Root stacks of logarithmic schemes and moduli of parabolic sheaves

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Outline

Moduli theory for parabolic sheaves, through root stacks of log schemes. (partly) joint work with Angelo Vistoli.

Parabolic sheaves

Log schemes and root stacks

Moduli theory

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Parabolic sheaves (on a curve)

Let X be a compact Riemann surface, and $x_1, \ldots, x_k \in X$.

Definition (Seshadri)

A **parabolic bundle** on X (with respect to $x_1, \ldots, x_k$) is an (algebraic) vector bundle E on X with additional data: for every point x_i there is a filtration $0 \subset F_{i,h_i} \subset \cdots \subset F_{i,2} \subset F_{i,1} = E_{x_i}$ of the fiber of E over x_i, and weights $0 \leq a_{i,1} < \cdots < a_{i,h_i} < 1$.

Sometimes the weights are not included in the definition, and introduced as an auxiliary gadget. In my case, they a fundamental part of the definition.

Narasimhan-Seshadri correspondence: for X a compact Riemann surface

$$\{\text{degree 0 stable vector bundles on } X\}$$
$$\updownarrow$$
$$\{\text{unitary irreducible representations of } \pi_1(X)\}.$$

Parabolic bundles were defined in order to generalize this correspondence to non-compact curves (i.e. $X \setminus \{x_1, \ldots, x_k\}$).

The filtration $0 \subset F_{i,h_i} \subset \cdots \subset F_{i,2} \subset F_{i,1} = E_{x_i}$ and the weights $a_{i,1}, \cdots, a_{i,h_i}$ come from eigenspaces and eigenvalues of the matrix corresponding to a small loop $\gamma \in \pi_1(X \setminus \{x_1, \ldots, x_k\})$ around the puncture x_i.

From now on I will assume that the weights $a_{i,k}$ are rational numbers. For example parabolic bundles arising from representations of the algebraic fundamental group $\widehat{\pi}_1(X \setminus \{x_1, \ldots, x_k\})$ always have rational weights.

If $D = x_1 + \cdots + x_k$, by taking inverse images along $E \to E|_D$, a parabolic bundle can be seen as

$$E(-D) \subset F_h \subset \cdots \subset F_1 = E$$

with weights $0 \leq a_1 < \cdots < a_h < 1$.

This definition makes sense for any variety X with an effective Cartier divisor $D \subseteq X$ (Maruyama-Yokogawa).

One defines morphisms, subsheaves, kernels, cokernels, etc.. and obtains a nice category of parabolic sheaves.

Parabolic sheaves are “best” defined on an arbitrary logarithmic scheme.

Log schemes

Log schemes were first introduced for arithmetic purposes, and afterwards spread to other areas of algebraic geometry, including moduli theory.

For example the moduli space of log-smooth curves recovers the Deligne-Mumford compactification $\overline{M}_{g,n}$. In this and other situations, log structures single out “good” degenerations of smooth objects.

Definition

A log scheme is a scheme X together with a sheaf of monoids A and a symmetric monoidal functor $A \to \mathrm{Div}_X$ where Div_X is the (fibered) category of line bundles with a section on X (generalized Cartier divisors).

More concretely, if P is a monoid a symmetric monoidal functor $L\colon P \to \mathrm{Div}(X)$ sends

$$p \mapsto (L_p, s_p)$$

where $L_p \in \mathrm{Pic}(X)$ and $s_p \in \Gamma(X, L_p)$, with isomorphisms

$$L_p \otimes L_q \cong L_{p+q}$$

carrying $s_p \otimes s_q$ to s_{p+q}.

The idea is to generalize the situation were we have an effective Cartier divisor $D \subseteq X$, that we see as a "boundary", and we are interested in $X \setminus D$.

In the case of (X, D) we can take the symmetric monoidal functor $\mathbb{N} \to \mathrm{Div}(X)$ sending 1 to $(\mathcal{O}_X(D), s)$ where s is the canonical section.

If D has r irreducible components $D_1, \ldots, D_r$ and is simple normal crossings we can "separate the components" with the functor $\mathbb{N}^r \to \mathrm{Div}(X)$ sending e_i to $(\mathcal{O}_X(D_i), s_i)$.
This gives the "correct" log structure (Iyer-Simpson, Borne) in this case.

