The motivic class of BG_2 and BSpin_n

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Abstract

In the late 2000s, Ekedahl introduced a variation of the standard Grothendieck ring of algebraic varieties which contains the classes of all (reasonable) algebraic stacks. This ring $K_0(\mathrm{Stk}/k)$ turns out to be a localization of the usual ring $K_0(\mathrm{Var}/k)$. There are many important open questions on this ring, one being whether a for a connected algebraic group G the class $\{\mathrm{BG}\}$ of its classifying stack satisfies the"expected class formul" $\{\mathrm{BG}\} = \{\mathrm{G}\}^{-1}$.

Counterexamples are believed to exist, but none has been found yet. We investigate the question for the algebraic groups G_2 and Spin_n, showing that G_2 and Spin_n for n lower than nine satisfy the formula. We reduce the question for general n to a question on the class of $\mathrm{B}\Delta_n$ for a particular finite group $\Delta_n \subset \mathrm{Spin}_n$ which is tied to many unexpected behaviors of the groups Spin_n for $n \geq 15$, such as the exponential growth of their essential dimension.

The Grothendieck ring of varieties

notation: k is a field with $\mathrm{char}(k) \neq 2$, containing $\sqrt{-1}$. All schemes and algebraic stacks will be over k. Given an algebraic group G acting on a scheme X we denote by $[X/\mathrm{G}]$ the corresponding quotient stack.

Grothendieck: ring $K_0(\mathrm{Var}/k)$ of isomorphism classes $\{X\}$ of algebraic varieties over k. Two relations:
1. Given an open subset U of X whose complement is V, we have $\{X\} = \{U\} + \{V\}$ (*scissor relation*).
2. Given varieties X, Y we have $\{X \times Y\} = \{X\}\{Y\}$ (*product relation*).

$K_0(\mathrm{Var}/k)$ is a unital ring, with $\{\mathrm{Spec}(k)\} = 1$. Using the two relations one can see that
- If $E \to X$ is a vector (or affine) bundle of rank d, we have $\{E\} = \{\mathbb{A}^d\}\{X\}$.
- If $T \to X$ is a G-torsor for a *special*[1] algebraic group G, we have $\{T\} = \{G\}\{X\}$.

The second result *does not* hold for a general algebraic group G, and in fact Ekedahl proves:

Theorem (Ekedahl)**.** *Let G be a connected and reductive non-special affine algebraic group. Then there is a G-torsor $T \to X$ such that* $\{T\} \neq \{G\}\{X\}$.

An important element of $K_0(\mathrm{Var}/k)$ is the *Lefschetz motive* $\mathbb{L}$, which is the class of $\mathbb{A}^1$. Two interesting results involving it:

Theorem (Larsen, Lunts)**.** *The quotient $K_0(\mathrm{Var}/k)/\{\mathbb{L}\}$ is in $1:1$ correspondence with the set of projective varieties up to stable birationality.*

Theorem (Borisov)**.** *The class $\mathbb{L}$ is a zero divisor in $K_0(\mathrm{Var}/k)$.*

The Grothendieck ring of algebraic stacks

Ekedahl, 2009: ring $K_0(\mathrm{Stk}/k)$ of isomorphism classes of algebraic stacks over k of finite type and with affine stabilizers. Had already appeared in various forms in works of Behrend, Dhillon and Toën.

Idea: formally add to $K_0(\mathrm{Var}/k)$ all isomorphism classes of algebraic stacks, and impose the usual scissor and product relations.

Problems: The classifying stack BG of G-torsors is topologically a point, so the scissor relation is useless. Vector bundles do not in general have the same class as the trivial one, as they may not be Zariski-locally trivial[2].

Solution: force the vector bundle formula as an additional relation, so that our new relations are
1. Given an open subset $\mathscr{U}$ of $\mathscr{X}$ whose complement is $\mathscr{V}$, we have $\{\mathscr{X}\} = \{\mathscr{U}\} + \{\mathscr{V}\}$ (*scissor relation*).
2. Given algebraic stacks $\mathscr{X}, \mathscr{Y}$ we have $\{\mathscr{X} \times \mathscr{Y}\} = \{\mathscr{X}\}\{\mathscr{Y}\}$ (*product relation*).
3. Given a vector bundle $\mathscr{E} \to \mathscr{X}$, we have $\{\mathscr{E}\} = \{\mathbb{A}^d\}\{\mathscr{X}\} = \mathbb{L}^d\{\mathscr{X}\}$ (*vector bundle relation*).

