{G}$-torsor $\mathcal{T}$. We are hoping to represent $\mathcal{T}$ as $\mathcal{I}som\,(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})$, and we know from remark 18.15 that if we twist $\mathcal{G} \oplus \mathbb{Z}_S$ by $\mathcal{I}som(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})$ we get $\mathcal{E}$. So if $\mathcal{T} \cong \mathcal{I}som\,(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})$, then $\mathcal{E}$ should be isomorphic to

$$(\mathcal{I}som(\mathcal{G} \oplus \mathbb{Z}_S, \mathcal{E})) \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathbb{Z}_S)\,.$$

But the first term in this tensor product is supposed to be $\mathcal{T}$, so we can hope to define $\mathcal{E}$ by the formula

$$\mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathbb{Z}_S)\,.$$

This leads us to considering the exact sequence

$$0 \to \mathcal{G} \to \mathcal{G} \oplus \mathbb{Z}_S \to \mathbb{Z}_S \to 0,$$

where we interpret each terms as a sheaf with a $\mathcal{G}$-action by declaring that $\mathcal{G}$ acts trivially on $\mathcal{G}$ and $\mathbb{Z}_S$ and as before on $\mathcal{G} \oplus \mathbb{Z}_S$. This is then an exact sequence of $\mathcal{G}$-sheaves in the obvious sense (and clearly it needs not split as a sequence of $\mathcal{G}$-sheaves, because sections $\mathbb{Z}_S \to \mathcal{G} \oplus \mathbb{Z}_S$ won't be $\mathcal{G}$-equivariant). Notice that by *trivial action* of $\mathcal{G}$ on itself here we really mean the action that fixes every point, not the natural multiplication action of $\mathcal{G}$ on itself.

Tensoring the above sequence by $\mathcal{T}$ we then get

$$\mathcal{T} \otimes_{\mathcal{G}} \mathcal{G} \to \mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathbb{Z}_S) \to \mathcal{T} \otimes_{\mathcal{G}} \mathbb{Z}_S \to 0;$$

notice that the first arrow is still injective (for example because the corresponding arrow in the original sequence has a section in the category of $\mathcal{G}$-sheaves; it's also easy to check this directly).

Now notice that the canonical projection $\mathcal{T} \times \mathbb{Z}_S \to \mathbb{Z}_S$ is $\mathcal{G}$-equivariant (since $\mathcal{G}$ acts trivially on $\mathbb{Z}_S$), so we have a canonical surjective morphism $\mathcal{T} \times_{\mathcal{G}} \mathbb{Z}_S \to \mathbb{Z}_S$. But since $\mathcal{T}$ is étale-locally isomorphic to $\mathcal{G}$ and isomorphisms can be checked étale-locally, this is actually an isomorphism. For the same reason, $\mathcal{T} \otimes_{\mathcal{G}} \mathcal{G}$ is isomorphic to $\mathcal{G}$. We have thus constructed an extension

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T} \otimes_{\mathcal{G}} \mathcal{G} \arrow[r] \arrow[d, "\sim"'] & \mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathbb{Z}_S) \arrow[r] \arrow[d] & \mathcal{T} \otimes_{\mathcal{G}} \mathbb{Z}_S \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathbb{Z}_S) \arrow[r] & \mathbb{Z}_S \arrow[r] & 0,
\end{tikzcd}$$

that is, an element of $\mathrm{Ext}^1(\mathbb{Z}_S, \mathcal{G})$. This describes a map

$$H^1_{\text{ét}}(S, \mathcal{G}) \to \mathrm{Ext}^1(\mathbb{Z}_S, \mathcal{G}).$$

To prove that these two constructions are quasi-inverse to each other it suffices to prove that we have a morphism

$$\mathcal{T} \to \mathit{Isom}(\mathcal{G} \oplus \mathbb{Z}_S,\ \mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathbb{Z}_S));$$

indeed such a morphism exists, and is simply given by

$$t \mapsto ((g, x) \mapsto t \otimes (g, x)).$$

As already remarked, any morphism of $\mathcal{G}$-torsors is an isomorphism, so we are done. □