Parabolic sheaves (on log schemes)

A parabolic sheaf on X with respect to D

$$E(-D) \subset F_h \subset \cdots \subset F_1 = E$$

with rational weights $0 \leq a_1 < \cdots < a_h < 1$ with common denominator n can be seen as a diagram

$$\begin{array}{cccccc} -1 & -a_h & \cdots & -a_2 & -a_1 & 0 \\ E \otimes \mathcal{O}(-D) \longrightarrow & F_h \longrightarrow & \cdots \longrightarrow & F_2 \longrightarrow & F_1 \longrightarrow & E \end{array}$$

of sheaves placed in the interval $[-1, 0]$, with maps going in the positive direction.

We can fill the gaps in $[-1,0]\cap\frac{1}{n}\mathbb{Z}$ by setting

$$E_{\frac{1}{n}k}=F_j \qquad \text{for} \qquad -a_j\leq\frac{k}{n}<-a_{j-1}$$

and extend the diagram to the whole $\frac{1}{n}\mathbb{Z}$ by setting

$$E_{\frac{1}{n}k+z}=E_{\frac{1}{n}k}\otimes\mathcal{O}_X(zD) \quad \text{for} \quad z\in\mathbb{Z} \quad \text{and} \quad -n<k\leq 0.$$

We obtain a functor

$$\frac{1}{n}\mathbb{Z}\to\mathrm{Qcoh}(X)$$

where there is an arrow $a\to b$ in $\frac{1}{n}\mathbb{Z}$ if and only if $a\leq b$ (i.e. there is $p\in\frac{1}{n}\mathbb{N}$ such that $a+p=b$).

Let X be a log scheme with log structure defined by $L\colon P \to \mathrm{Div}(X)$, and choose a denominator $n \in \mathbb{N}$ ($\sim$ common denominator of the weights).
Denote by $\frac{1}{n}P^{\mathrm{wt}}$ the category with objects the elements of $\frac{1}{n}P^{\mathrm{gp}}$ and arrows $a \to b$ elements $p \in \frac{1}{n}P$ such that $a + p = b$.

Definition (Borne-Vistoli)

A parabolic sheaf on X with weights in $\frac{1}{n}P$ is a functor $E\colon \frac{1}{n}P^{\mathrm{wt}} \to \mathrm{Qcoh}(X)$ together with isomorphisms $E_{a+p} \cong E_a \otimes L_p$ for any $a \in \frac{1}{n}P^{\mathrm{gp}}$ and $p \in P$, that satisfy some compatibility properties.

Example

If the log structure on X is given by a snc divisor $D = D_1 + D_2$ with 2 irreducible components and we take $n = 2$, then a parabolic sheaf can be seen as

$$
\begin{array}{ccccc}
E_{(0,0)} \otimes \mathcal{O}_X(-D_1) & \longrightarrow & E_{(-\frac{1}{2},0)} & \longrightarrow & E_{(0,0)} \\
\uparrow & & \uparrow & & \uparrow \\
E_{(0,-\frac{1}{2})} \otimes \mathcal{O}_X(-D_1) & \longrightarrow & E_{(-\frac{1}{2},-\frac{1}{2})} & \longrightarrow & E_{(0,-\frac{1}{2})} \\
\uparrow & & \uparrow & & \uparrow \\
E_{(0,0)} \otimes \mathcal{O}_X(-D) & \longrightarrow & E_{(-\frac{1}{2},0)} \otimes \mathcal{O}_X(-D_2) & \longrightarrow & E_{(0,0)} \otimes \mathcal{O}_X(-D_2)
\end{array}
$$

in the “negative unit square”, and extended outside by tensoring with powers of $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$.

Root stacks (Olsson, Borne-Vistoli)

Take a log scheme X with log structure $L\colon A \to \mathrm{Div}_X$, and $n \in \mathbb{N}$. We have an inclusion of sheaves of monoids $A \subseteq \frac{1}{n}A$.

We define the n-th root stack $\sqrt[n]{X}$ as the stack that parametrizes liftings $\frac{1}{n}A \to \mathrm{Div}_X$ of $L\colon A \to \mathrm{Div}_X$ ("roots" of the log structure).

