Parabolic sheaves, root stacks and the Kato-Nakayama space

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Outline

Parabolic sheaves as sheaves on "stacks of roots", and log geometry.

Partly joint with A. Vistoli, and Carchedi-Scherotzke-Sibilla.

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Parabolic sheaves

Log schemes and (in)finite root stacks

Kato-Nakayama space and real roots

Parabolic sheaves (on a curve)

Let X be a compact Riemann surface.

Narasimhan-Seshadri correspondence: there is a bijection

{unitary irreducible representations of $\pi_1(X)$ } \uparrow {degree 0 stable (holomorphic) vector bundles on *X*}. (via local systems)

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What about the non-compact case?

Let $x_1, \ldots, x_k \in X$, and consider $X \setminus \{x_1, \ldots, x_k\}$.

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(Mehta-Seshadri, Deligne)

The "parabolic" structure is meant to encode the action of the small loops around the punctures.

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of the fiber of *E* over x_i , and weights $0 \le a_{i,1} < \cdots < a_{i,h_i} < 1$.

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~ eigenspaces and eigenvalues of the matrix corresponding to a small loop $\gamma \in \pi_1(X \setminus \{x_1, \ldots, x_k\})$ around the puncture x_i .

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If $D = x_1 + \cdots + x_k$ (divisor on *X*), by taking inverse images along $E \to E|_D$, a parabolic bundle can be seen as

$$E(-D) \subset F_h \subset \cdots \subset F_1 = E$$

with weights $0 \le a_1 < \cdots < a_h < 1$.

We can generalize and allow sheaves and maps

$$E\otimes \mathcal{O}(-D) o F_h o \cdots o F_1 = E$$

whose composition $E(-D) \rightarrow E$ is multiplication by the section 1_D of the line bundle $\mathcal{O}(D)$, and weights $0 \le a_1 < \cdots < a_h < 1$.

This definition makes sense for any variety X with an effective Cartier divisor $D \subseteq X$ (Maruyama-Yokogawa).

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One defines morphisms, subsheaves, kernels, cokernels, etc.. ~> a nice category of parabolic sheaves.

Parabolic sheaves are "best" defined on an arbitrary logarithmic scheme.

Log schemes (K. Kato, Fontaine-Illusie)

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More concretely: if *P* is a monoid, a symmetric monoidal functor *L*: $P \rightarrow \text{Div}(X)$ sends

$$p\mapsto (L_p,s_p)$$

with isomorphisms

$$L_p \otimes L_q \cong L_{p+q}$$

carrying $s_p \otimes s_q$ to s_{p+q} .

If $D \subseteq X$ is an eff. Cartier divisor we get a log scheme (X, D): take the symmetric monoidal functor $\mathbb{N} \to \text{Div}(X)$ sending 1 to $(\mathcal{O}(D), 1_D)$.

If *D* has *r* irreducible components D_1, \ldots, D_r and is simple normal crossings you might want to "separate the components" with the functor $\mathbb{N}^r \to \text{Div}(X)$ sending e_i to $(\mathcal{O}(D_i), 1_{D_i})$ (lyer-Simpson, Borne). If $D \subseteq X$ is an eff. Cartier divisor we get a log scheme (X, D): take the symmetric monoidal functor $\mathbb{N} \to \text{Div}(X)$ sending 1 to $(\mathcal{O}(D), 1_D)$.

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Example: start with X non-proper, compactify to $X \subseteq \overline{X}$ with SNC complement $D = \overline{X} \setminus X = D_1 \cup \ldots \cup D_r$,

and take $(\overline{X}, (D_1, \ldots, D_r))$.

How to think about this

To visualize the log scheme $(X, L: A \rightarrow Div_X)$, think about the stalks of the sheaf *A*.

There is a largest open subset $U \subseteq X$ where $A_p = 0$ (might be empty). In the "divisorial" case, $U = X \setminus D$.

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To visualize the log scheme $(X, L: A \rightarrow \text{Div}_X)$, think about the stalks of the sheaf *A*.

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More generally A is locally constant on a stratification (\sim discrete data).

Example: $X = \mathbb{A}^2$, $D = \{xy = 0\}$. The stalks of the sheaf A are

0	on	$\mathbb{A}^2 \setminus \{xy = 0\}$
\mathbb{N}	on	${xy = 0} \setminus {(0, 0)}$
N2	on	{ (0 , 0)}.