Theorem 18.19. *We have*

$$H^1_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}) = \left\{ \begin{matrix} \textit{sheaves of sets } \mathcal{F} \textit{ on the étale site } X_{\text{ét}} \\ \textit{which are } (\mathbb{Z}/n\mathbb{Z})_X\textit{-torsors} \end{matrix} \right\} \Big/ \cong$$
$$= \left\{ \begin{matrix} \textit{morphisms } Y \to X \textit{ which are finite étale together} \\ \textit{with a free } \mathbb{Z}/n\mathbb{Z} \textit{ action such that } X = Y/(\mathbb{Z}/n\mathbb{Z}). \end{matrix} \right\} \Big/ \cong .$$

Proof. We give a sketch, omitting some of the verifications.

The first identification is a direct consequence of theorem 18.16. For the second, a morphism $p : Y \to X$ that is finite étale and such that $X = Y/(\mathbb{Z}/n\mathbb{Z})$ certainly gives a $\mathbb{Z}/n\mathbb{Z}$-torsor by considering the sheaf of sections,

$$\mathcal{F}(U) = \{ f : U \to Y \times_X U \mid pf = \mathrm{id}_U \}.$$

It is clear by assumption that the action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathcal{F}$ is free and transitive; we only need to check that upon restriction to a sufficiently small étale covering $(U_i \to X)_{i \in I}$ we have $\mathcal{F}(U_i) \neq \emptyset$ for every $i \in I$. But this is obvious by taking $Y \to X$ itself as an étale covering: indeed in this case $\mathcal{F}(Y) = \{ f : Y \to Y \times_X Y \mid pf = \mathrm{id}_Y \}$ is nonempty, since it contains the diagonal.

For the other direction, let $(U_i \to U)_{i \in I}$ be an étale covering that trivialises the $(\mathbb{Z}/n\mathbb{Z})_X$-torsor $\mathcal{F}$. We get local isomorphisms $\varphi_i : \mathcal{F}|_{U_i} \cong U_i \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$, which induce isomorphisms $\varphi_{ij} : (U_i \times_X U_j) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \cong (U_i \times_X U_j) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$. These φ_{ij} are not necessarily the identity, but (being given by a restriction of $\varphi_j \varphi_i^{-1}$) they satisfy $\varphi_{ik} = \varphi_{jk} \varphi_{ij}$ upon restriction to $U_i \times_X U_j \times_X U_k$ for all $i, j, k \in I$. But this is precisely an fpqc descent datum for affine schemes (see theorem 12.14 and the remark following it), so that we obtain an affine scheme $Y \to X$ such that $Y|_{U_i}$ represents $\mathcal{F}|_{U_i}$. Since both h_Y and $\mathcal{F}$ are sheaves, this proves that they are equal. Finally, one sees that the $(\mathbb{Z}/n\mathbb{Z})_X$-action on $\mathcal{F}$ translates into a free $\mathbb{Z}/n\mathbb{Z}$ action on Y with $Y/(\mathbb{Z}/n\mathbb{Z}) = X$: indeed this can be tested on an étale open cover, so it can be tested on the trivialising cover $(U_i \to X)$, for which it is essentially obvious. □

18.3 Poincaré duality for curves

Throughout this section let X be a smooth projective curve over an algebraically closed field k. We give a brief account of how to state and prove Poincaré duality for the étale cohomology of X; recalled that this was needed to prove the functional equation part of the Weil conjectures, which in turn played a pivotal role in the proof of the Riemann hypothesis for curves (see the proof of lemma 5.3). However, we should point out that there are more direct approaches to the proof of the Weil conjectures for curves, in which Poincaré duality is replaced by Serre duality (which in the case of curves is reasonably elementary).