In other words, for a scheme $T \to X$, the groupoid $\sqrt[n]{X}(T)$ is the category of symmetric monoidal functors $\frac{1}{n}A_T \to \mathrm{Div}_T$ that restrict to $L_T\colon A_T \to \mathrm{Div}_T$ along the inclusion $A_T \subseteq \frac{1}{n}A_T$.

- If the log structure is induced by an irreducible Cartier divisor $D \subseteq X$, the stack $\sqrt[n]{X}$ parametrizes n-th roots of the divisor D.
- Locally, if $X = \operatorname{Spec} R$ and the divisor is given by $f \in R$, then the n-th root stack is the quotient $[\operatorname{Spec}(R[x]/(x^n - f))/\mu_n]$.
- If X is a compact Riemann surface, $\sqrt[n]{X}$ is an orbifold with coarse moduli space X, and stabilizer $\mathbb{Z}/n\mathbb{Z}$ over the punctures x_i.

Root stacks are tame Artin stacks, Deligne–Mumford in certain cases (for example if $\mathrm{char}(k) = 0$).

Theorem (Borne-Vistoli)

Let X be a coherent log scheme with log structure $A \to \mathrm{Div}_X$. There is an equivalence between parabolic sheaves on X with weights in $\frac{1}{n}A$ and quasi-coherent sheaves on the root stack $\sqrt[n]{X}$.

The infinite root stack

(with A. Vistoli)

If $n \mid m$, there is a projection morphism $\sqrt[m]{X} \to \sqrt[n]{X}$ that corresponds to raising to the $\frac{m}{n}$-th power. This gives a projective system of algebraic stacks.

Definition

The infinite root stack of X is the inverse limit $\sqrt[\infty]{X} = \varprojlim_n \sqrt[n]{X}$.

The stack $\sqrt[\infty]{X}$ parametrizes roots of all orders simultaneously. Furthermore, it carries all (finitely presented) parabolic sheaves with rational weights on X.

If X is a compact Riemann surface with the log structure induced by the divisor $D = x_1 + \cdots + x_k$, the infinite root stack $\sqrt[\infty]{X}$ looks like X outside of D, and there is a stabilizer group $\widehat{\mathbb{Z}}$ at each of the points x_i.

The infinite root stack is some kind of algebraic incarnation of the Kato-Nakayama space $X^{\log}$ of a log analytic space, and the geometry of $\sqrt[\infty]{X}$ is closely related to the log geometry of X. For example if $\sqrt[\infty]{X} \cong \sqrt[\infty]{Y}$ then $X \cong Y$ as log schemes.

Moduli of parabolic sheaves (with fixed weights)

Moduli spaces for parabolic sheaves were constructed by

- Mehta-Seshadri in the case of curves,
- Maruyama-Yokogawa for a variety with an effective Cartier divisor.

They generalize the classical GIT construction by defining an appropriate notion of (semi-)stability.

To generalize these results we start from the equivalence

$$\{\text{parabolic sheaves}\} \cong \{\text{quasi-coherent sheaves on a root stack}\}$$

to reduce to moduli of coherent sheaves on an algebraic stack.

Nironi's theory

Nironi developed a moduli theory for coherent sheaves on a class of DM stacks.
Assumptions: fix a denominator $n \in \mathbb{N}$, assume that $\sqrt[n]{X}$ is Deligne–Mumford and that the scheme X (which is its coarse moduli space) is projective.

Idea: define a Hilbert polynomial (and thus a slope) by using the polarization on X.
Simply taking π_* (where $\pi\colon \sqrt[n]{X} \to X$) is not sufficient. Before taking the pushforward one has to twist by a generating sheaf on $\sqrt[n]{X}$.

A generating sheaf is a locally free sheaf of finite rank $\mathcal{E}$ on $\sqrt[n]{X}$ such that for every point x of $\sqrt[n]{X}$, the fiber $\mathcal{E}_x$ contains every irreducible representation of the stabilizer group $\operatorname{stab}(x)$ of $\sqrt[n]{X}$ at x.

To write down a generating sheaf, assume that the log structure of X is given by a functor $P \to \operatorname{Div}(X)$ (a “global chart”). The construction is inspired by comparison with the case of Maruyama and Yokogawa.

One defines a (modified) Hilbert polynomial

$$P_{\mathcal{E}}(F) = P(\pi_*(F \otimes \mathcal{E}^\vee)) \in \mathbb{Z}[x]$$

where P in the RHS is the usual Hilbert polynomial on X with respect to some fixed polarization.