Parabolic sheaves (on log schemes)

A parabolic sheaf on X with respect to D

$$E \otimes \mathcal{O}(-D) \to F_h \to \cdots \to F_1 = E$$

with rational weights $0 \le a_1 < \cdots < a_h < 1$ with common denominator *n*

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can be seen as a diagram

$$-1$$
 $-a_h$ \cdots $-a_2$ $-a_1$ 0

$$E \otimes \mathcal{O}(-D) \longrightarrow F_h \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow E$$

of sheaves placed in the interval [-1, 0], with maps going in the positive direction.

We can fill the (possible) "gaps" in $[-1, 0] \cap \frac{1}{n}\mathbb{Z}$ by "looking at the sheaf on the left", and

extend out of [-1,0] by tensoring with powers of $\mathcal{O}(D)$.

(so that $E_{q+1} \cong E_q \otimes \mathcal{O}(D)$ for every $q \in \frac{1}{n}\mathbb{Z}$)

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We obtain a functor

$$\frac{1}{n}\mathbb{Z}\to\operatorname{Qcoh}(X)$$

where there is one arrow $a \to b$ in $\frac{1}{n}\mathbb{Z}$ if and only if $a \le b$ (i.e. there is $p \in \frac{1}{n}\mathbb{N}$ such that a + p = b).

Let *X* be a log scheme with log structure $L: P \to Div(X)$, and choose an index $n \in \mathbb{N}$ (~ common denominator of the weights).

Denote by $\frac{1}{n}P^{\text{wt}}$ the category with objects the elements of $\frac{1}{n}P^{gp}$ and arrows $a \rightarrow b$ elements $p \in \frac{1}{n}P$ such that a + p = b.

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Definition (Borne-Vistoli)

A parabolic sheaf on X with weights in $\frac{1}{n}P$ is a functor $E: \frac{1}{n}P^{\text{wt}} \rightarrow \text{Qcoh}(X)$ together with isomorphisms

$$E_{a+p} \cong E_a \otimes L_p$$
 for any $a \in \frac{1}{n} P^{gp}$ and $p \in P$

(that satisfy some compatibility properties).

Example

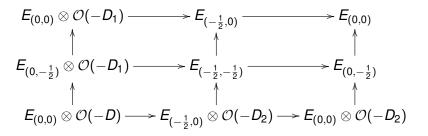
If the log structure on X is given by a snc divisor $D = D_1 + D_2$ with 2 irreducible components and we take n = 2,

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Example

If the log structure on X is given by a snc divisor $D = D_1 + D_2$ with 2 irreducible components and we take n = 2,

then a parabolic sheaf can be seen as



in the "negative unit square", and extended outside by tensoring with powers of $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$.

Root stacks (Olsson, Borne-Vistoli)

Take a log scheme X with log structure $L: A \rightarrow \text{Div}_X$, and $n \in \mathbb{N}$.

The *n*-th root stack $\sqrt[n]{X}$ parametrizes liftings



where $\wedge n$: $\text{Div}_X \rightarrow \text{Div}_X$ is given by

 $(L, s) \mapsto (L^{\otimes n}, s^{\otimes n}).$

If the log structure is induced by an irreducible Cartier divisor D ⊆ X, the stack ⁿ√X parametrizes *n*-th roots of the divisor D.

That is, pairs (L, s) such that $(L, s)^{\otimes n} \cong (\mathcal{O}(D), 1_D)$.

▶ If *X* is a compact Riemann surface and $D = x_1 + ... + x_k$, then $\sqrt[n]{X}$ is an orbifold with coarse moduli space *X*, and stabilizer $\mathbb{Z}/n\mathbb{Z}$ over the punctures x_i . Root stacks are tame Artin stacks, Deligne–Mumford in good cases (for example if char(k) = 0).

Theorem (Borne-Vistoli)

Let X be a log scheme with log structure $A \rightarrow \text{Div}_X$. There is an equivalence between

parabolic sheaves on X with weights in $\frac{1}{n}A$, and

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The "pieces" E_a of the parabolic sheaves are obtained (roughly) as eigensheaves for the action of the stabilizers of $\sqrt[n]{X}$.

The infinite root stack

(with A. Vistoli)

If $n \mid m$, there is a projection morphism

$$\sqrt[m]{X} \to \sqrt[n]{X}$$

that corresponds to raising to the $\frac{m}{n}$ -th power. This gives a projective system of algebraic stacks.

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Definition The infinite root stack of X is the inverse limit $\sqrt[\infty]{X} = \lim_{n \to \infty} \sqrt[n]{X}$.