18.3.1 Construction of the pairing

Let n be an integer prime to the characteristic of k. By a **finite locally constant sheaf killed by** n we mean a sheaf of abelian groups (on $X_{\text{ét}}$) for which there is a finite étale morphism $Y \to X$ such that $\mathcal{F}|_Y$ is the sheaf defined by a finite $\mathbb{Z}/n\mathbb{Z}$-module. We denote by $\mathbf{Ab}_n(X_{\text{ét}})$ the subcategory of $\mathbf{Ab}(X_{\text{ét}})$ consisting of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules. One checks that this category – just like the larger category $\mathbf{Ab}(X_{\text{ét}})$ – has enough injectives, and that the cohomology groups $H^r(X, \mathcal{F})$ computed in $\mathbf{Ab}_n(X_{\text{ét}})$ or in $\mathbf{Ab}(X_{\text{ét}})$ agree[35] for all $\mathcal{F} \in \mathbf{Ab}_n(X_{\text{ét}})$.

Definition 18.20. Given $\mathcal{F} \in \mathbf{Ab}_n(X_{\text{ét}})$ we set

$$\check{\mathcal{F}}(1) := \underline{\mathrm{Hom}}\,(\mathcal{F}, \mu_n)\,.$$

One can see that for $\mathcal{F} \in \mathbf{Ab}_n(X_{\text{ét}})$ we have

$$H^r\left(X_{\text{ét}}, \check{\mathcal{F}}(1)\right) \cong \mathrm{Ext}^r_{X,n}(\mathcal{F}, \mu_n)$$

Proof. For sheaves $\mathcal{F}, \mathcal{G}$ on $X_{\text{ét}}$ define $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ to be the sheaf on $X_{\text{ét}}$ whose sections on U are homomorphisms $\mathcal{F}|_U \to \mathcal{G}|_U$. Let now $\mathcal{F}_0$ be a fixed (finite locally constant) sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Then $\underline{\mathrm{Hom}}(\mathcal{F}_0, -)$ is a left-exact functor (check on stalks) from the category $\mathbf{Ab}_n(X_{\text{ét}})$ to itself; its derived functors are denoted by $\underline{\mathrm{Ext}}^r_{X,n}\,(\mathcal{F}_0, -)$. By taking global sections and using the spectral sequence for composed functors we get

$$H^r\left(X_{\text{ét}}, \underline{\mathrm{Ext}}^s_{X,n}(\mathcal{F}_0, \mathcal{G})\right) \Rightarrow \mathrm{Ext}^{r+s}_{X,n}(\mathcal{F}_0, \mathcal{G})$$

However, the stalks of $\underline{\mathrm{Ext}}^s_{X,n}(\mathcal{F}_0, \mathcal{G})$ at a geometric point $\overline{y}$ are given by

$$\underline{\mathrm{Ext}}^s_{X,n}(\mathcal{F}_0, \mathcal{G})_{\underline{y}} = \underline{\mathrm{Ext}}^s_{X,n}((\mathcal{F}_0)_{\underline{y}}, (\mathcal{G})_{\underline{y}}) = \underline{\mathrm{Ext}}^s_{\mathbb{Z}/n\mathbb{Z}}((\mathbb{Z}/n\mathbb{Z})^t, (\mathcal{G})_{\underline{y}}) = 0 \text{ for } s > 0,$$

where the second equality follow because $\mathcal{F}_0$ is locally constant and the last one follows because $(\mathbb{Z}/n\mathbb{Z})^t$ is free as a $\mathbb{Z}/n\mathbb{Z}$-module. Thus the spectral sequence (for $\mathcal{G} = \mu_n$) gives exactly

$$H^r\left(X_{\text{ét}}, \underline{\mathrm{Hom}}(\mathcal{F}_0, \mu_n)\right) = \mathrm{Ext}^r_{X,n}\,(\mathcal{F}, \mu_n)$$

as claimed. □

Recall that in any abelian category the elements of $\mathrm{Ext}^r(A, B)$ can be identified with (isomorphism classes of) r-fold extensions

$$0 \to B \to E_1 \to \cdots \to E_r \to A \to 0.$$

We can now define the Poincaré pairing.