In the case (X, D), the Hilbert polynomial of the parabolic sheaf

$$\begin{array}{ccccccccccc} -1 & & -\frac{n-1}{n} & & \cdots & & -\frac{2}{n} & & -\frac{1}{n} & & 0 \\ E \otimes \mathcal{O}(-D) = F_n & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & E \end{array}$$

is

$$P_{\mathcal{E}}(F) = P(F_1) + P(F_2) + \cdots + P(F_n).$$

Semi-stability

We have a (generalized) slope

$$p_{\mathcal{E}}(F) = \frac{P_{\mathcal{E}}(F)}{\alpha^d(F)}$$

where $\alpha^d(F)$ is the leading coefficient of $P_{\mathcal{E}}(F)$, and a notion of stability.

Definition

A parabolic sheaf $F \in \mathrm{Coh}(\sqrt[n]{X})$ is (semi-)stable if for any subsheaf $G \subseteq F$ we have

$$p_{\mathcal{E}}(G)\ (\leq)\ p_{\mathcal{E}}(F).$$

This satisfies the usual properties of (semi-)stability of coherent sheaves on a projective variety. Fix the Hilbert polynomial $H \in \mathbb{Z}[x]$.

Theorem

The stack $\mathcal{M}_H^{ss}$ parametrizing semi-stable parabolic sheaves with Hilbert polynomial H is a finite type Artin stack, with a projective good moduli space M_H^{ss}.
The open substack $\mathcal{M}_H^{s}$ parametrizing stable sheaves with Hilbert polynomial H has a coarse moduli space M_H^{s}, an open subscheme of M_H^{ss}.

Moduli of parabolic sheaves (with varying weights)

The next step is to let the weights vary arbitrarily, without bounding the denominators.
In other words, we consider moduli of sheaves on the infinite root stack $\sqrt[\infty]{X}$.

Natural approach: take a limit of the moduli theories on the finite-level root stacks $\sqrt[n]{X}$.
Question: how does (semi-)stability behave under pullback along $\sqrt[m]{X} \to \sqrt[n]{X}$ for $n \mid m$?

In general semi-stability is not preserved.

Semi-stability is preserved if we assume that X has a global chart $P \to \mathrm{Div}(X)$ where P is simplicial and we use a slightly different system of finite-level root stacks.

Assume for simplicity that $P \cong \mathbb{N}^r$ is free, that the log structure is generically trivial and that we are considering torsion-free parabolic sheaves.

Theorem

In this case pullback along $\sqrt[m]{X} \to \sqrt[n]{X}$ preserves semi-stability and stability, and the slope.
Moreover the induced maps between the corresponding moduli stacks and spaces are open and closed immersions.

Moduli theory on $\sqrt[\infty]{X}$

Using the preceding result we can give a notion of (semi-)stability for finitely presented sheaves on $\sqrt[\infty]{X}$. We use the fact that $\mathrm{FP}(\sqrt[\infty]{X}) = \varinjlim_n \mathrm{FP}(\sqrt[n]{X})$.

Definition

The slope $p(F)$ of $F \in \mathrm{FP}(\sqrt[\infty]{X})$ is the slope of any $F_n \in \mathrm{FP}(\sqrt[n]{X})$ that pulls back to F.
A finitely presented sheaf F on $\sqrt[\infty]{X}$ is (semi-)stable if for every finitely presented subsheaf $G \subseteq F$ we have

$$p(G)\ (\leq)\ p(F).$$

We can take a direct limit of the moduli stacks $\mathcal{M}_n^{ss}$ and spaces M_n^{ss} at finite level and obtain the following result.

Theorem

The stack $\mathcal{M}^{ss}$ parametrizing semi-stable parabolic sheaves with rational weights is an Artin stack, locally of finite type. Moreover it is isomorphic to the direct limit $\varinjlim_n \mathcal{M}_n^{ss}$, and it has a good moduli space $M^{ss} = \varinjlim_n M_n^{ss}$.
The open substack $\mathcal{M}^s \subseteq \mathcal{M}^{ss}$ parametrizing stable sheaves has a coarse moduli space M^s, which is an open subscheme of M^{ss}.

Thank you for your attention!