With these relations it turns out that $K_0(\mathrm{Stk}/k)$ is a localization of $K_0(\mathrm{Var}/k)$:

Theorem (Ekedahl)**.** *Let $\Phi_{\mathbb{L}}$ be the submonoid of $\mathbb{Z}[\mathbb{L}]$ generated by $\mathbb{L}$ and all cyclotomic polynomials in $\mathbb{L}$. Then*

$$K_0(\mathrm{Stk}/k) = \Phi_{\mathbb{L}}^{-1} K_0(\mathrm{Var}/k).$$

We will try to explain why the theorem works and give an idea how to do computations on $K_0(\mathrm{Stk}/k)$. Our main source of vector bundles to apply the relations comes from the following fact:

Lemma. *Let V be a representation of an algebraic group G. Then the projection $[V/\mathrm{G}] \to \mathrm{BG}$ is a vector bundle.*

Sample computations: $\mathbb{G}_m$ multiplicative group acting on $\mathbb{A}^1$. By the lemma $\left[\mathbb{A}^1/\mathbb{G}_m\right] \to \mathrm{B}\mathbb{G}_m$ is a vector bundle of rank one. Then

$$\left\{\left[\mathbb{A}^1/\mathbb{G}_m\right]\right\} = \mathbb{L}\{\mathrm{B}\mathbb{G}_m\} \text{ (vector bundle relation)}$$

$$\left\{\left[\mathbb{A}^1/\mathbb{G}_m\right]\right\} = \{[\mathbb{G}_m/\mathbb{G}_m]\} + \{[0/\mathbb{G}_m]\} = 1 + \{\mathrm{B}\mathbb{G}_m\} \text{ (scissor relation).}$$

Putting the two equations together we get

$$\mathbb{L}\{\mathrm{B}\mathbb{G}_m\} = 1 + \{\mathrm{B}\mathbb{G}_m\} \Rightarrow (\mathbb{L}-1)\{\mathrm{B}\mathbb{G}_m\} = 1 \Rightarrow \{\mathrm{B}\mathbb{G}_m\} = (\mathbb{L}-1)^{-1} = \{\mathbb{G}_m\}^{-1}.$$

So the class of $\mathrm{B}\mathbb{G}_m$ is equal to the inverse of the class of $\mathbb{G}_m$.

$\mu_q \subset \mathbb{G}_m$ group of q-th roots of unit. Again we have it act on $\mathbb{A}^1$ through the action of $\mathbb{G}_m$. Then

$$\left\{\left[\mathbb{A}^1/\mu_q\right]\right\} = \mathbb{L}\{\mathrm{B}\mu_q\} \text{ (vector bundle relation)}$$

$$\left\{\left[\mathbb{A}^1/\mu_q\right]\right\} = \{[\mathbb{G}_m/\mu_q]\} + \{[0/\mathbb{G}_m]\} = \{\mathbb{G}_m\} + \{\mathrm{B}\mu_q\} = \mathbb{L} - 1 + \{\mathrm{B}\mu_q\} \text{ (scissor relation).}$$

as the quotient $\mathbb{G}_m/\mu_q$ is equal to $\mathbb{G}_m$ itself. Putting the two together we get

$$\mathbb{L}\{\mathrm{B}\mu_q\} = \mathbb{L} - 1 + \{\mathrm{B}\mu_q\} \Rightarrow (\mathbb{L}-1)\{\mathrm{B}\mu_q\} = \mathbb{L} - 1 \Rightarrow \{\mathrm{B}\mu_q\} = 1$$

so the class of $\mathrm{B}\mu_q$ is equal to the class of a point. With similar methods one can compute the class of any $\mathbb{GL}_n$-torsor:

Theorem (Ekedahl)**.** *Let $\mathscr{E} \to \mathscr{X}$ be a $\mathbb{GL}_n$-torsor. We have*

$$\{\mathscr{E}\} = \{\mathbb{GL}_n\}^{-1}\{\mathscr{X}\} = ((\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \ldots (\mathbb{L}^n - \mathbb{L}^{n-1}))^{-1}\{\mathscr{X}\}.$$

In particular $\{\mathrm{B}\mathbb{GL}_n\} = \{\mathbb{GL}_n\}^{-1}$.

Let G be an affine algebraic group, consider a representation $\mathrm{G} \to \mathbb{GL}_n$.The quotient $X = \mathbb{GL}_n/\mathrm{G}$ is a variety. We have an isomorphism $\mathrm{BG} = [X/\mathbb{GL}_n]$. The map $X \to [X/\mathbb{GL}_n]$ is a $\mathbb{GL}_n$ torsor, so

$$\{\mathrm{BG}\} = \{\mathbb{GL}_n\}^{-1}\{X\} = (\mathbb{L}^n - 1)^{-1} \ldots (\mathbb{L}^n - \mathbb{L}^{n-1})^{-1}\{X\}$$

that is, we can express the class of BG as an element in $K_0(\mathrm{Var}/k)$ divided by a power of $\mathbb{L}$ and some cyclotomic polynomials in $\mathbb{L}$. Extending this method the same is shown for all (reasonable) algebraic stacks.