Definition 18.21. For any finite locally constant sheaf $\mathcal{F}$ and for every $r = 0, 1, 2$ there is a canonical pairing

$$H^r(X, \mathcal{F}) \times H^{2-r}(X, \check{\mathcal{F}}(1)) \to H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

The definition goes as follows. Notice that what we want is, for every element of $H^{2-r}(X, \check{\mathcal{F}}(1))$, a homomorphism $H^r(X_{\text{ét}}, \mathcal{F}) \to H^2(X_{\text{ét}}, \mu_n)$.

1. for $r = 2$, elements of the group $H^{2-r}(X, \check{\mathcal{F}}(1))$ are simply sheaf homomorphisms $\mathcal{F} \to \mu_n$, which (by functoriality) can be evaluated on $H^2(X, \mathcal{F})$ to give an element in $H^2(X, \mu_n)$.
2. for $r = 1$, we can interpret $H^{2-r}(X, \check{\mathcal{F}}(1))$ as $\mathrm{Ext}^1_{X,n}(\mathcal{F}, \mu_n)$, so that a class in $H^{2-r}(X, \check{\mathcal{F}}(1))$ is represented by an extension

$$0 \to \mu_n \to E_1 \to \mathcal{F} \to 0.$$

Taking cohomology of this sequence of sheaves we get a map $H^1(X_{\text{ét}}, \mathcal{F}) \to H^2(X_{\text{ét}}, \mu_n)$, which is what we wanted.

[35] to prove this, one can show that the forgetful functor $\mathbf{Ab}_n(X_{\text{ét}}) \to \mathbf{Ab}(X_{\text{ét}})$ sends injective objects to acyclic ones

3. for $r = 0$, elements of $H^{2-r}(X, \check{\mathcal{F}}(1))$ are represented by biextensions

$$0 \to \mu_n \to E_1 \to E_2 \to \mathcal{F} \to 0,$$

from which we get two exact sequences

$$0 \to \mu_n \to E_1 \to C_1 \to 0, \quad 0 \to C_1 \to E_2 \to \mathcal{F} \to 0.$$

We then get boundary maps $H^0(X_{\text{ét}}, \mathcal{F}) \to H^1(X_{\text{ét}}, C_1)$ and $H^1(X_{\text{ét}}, C_1) \to H^2(X_{\text{ét}}, \mu_n)$. The composition of these two boundary maps is the sought-after homomorphism

$$H^0(X_{\text{ét}}, \mathcal{F}) \to H^2(X_{\text{ét}}, \mu_n).$$

18.3.2 Statement and proof of Poincaré duality

Theorem 18.22 (Poincaré duality)**.** *For any finite locally constant sheaf $\mathcal{F}$ killed by n the canonical pairing*

$$H^r(X, \mathcal{F}) \times H^{2-r}(X, \check{\mathcal{F}}(1)) \to H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

defined above is perfect.

Remark 18.23. Applying this with $r = 1$ and $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$ and twisting both sides by μ_n we get a nondegenerate pairing

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \to \mu_n;$$

we have already identified $H^1(X, \mu_n)$ with $\operatorname{Pic}(X)(k)[n]$, and one can see that the pairing thus obtained is (up to sign) the Weil pairing on the Jacobian of X.

Sketch of proof. **Step 1. Behaviour under finite maps.** Let $\pi : X' \to X$ be finite and let $\mathcal{G}$ be a locally constant finite sheaf killed by n on X'. Then the theorem is true for $\mathcal{G}$ if and only if it is true for $\pi_*\mathcal{G}$: this is true because by proposition 17.3 we have $H^r(X', \mathcal{G}) = H^r(X, \pi_*\mathcal{G})$, and a similar statement holds for H^{2-r}.

