Infinite root stacks of logarithmic schemes and moduli of parabolic sheaves

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Outline

Two aspects:

- the infinite root stack of a log scheme
- moduli theory of parabolic sheaves.

Log structures and parabolic sheaves

Infinite root stacks

Moduli of parabolic sheaves

Outline

Two aspects:

- the infinite root stack of a log scheme
- moduli theory of parabolic sheaves.

Parabolic sheaves (on a curve)

Let X be a smooth projective curve over $\mathbb{C}$, and $x_1, \ldots, x_k \in X$.

Definition (Seshadri)

A parabolic bundle on X is a vector bundle E on X with additional data: for every point x_i there is a filtration $0 \subset F_{i,h_i} \subset \cdots \subset F_{i,2} \subset F_{i,1} = E_{x_i}$ of the fiber of E over x_i, and weights $0 \leq a_{i,1} < \cdots < a_{i,h_i} < 1$.

Parabolic bundles generalize to the non-compact case the correspondence

$$\{\text{degree 0 poly-stable vector bundles on } X\}$$
$$\updownarrow$$
$$\{\text{unitary representations of } \pi_1(X)\}$$

If $D = x_1 + \cdots + x_k$, by taking inverse images along $E \to E|_D$, a parabolic bundle can be seen as

$$E(-D) \subset F_h \subset \cdots \subset F_1 = E$$

with weights $0 \leq a_1 < \cdots < a_h < 1$.

This definition makes sense for X a variety with an effective Cartier divisor $D \subseteq X$ (Maruyama-Yokogawa).
We assume that the weights are rational numbers.

One defines morphisms, subsheaves, kernels, cokernels, etc.. obtaining a category of parabolic sheaves.

Log structures

A log scheme is a scheme X together with a sheaf of monoids A and a symmetric monoidal functor $A \to \mathrm{Div}_X$ where Div_X is the (fibered) category of line bundles with a section on X.

More concretely if P is a monoid, a symmetric monoidal functor $L\colon P \to \mathrm{Div}(X)$ sends

$$p \mapsto (L_p, s_p)$$

where $L_p \in \mathrm{Pic}(X)$ and $s_p \in \Gamma(X, L_p)$, with isomorphisms

$$L_p \otimes L_q \cong L_{p+q}$$

carrying $s_p \otimes s_q$ to s_{p+q}.

Example

If X is a variety and $D \subseteq X$ an effective Cartier divisor, we can take the symmetric monoidal functor $\mathbb{N} \to \mathrm{Div}(X)$ sending 1 to $(\mathcal{O}_X(D), s)$ where s is the canonical section. This induces a log structure on X.

If D has r irreducible components $D_1, \ldots, D_r$ and is simple normal crossings we can "separate the components" with the functor $\mathbb{N}^r \to \mathrm{Div}(X)$ sending e_i to $(\mathcal{O}_X(D_i), s_i)$.
This gives the "correct" log structure (Iyer-Simpson, Borne).

The idea is that to study $X \setminus D$ we reduce to studying X (e.g. $X \subseteq \overline{X}$ with $\overline{X}$ proper), but we need to keep track of what happens along D, and we do so with the log structure.

A parabolic sheaf on X with respect to D

$$E(-D) \subset F_h \subset \cdots \subset F_1 = E$$

with rational weights $0 \leq a_1 < \cdots < a_h < 1$ with common denominator n can be seen as a diagram

$$\begin{array}{cccccc} -1 & -a_h & \cdots & -a_2 & -a_1 & 0 \\ E \otimes \mathcal{O}(-D) \longrightarrow & F_h \longrightarrow & \cdots \longrightarrow & F_2 \longrightarrow & F_1 \longrightarrow & E \end{array}$$

of sheaves placed in the interval $[-1, 0]$, with maps going in the positive direction.

We can fill the gaps in $[-1, 0]$ by setting

$$E_{\frac{1}{n}k} = F_j \quad \text{for} \quad -a_j \leq \frac{k}{n} < -a_{j-1}$$

and extend the diagram to the whole $\frac{1}{n}\mathbb{Z}$ by setting

$$E_{\frac{1}{n}k+z} = E_{\frac{1}{n}k} \otimes \mathcal{O}_X(zD) \quad \text{for} \quad z \in \mathbb{Z} \quad \text{and} \quad -n < k \leq 0.$$

We obtain a functor

$$\frac{1}{n}\mathbb{Z} \to \mathrm{Qcoh}(X)$$

where there is an arrow $a \to b$ in $\frac{1}{n}\mathbb{Z}$ if and only if $a \leq b$ (i.e. there is $p \in \frac{1}{n}\mathbb{N}$ such that $a + p = b$).

