(Infinite) root stacks of log schemes

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Logarithmic geometry[1]

Log scheme = scheme X + "extra data".

The "extra data" typically

- keeps track of a "boundary" in X, or
- contains infinitesimal information about a family, of which X is a fiber

and involves monoids ($\to$ toric geometry, combinatorics, etc.).

Example: start with a smooth non-proper variety Y/k, compactify it to $Y \subseteq X$, and say $D = X \setminus Y$ is a NC (normal crossings) divisor.

X is nicer to work with, but what we actually care about is Y. Idea: encode the stuff that we added (the divisor D) in the geometry.

One way to do it: take
$M(U) = \{f \in \mathcal{O}_X(U) \mid f \text{ is invertible outside of } D\}$.
This is a subsheaf of **monoids** of $\mathcal{O}_X$ containing $\mathcal{O}_X^\times$, that somehow keeps track of the boundary D.

(take two) Log scheme = scheme X + sheaf of monoids M and a map $\alpha\colon M \to \mathcal{O}_X$, that identifies the units.

One can develop (some) **EGA-style** algebraic geometry with log schemes. For example:

- one can define a sheaf $\Omega_X^{\log}$ of **"logarithmic differentials"**, related to log derivations and deformation theory
- in the example above $\Omega_X^{\log} = \Omega_X^1(\log D)$ is the sheaf of **logarithmic 1-forms**
- one can define a whole log de Rham complex $\Omega_X^\bullet(\log D)$, and in good cases this computes the cohomology of $X \setminus D = Y$.

Moreover, in this spirit, there is a notion of log smoothness that extends the classical notion.

Some morphisms that aren't classically smooth become log smooth when the spaces are equipped with appropriate log structures.

Roughly, this is related to $\Omega_X^{\log}$ as classical smoothness is related to Ω_X.

Example: log smooth curves[1]

We all know the moduli space (stack) of smooth curves $\mathcal{M}_{g,n}$ of genus g with n marked points. It's not compact. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ has **nodal curves** in the boundary $\partial\overline{\mathcal{M}}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ (that is a NC divisor... you probably see where this is going).

Back to log geometry, there is a notion of (stable) **log smooth log curve**, and of families of those over log schemes. This defines a moduli functor $\mathcal{L}og\mathcal{M}_{g,n}\colon (\mathrm{Logsch})^{\mathrm{op}} \to (\mathrm{Groupoids})$ over log schemes.

(g is the genus of the underlying curve, n is another invariant, corresponding of course to marked points)

Magic:

- the underlying curve of every log smooth log curve is nodal, and every family of nodal curves admits "canonical" log structures (on the base and the total space) for which it becomes log smooth
- the boundary divisor $\partial\overline{\mathcal{M}}_{g,n}$ gives a log structure on $\overline{\mathcal{M}}_{g,n}$. The resulting **log stack** $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ represents the functor $\mathcal{L}og\mathcal{M}_{g,n}$, in the sense that there is an equivalence

$$\mathcal{L}og\mathcal{M}_{g,n} \cong \mathrm{Hom}(-, (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})).$$

So log smoothness "automatically singles out" nodal curves. More generally

log smoothness selects "good" degenerations.

This philosophy was applied to other cases (K3 surfaces, abelian varieties, toric Hilbert schemes, etc.).

Root stacks[4]

Say $D \subseteq X$ is an effective Cartier divisor (giving a log structure on X). An ***n*-th root** of D is a line bundle L with a global section s such that $L^{\otimes n} \cong \mathcal{O}(D)$, with $s^{\otimes n}$ corresponding to the canonical section s_D.

There is a stack $\sqrt[n]{X, D}$ over X that functorially parametrizes n-th roots of D. The map $\sqrt[n]{X, D} \to X$ is an isomorphism outside of D, and a μ_n-gerbe along D.

For example say $X = \mathbb{A}^1$ and D is the origin. Then the n-th root stack is the quotient $[\mathbb{A}^1/\mu_n] \to \mathbb{A}^1$, where $\mathbb{A}^1 \to \mathbb{A}^1$ is $z \mapsto z^n$ and μ_n acts by multiplication.