Step 2. Reformulation of the problem. Denote by $T^r(X_{\text{ét}}, \mathcal{F}) := H^{2-r}(X_{\text{ét}}, \check{\mathcal{F}}(1))^\vee$, where $^\vee$ denotes the dual in the sense of $\operatorname{Hom}(-, \mathbb{Z}/n\mathbb{Z})$. Then the statement is that the map $\phi^r(X_{\text{ét}}, \mathcal{F}) : H^r(X_{\text{ét}}, \mathcal{F}) \to T^r(X_{\text{ét}}, \mathcal{F})$ induced by the pairing in the statement is an isomorphism.

Observe that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, then we get an exact sequence

$$\cdots \to H^r(X_{\text{ét}}, \mathcal{F}') \to H^r(X_{\text{ét}}, \mathcal{F}) \to H^r(X_{\text{ét}}, \mathcal{F}'') \to H^{r+1}(X_{\text{ét}}, \mathcal{F}') \to \cdots$$

But this is not the only operation we can perform: indeed the sequence above also induces an exact sequence

$$0 \to \check{\mathcal{F}}''(1) \to \check{\mathcal{F}}(1) \to \check{\mathcal{F}}'(1) \to 0,$$

because $\operatorname{Hom}(-, \mu_n)$ is exact (look at stalks and recall that $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}/n\mathbb{Z}$-module). By taking cohomology of *this* sequence and dualising again (which is exact, because $\operatorname{Hom}_{\mathbb{Z}/n\mathbb{Z}}(-, \mathbb{Z}/n\mathbb{Z})$ preserves exactness of sequences of $\mathbb{Z}/n\mathbb{Z}$-modules), we get an exact sequence

$$\cdots \to T^r(X_{\text{ét}}, \mathcal{F}') \to T^r(X_{\text{ét}}, \mathcal{F}) \to T^r(X_{\text{ét}}, \mathcal{F}'') \to T^{r+1}(X_{\text{ét}}, \mathcal{F}') \to \cdots$$

Step 3. Proof for $r = 0$ and $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$. In this case

$$H^0(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) = \operatorname{Ext}^0_{X,n}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} \text{ and } H^2(X_{\text{ét}}, \check{\mathcal{F}}(1)) = H^2(X_{\text{ét}}, \mu_n) = \mathbb{Z}/n\mathbb{Z}.$$

The coproduct then gives $H^2(X_{\text{ét}}, \check{\mathcal{F}}(1)) = \operatorname{Ext}^2_{X,n}(\mathbb{Z}/n\mathbb{Z}, \mu_n)$ its natural structure of module over $\operatorname{Ext}^0_{X,n}(\mathbb{Z}/n\mathbb{Z}, \mu_n)$, and is therefore nondegenerate.

Step 4. Proof for $r = 0$ and $\mathcal{F}$ locally constant. Let $X' \to X$ be a finite étale cover such that $\mathcal{F}|_{X'}$ is constant. There is an exact sequence of sheaves on X

$$0 \to \mathcal{F} \to \pi_*\mathcal{F}' \to \mathcal{F}'' \to 0$$

induced from applying π_* to $\mathcal{F}|_{X'} \hookrightarrow \mathcal{F}'$ and then noticing that $\mathcal{F}$ embeds in $\pi_*\pi^*\mathcal{F}$. The cokernel $\mathcal{F}''$ is also locally constant. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d, "\cong"] & H^0(X_{\text{ét}}, \mathcal{F}) \arrow[r] \arrow[d] & H^0(X_{\text{ét}}, \pi_*\mathcal{F}) \arrow[r] \arrow[d, "\cong"] & H^0(X_{\text{ét}}, \mathcal{F}'') \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & T^0(X_{\text{ét}}, \mathcal{F}) \arrow[r] & T^0(X_{\text{ét}}, \pi_*\mathcal{F}) \arrow[r] & T^0(X_{\text{ét}}, \mathcal{F}'').
\end{tikzcd}$$

The five lemma implies that the third vertical arrow is injective; since this is true for all finite locally constant $\mathcal{F}$'s, also the fifth vertical arrow is injective. Another application of the five lemma then implies that the middle vertical arrow is an isomorphism. Notice that $H^0(X_{\text{ét}}, \mathcal{F})$ is finite, since it embeds in the finite group $H^0(X_{\text{ét}}, \pi_*\mathcal{F}') = H^0(X'_{\text{ét}}, \mathcal{F}') = (\mathbb{Z}/n\mathbb{Z})^t$.