Parabolic sheaves (on a log scheme)

Let X be a log scheme with log structure defined by $L\colon P \to \mathrm{Div}(X)$, and choose a denominator $n \in \mathbb{N}$ ($\sim$ common denominator of the weights).
Denote by $\frac{1}{n}P^{\mathrm{wt}}$ the category with objects the elements of $\frac{1}{n}P^{\mathrm{gp}}$ and arrows $a \to b$ elements $p \in \frac{1}{n}P$ such that $a + p = b$.

Definition (Borne-Vistoli)

A parabolic sheaf on X with weights in $\frac{1}{n}P$ is a functor $E\colon \frac{1}{n}P^{\mathrm{wt}} \to \mathrm{Qcoh}(X)$ together with isomorphisms $E_{a+p} \cong E_a \otimes L_p$ for any $a \in \frac{1}{n}P^{\mathrm{gp}}$ and $p \in P^{\mathrm{gp}}$, that satisfy some compatibility properties.

For example, for any $a \in \frac{1}{n}P^{\mathrm{gp}}$ and $p \in P$, the map $E_a \to E_{a+p} \cong E_a \otimes L_p$ corresponding to p is given by multiplication by the section s_p of L_p.

We can visualize a parabolic sheaf $E\colon \frac{1}{n}P^{\mathrm{wt}} \to \mathrm{Qcoh}(X)$ as a "diagram" of sheaves: we have to imagine the lattice $\frac{1}{n}P^{\mathrm{gp}}$, place a sheaf in every point of the lattice, and maps between them, going in the "positive direction".

If the log structure on X is given by an effective Cartier divisor $D = D_1 + D_2$ and we take $n = 2$, then a parabolic sheaf can be seen as

$$
\begin{array}{ccccc}
E_{(0,0)} \otimes \mathcal{O}_X(-D_1) & \longrightarrow & E_{(-\frac{1}{2},0)} & \longrightarrow & E_{(0,0)} \\
\uparrow & & \uparrow & & \uparrow \\
E_{(0,-\frac{1}{2})} \otimes \mathcal{O}_X(-D_1) & \longrightarrow & E_{(-\frac{1}{2},-\frac{1}{2})} & \longrightarrow & E_{(0,-\frac{1}{2})} \\
\uparrow & & \uparrow & & \uparrow \\
E_{(0,0)} \otimes \mathcal{O}_X(-D) & \longrightarrow & E_{(-\frac{1}{2},0)} \otimes \mathcal{O}_X(-D_2) & \longrightarrow & E_{(0,0)} \otimes \mathcal{O}_X(-D_2)
\end{array}
$$

in the “negative unit square”, and extended outside by tensoring with powers of $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$.

Root stacks (Olsson, Borne-Vistoli)

Take a log scheme X with log structure $L\colon A \to \mathrm{Div}_X$, and $n \in \mathbb{N}$. We can define the n-th root stack $\sqrt[n]{X}$ as the stack that parametrizes liftings of L to $\frac{1}{n}A \to \mathrm{Div}_X$.

- If the log structure is induced by an irreducible Cartier divisor $D \subseteq X$, the stack $\sqrt[n]{X}$ parametrizes n-th roots of the divisor D.
- Locally, if $X = \operatorname{Spec} R$ and the divisor is given by $f \in R$, then the n-th root stack is the quotient $[\operatorname{Spec}(R[x]/(x^n - f))/\mu_n]$.

Root stacks are tame Artin stacks, Deligne–Mumford in certain cases (for example if $\mathrm{char}(k) = 0$).

Theorem (Borne-Vistoli)

Let X be a coherent log scheme with log structure $A \to \mathrm{Div}_X$. There is an equivalence between parabolic sheaves on X with weights in $\frac{1}{n}A$ and quasi-coherent sheaves on the root stack $\sqrt[n]{X}$.

The infinite root stack

(with A. Vistoli)

If $n \mid m$, there is a projection morphism $\sqrt[m]{X} \to \sqrt[n]{X}$ that corresponds to raising to the $\frac{m}{n}$-th power. This gives a projective system of algebraic stacks.