The stack $\sqrt[\infty]{X}$ parametrizes compatible systems of roots of all orders. It is not algebraic, but it has local presentations as a quotient stack.

If *X* is a compact Riemann surface with the log structure induced by the divisor $D = x_1 + \cdots + x_k$, the infinite root stack $\sqrt[\infty]{X}$

- looks like X outside of D, and
- there is a stabilizer group $\widehat{\mathbb{Z}}$ at each of the points x_i .

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Theorem (-, Vistoli)

There is an equivalence between quasi-coherent sheaves on $\sqrt[\infty]{X}$ and parabolic sheaves on X with arbitrary rational weights. (\rightsquigarrow moduli spaces for parabolic sheaves)

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Theorem (-, Vistoli)

Every isomorphism $\sqrt[\infty]{X} \cong \sqrt[\infty]{Y}$ of stacks comes from a unique isomorphism of log schemes $X \cong Y$.

The Kato-Nakayama space

From now on consider schemes locally of finite type over \mathbb{C} . Let *X* be a log scheme.

There is an "underlying topological space" X_{log} with a surjective map $\tau : X_{log} \rightarrow X$.

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The (reduced) fiber of $\sqrt[\infty]{X} \to X$ over x is $B\widehat{\mathbb{Z}}^k$, where k is the same number.

Note that $S^1 = B\mathbb{Z}$, and so $\widehat{(S^1)^k} \cong B\widehat{\mathbb{Z}}^k$.

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Theorem (Carchedi, Scherotzke, Sibilla, -) There is a canonical map of topological stacks

$$\Phi_X \colon X_{log} \to \sqrt[\infty]{X_{top}}$$

that induces an equivalence upon profinite completion.

The description of the map is easier if one interprets X_{log} itself as parametrizing "roots" of a certain kind.

X_{log} as a root stack

As the stack $\sqrt[n]{X}$ parametrizes

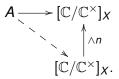


X_{log} as a root stack

As the stack $\sqrt[n]{X}$ parametrizes



it turns out $\sqrt[n]{X_{top}}$ parametrizes

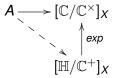


(note $\operatorname{Div}_X \sim [\mathbb{A}^1/\mathbb{G}_m]_X$).

A way to map to something that dominates every morphism $\land n: [\mathbb{C}/\mathbb{C}^{\times}]_X \to [\mathbb{C}/\mathbb{C}^{\times}]_X$ is to (in some sense) extract a logarithm.

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Consider the stack $X_{\mathbb{H}}$ that parametrizes

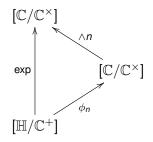


where $\mathbb{H}=\mathbb{R}_{\geq 0}\times\mathbb{R}$ and exp is induced by $\mathbb{H}\to\mathbb{C}$ given by

$$(x,y)\mapsto x\cdot e^{iy}$$

and by the exponential $\mathbb{C}^+ \to \mathbb{C}^{\times}.$

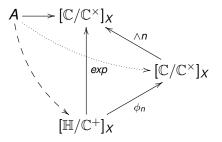
For every *n* we have a factorization



where $\phi_n \colon [\mathbb{H}/\mathbb{C}^+] \to [\mathbb{C}/\mathbb{C}^\times]$ is given by $\mathbb{H} \to \mathbb{C}$

$$(x,y)\mapsto (\sqrt[n]{x},y/n)\mapsto \sqrt[n]{x}\cdot e^{irac{y}{n}}$$
and by $\mathbb{C}^+ o \mathbb{C}^ imes$ given by $z\mapsto e^{rac{z}{n}}$.

Now the diagram



gives a natural transformation $X_{\mathbb{H}} \to \sqrt[n]{X_{top}}$.

These are compatible and give $X_{\mathbb{H}} \to \sqrt[\infty]{X_{top}} = \varprojlim_n \sqrt[n]{X_{top}}$.

Theorem ((in progress) -, Vistoli) The topological space X_{log} represents the stack $X_{\mathbb{H}}$.

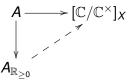
The morphism $X_{log} \rightarrow \sqrt[\infty]{X_{top}}$ that we obtain is the one mentioned before.

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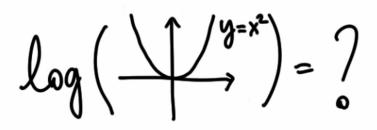
This is related to

real roots of the log structure, i.e. diagrams



parabolic sheaves with real weights.

Thank you for your attention!



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