Step 5. Proof for $r = 1$ and $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$. By theorem 18.19, a class $s \in H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ corresponds to a $\mathbb{Z}/n\mathbb{Z}$-torsor $\pi : Y \to X$. Upon pullback to Y, the torsor trivialises, and therefore so does the class s. Let as above $\mathcal{F}''$ be the cokernel of $\mathbb{Z}/n\mathbb{Z} \to \pi_*(\mathbb{Z}/n\mathbb{Z})$. We have a commutative diagram

$$\begin{tikzcd}
H^0(X_{\text{ét}}, \pi_*\mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^0(X_{\text{ét}}, \mathcal{F}'') \arrow[r] \arrow[d, "\cong"] & H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d] & H^1(X_{\text{ét}}, \pi_*\mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d] & H^1(X_{\text{ét}}, \mathcal{F}'') \arrow[d] \\
T^0(X_{\text{ét}}, \pi_*\mathbb{Z}/n\mathbb{Z}) \arrow[r] & T^0(X_{\text{ét}}, \mathcal{F}'') \arrow[r] & T^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] & T^1(X_{\text{ét}}, \pi_*\mathbb{Z}/n\mathbb{Z}) \arrow[r] & T^1(X_{\text{ét}}, \mathcal{F}'');
\end{tikzcd}$$

suppose that our $s \in H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ maps to 0 in $T^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$. Since by construction it maps to 0 in $H^1(X_{\text{ét}}, \pi_*\mathbb{Z}/n\mathbb{Z}) = H^1(Y_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$, a simple diagram chase shows that it must be 0, so the middle vertical arrow is injective. Since we computed both groups to have the same order n^{2g}, it must be an isomorphism.

Step 6. Proof for $r = 1$ and $\mathcal{F}$ locally constant. As in step 4, one gets a commutative diagram

$$\begin{tikzcd}
H^0(X_{\text{ét}}, \pi_*\mathcal{F}) \arrow[r] \arrow[d, "\cong"] & H^0(X_{\text{ét}}, \mathcal{F}'') \arrow[r] \arrow[d, "\cong"] & H^1(X_{\text{ét}}, \mathcal{F}) \arrow[r] \arrow[d] & H^1(X_{\text{ét}}, \pi_*\mathcal{F}) \arrow[r] \arrow[d, "\cong"] & H^1(X_{\text{ét}}, \mathcal{F}'') \arrow[d] \\
T^0(X_{\text{ét}}, \pi_*\mathcal{F}) \arrow[r] & T^0(X_{\text{ét}}, \mathcal{F}'') \arrow[r] & T^1(X_{\text{ét}}, \mathcal{F}) \arrow[r] & T^1(X_{\text{ét}}, \pi_*\mathcal{F}) \arrow[r] & T^1(X_{\text{ét}}, \mathcal{F}'').
\end{tikzcd}$$

which (by applying the five lemma) gives first the injectivity of $H^1(X_{\text{ét}}, \mathcal{F}) \to T^1(X_{\text{ét}}, \mathcal{F})$, then the injectivity of the last vertical arrow, and finally the fact that the middle vertical arrow is an isomorphism.

Step 7. Proof for $r = 2$. Since $\mathcal{G} = \check{\mathcal{F}}(1)$ implies $\check{\mathcal{G}}(1) \cong \mathcal{F}$, the situation is symmetric, so this case reduces to the case $r = 0$ that we treated above.

□

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