Definition

The infinite root stack of X is the inverse limit $\sqrt[\infty]{X} = \varprojlim_n \sqrt[n]{X}$.

The stack $\sqrt[\infty]{X}$ parametrizes roots of all orders simultaneously.

Example

If X is a smooth projective curve over $\mathbb{C}$ with the log structure induced by the divisor $D = x_1 + \cdots + x_k$, the infinite root stack $\sqrt[\infty]{X}$ looks like X outside of D, and there is a stabilizer group $\widehat{\mathbb{Z}}$ at each of the points x_i.

The infinite root stack is some kind of algebraic incarnation of the Kato-Nakayama space $X^{\log}$ of a log analytic space, which is constructed by replacing every point of X by $(S^1)^r$, where r is the rank of the log structure at x.
Here S^1 plays the role of $B\mathbb{Z}$, and $B\widehat{\mathbb{Z}}$ is an algebraic approximation of it.

The infinite root stack is not algebraic, but it is (étale) locally on X a quotient of an affine scheme by a diagonalizable group (not of finite type), and thus it has an fpqc presentation.

The geometry of $\sqrt[\infty]{X}$ is closely related to the log geometry of X.

Results

Theorem

There is a (functorial) reconstruction procedure that expresses the log structure of X in terms of $\sqrt[\infty]{X}$. In particular if $\sqrt[\infty]{X} \cong \sqrt[\infty]{Y}$ then $X \cong Y$ as log schemes.

Theorem

The functor $(Y \to X) \mapsto (\sqrt[\infty]{Y} \to \sqrt[\infty]{X})$ from the Kummer-flat site of X to the small fppf site of $\sqrt[\infty]{X}$ induces an equivalence between the corresponding topoi.

Theorem

Quasi-coherent sheaves on $\sqrt[\infty]{X}$ correspond to parabolic sheaves with arbitrary rational weights on X.

So finitely presented sheaves on the Kummer-flat site of X = finitely presented parabolic sheaves with rational weights on X.

Moduli of parabolic sheaves (with fixed weights)

Moduli spaces for parabolic sheaves were constructed by

- Mehta-Seshadri in the case of curves,
- Maruyama-Yokogawa for a variety with an effective Cartier divisor.

They generalize the classical GIT construction by defining an appropriate notion of (semi-)stability.

To generalize these results we start from the equivalence

$$\{\text{parabolic sheaves}\} \cong \{\text{quasi-coherent sheaves on a root stack}\}$$

to reduce to moduli of coherent sheaves on an algebraic stack.

Nironi's theory

Nironi developed a moduli theory for coherent sheaves on a class of DM stacks.
Assumptions: fix a denominator $n \in \mathbb{N}$, assume that $\sqrt[n]{X}$ is Deligne–Mumford and that the scheme X (which is its coarse moduli space) is projective.

Idea: define a Hilbert polynomial (and thus a slope) by using the polarization on X.
Simply taking π_* (where $\pi\colon \sqrt[n]{X} \to X$) is not sufficient. Before taking the pushforward one has to twist by a generating sheaf on $\sqrt[n]{X}$.

A generating sheaf is a locally free sheaf of finite rank $\mathcal{E}$ on $\sqrt[n]{X}$ such that for every point x of $\sqrt[n]{X}$, the fiber $\mathcal{E}_x$ contains every irreducible representation of the stabilizer group $\mathrm{stab}(x)$ of $\sqrt[n]{X}$ at x.

To write down a generating sheaf, assume that X has a global chart $P \to \mathrm{Div}(X)$. The construction is inspired by comparison with the case of Maruyama and Yokogawa.

Example

If the log scheme is given by a projective variety with an irreducible effective Cartier Divisor $D \subseteq X$, we take as generating sheaf on $\sqrt[n]{X}$ the sheaf

$$\mathcal{E} = \mathcal{O}(\mathcal{D}) \oplus \mathcal{O}(2\mathcal{D}) \oplus \cdots \oplus \mathcal{O}(n\mathcal{D}),$$

where $\mathcal{D}$ is the n-th root of the divisor D on the root stack $\sqrt[n]{X}$.

This generating sheaf gives back the theory of Maruyama and Yokogawa.