More generally, a log scheme X has root stacks $\sqrt[n]{X}$ for any n. Taking these roots somehow **"magnifies"** the log structure. Take this to the extreme: $\varprojlim_n \sqrt[n]{X} =: \sqrt[\infty]{X}$.

This "infinite root stack" is not algebraic but has a flat atlas and local quotient stack presentations. It's still some kind of orbifold over X, although with huge stabilizers.

$\sqrt[\infty]{X}$ "embodies" aspects the log geometry of X, turning them into "plain" geometry.

Intuitively, having every possible root completely determines the "infinitesimal information" given by the log structure. In fact:

Theorem[4]: one can reconstruct X starting from $\sqrt[\infty]{X}$. In particular if $\sqrt[\infty]{X} \cong \sqrt[\infty]{Y}$ as stacks, then $X \cong Y$ as log schemes.

Another fact: **parabolic sheaves** on X can be seen as plain quasi-coherent sheaves on its root stacks. This can be exploited to construct moduli spaces of parabolic sheaves[3].

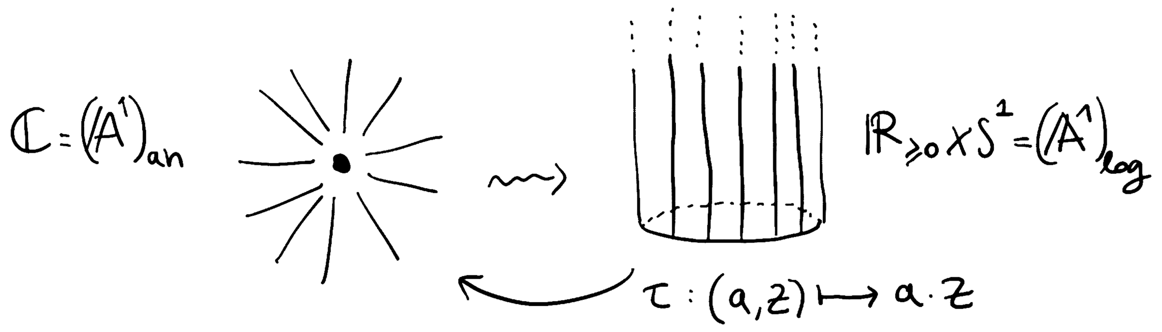
Root stacks vs Kato-Nakayama[2]

If X is a log scheme over $\mathbb{C}$, the **Kato-Nakayama space** $X_{\log}$ is a topological space that can be seen as an "underlying topological space" of the log scheme. The projection

$$\tau\colon X_{\log} \to X_{\mathrm{an}}$$

has fiber over x given by $(S^1)^r$, where r is the "rank" of the log structure at x.

Example: if $X = \mathbb{A}^1$ with the log structure given by the origin, then $X_{\log} \cong \mathbb{R}_{\geq 0} \times S^1$ is a cylinder.



The reduced fiber of $\sqrt[\infty]{X} \to X$ over x is $\mathrm{B}(\widehat{\mathbb{Z}})^r$, where r is the rank of the log structure at x again. Note that $S^1 \approx \mathrm{B}\mathbb{Z}$, and $\widehat{S^1} \approx \mathrm{B}\widehat{\mathbb{Z}}$.
This is not a coincidence.

Theorem[2]: there is a morphism of *topological stacks* $X_{\log} \to (\sqrt[\infty]{X})_{\mathrm{top}}$ that induces an equivalence

$$\widehat{X_{\log}} \cong \widehat{(\sqrt[\infty]{X})_{\mathrm{top}}}$$

between *profinite completions*.

This suggests to define the profinite homotopy type of a log scheme X as the profinite étale homotopy type (à la Artin-Mazur) of $\sqrt[\infty]{X}$, which coincides with $\widehat{(\sqrt[\infty]{X})_{\mathrm{top}}}$ over $\mathbb{C}$, but is defined even outside of this case.

References

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[3] M. Talpo, *Moduli of parabolic sheaves on a polarized logarithmic scheme*, `arxiv:1410.2212`, to appear in TransAMS.

[4] M. Talpo and A. Vistoli, *Infinite root stacks and quasi-coherent sheaves on log schemes*, `arxiv:1410.1164`.