One defines a (modified) Hilbert polynomial

$$P_{\mathcal{E}}(F) = P(\pi_*(F \otimes \mathcal{E}^\vee)) \in \mathbb{Z}[x]$$

where P in the RHS is the usual Hilbert polynomial on X with respect to some fixed polarization.
In the case of a projective variety with an irreducible effective Cartier divisor $D \subseteq X$, the Hilbert polynomial of the parabolic sheaf

$$\begin{array}{cccccc} -1 & -\frac{n-1}{n} & \cdots & -\frac{2}{n} & -\frac{1}{n} & 0 \\ E \otimes \mathcal{O}(-D) = F_n \longrightarrow & F_{n-1} \longrightarrow & \cdots \longrightarrow & F_2 \longrightarrow & F_1 \longrightarrow & E \end{array}$$

is

$$P_{\mathcal{E}}(F) = P(F_1) + P(F_2) + \cdots + P(F_n).$$

Semi-stability

We have a (generalized) slope

$$p_{\mathcal{E}}(F) = \frac{P_{\mathcal{E}}(F)}{\alpha^d(F)}$$

where $\alpha^d(F)$ is the leading coefficient of $P_{\mathcal{E}}(F)$, and a notion of stability.

Definition

A parabolic sheaf $F \in \mathrm{Coh}(\sqrt[n]{X})$ is (semi-)stable if for any subsheaf $G \subseteq F$ we have

$$p_{\mathcal{E}}(G) \ (\leq) \ p_{\mathcal{E}}(F).$$

This satisfies the usual properties of (semi-)stability of coherent sheaves on a projective variety. Fix the Hilbert polynomial $H \in \mathbb{Z}[x]$.

Theorem

The stack $\mathcal{M}_H^{ss}$ parametrizing semi-stable parabolic sheaves with Hilbert polynomial H is a finite type Artin stack, with a projective good moduli space M_H^{ss}.
The open substack $\mathcal{M}_H^{s}$ parametrizing stable sheaves with Hilbert polynomial H has a coarse moduli space M_H^{s}, an open subscheme of M_H^{ss}.

Moduli of parabolic sheaves (with varying weights)

The next step is to let the weights vary arbitrarily, without bounding the denominators.
In other words, we consider moduli of sheaves on the infinite root stack $\sqrt[\infty]{X}$.

Natural approach: take a limit of the moduli theories on the finite-level root stacks $\sqrt[n]{X}$.
Question: how does (semi-)stability behave under pullback along $\sqrt[m]{X} \to \sqrt[n]{X}$ for $n \mid m$?

In general semi-stability is not preserved.

Semi-stability is preserved if we assume that X has a global chart $P \to \mathrm{Div}(X)$ where P is simplicial and we use a slightly different system of finite-level root stacks.

Assume for simplicity that $P \cong \mathbb{N}^r$ is free, that the log structure is generically trivial and that we are considering torsion-free parabolic sheaves.

Theorem

In this case pullback along $\sqrt[m]{X} \to \sqrt[n]{X}$ *preserves semi-stability and stability, and the slope.*
Moreover the induced maps between the corresponding moduli stacks and spaces are open and closed immersions.

Moduli theory on $\sqrt[\infty]{X}$

Using the preceding result we can give a notion of (semi-)stability for finitely presented sheaves on $\sqrt[\infty]{X}$. We use the fact that $\mathrm{FP}(\sqrt[\infty]{X}) = \varinjlim_n \mathrm{FP}(\sqrt[n]{X})$.

Definition

The slope $p(F)$ of $F \in \mathrm{FP}(\sqrt[\infty]{X})$ is the slope of any $F_n \in \mathrm{FP}(\sqrt[n]{X})$ that pulls back to F.
A finitely presented sheaf F on $\sqrt[\infty]{X}$ is (semi-)stable if for every finitely presented subsheaf $G \subseteq F$ we have

$$p(G) \ (\leq) \ p(F).$$

We can take a direct limit of the moduli stacks $\mathcal{M}_n^{ss}$ and spaces M_n^{ss} at finite level and obtain the following result.

Theorem

The stack $\mathcal{M}^{ss}$ parametrizing semi-stable parabolic sheaves with rational weights is an Artin stack, locally of finite type. Moreover it is isomorphic to the direct limit $\varinjlim_n \mathcal{M}_n^{ss}$, and it has a good moduli space $M^{ss} = \varinjlim_n M_n^{ss}$.
The open substack $\mathcal{M}^s \subseteq \mathcal{M}^{ss}$ parametrizing stable sheaves has a coarse moduli space M^s, which is an open subscheme of M^{ss}.

Thank you for bearing with me!