The expected class formula

It's a theorem by Ekedahl that in general if G is special, we have $\{\mathscr{E}\} = \{G\}^{-1}\{\mathscr{X}\}$ for a G-torsor $\mathscr{E}\mathscr{X}$. For a general connected G it's not true that any torsor has the same class as a trivial torsor, so we expect this not to hold for $\mathrm{Spec}(k) \to \mathrm{BG}$, which is the"most complicated" torsor.

The *extended Euler characteristic*, which is a morphism from $K_0(\mathrm{Var}/k)$ to a certain Grothendieck ring of Galois representations $\hat{K}_0(\mathrm{Coh}_k)$, gives us an *expected class* for BG, which in particular is
- 1 if G is a finite group, and
- $\{\mathrm{G}\}^{-1}$ if G is connected.

Question: are there algebraic groups which do not satisfy the formula in the two cases?

Finite groups: positive answers are known. The unramified Brauer group is an obstruction. Note that the unramified Brauer group is also an obstruction to the much more famous Noether's problem[3], showing that the two problems might be related. The "simplest" positive answer, surprisingly, is for $\mathrm{G} = \mathbb{Z}/47\mathbb{Z}$, when the field k does not contain a 47th rooth of 1.[4]

Connected groups: no positive answer is known. The groups PGL_2 and PGL_3 are known to satisfy the formula due to Bergh, and the groups O_n and SO_n are known to satisfy the formula for all n due to Dhillon, Young, Talpo and Vistoli.

The Spin_n family and G_2

Let V be an n-dimensional vector space and let q be the split quadratic form defined by

$$\begin{cases} q(x) = x_1 x_{m+1} + x_2 x_{m+2} + \ldots + x_m x_{2m} & \text{when } n = 2m \\ q(x) = x_1 x_{m+1} + \ldots + x_m x_{2m} + x_{2m+1}^2 & \text{when } n = 2m+1. \end{cases}$$

and let $\mathrm{O}(q) := \mathrm{O}_n$ be the subgroup of $\mathbb{GL}_n$ fixing it. It has two connected components, and we denote by SO_n the connected component of the identity.

The algebraic group Spin_n maps $2:1$ to SO_n. The map is a group homomorphism. If $k = \mathbb{C}$ the map is the universal covering of SO_n. The Spin_n family exhibits unusual behavior regarding essential dimension[5]

Theorem (Brosnan, Reichstein, Vistoli)**.** *For $n \geq 15$ the essential dimension of Spin_n increases exponentially.*

Making it a natural candidate to give a negative answer to Noether's problem

Conjecture (Merkurjev)**.** The group Spin_n should provide a negative answer to Noether's problem for connected groups when $n \geq 15$.

The authors conjecture that the same should happen for the expected class formula

Conjecture (P-T)**.** The group Spin_n should provide a negative answer to the expected class formula for connected groups when $n \geq 15$.

Many of the unusual behaviors of the Spin_n family seem to depend on a particular finite subgroup of order 2^n

$$\Delta_n \subset \mathrm{Spin}_n$$

which is the inverse image of diagonal matrices in SO_n. Essential dimension of Δ_n provides an exponential lower bound to that of Spin_n and Noether's problem for Spin_n is equivalent to Noether's problem for Δ_n.

The algebraic group G_2 is the automorphism group of the complexified Octonion algebra $\mathbb{O}\otimes\mathbb{C}$. Its Lie counterpart, the automorphism groups of the Octonions, is the smallest exceptional Lie group. When $k = \mathbb{C}$ there is a fibration $\mathrm{Spin}_7 \to \mathrm{S}^6$ with fiber G_2, and in general we can see G_2 as a subgroup of Spin_7 by considering the stabilizer of a vector $v \in k^7$ with $q(v) = 1$.

Results

Our main result is reducing the problem of computing the class BSpin_n to the same problem for the finite group Δ_n:

Theorem (P-T)**.** *The formula $\{\mathrm{BSpin}_n\} = \{\mathrm{Spin}_n\}^{-1}$ holds for all $n \leq N$ if and only if the formula $\{\mathrm{B}\Delta_n\} = 1$ holds for all $n \leq N$.*

More precisely, if $\{\mathrm{B}\Delta_N\} \neq 1$ and $\{\mathrm{B}\Delta_n\} = 1$ for $n < N$ then $\{\mathrm{BSpin}_N\} \neq \{\mathrm{Spin}_N\}^{-1}$

Moreover we computed the classes for some low dimensional cases, including the algebraic group G_2, which appears in the computation for Spin_7.

Theorem (P-T)**.** *We have $\{\mathrm{BG}_2\} = \{\mathrm{G}_2\}^{-1}$ and $\{\mathrm{BSpin}_n\} = \{\mathrm{Spin}_n\}^{-1}$ for $n \leq 8$. In particular, this also shows that $\{\mathrm{B}\Delta_n\} = 1$ for $n \leq 8$.*

Forthcoming Research

Ekedahl: invariants $\mathrm{H}^i : K_0(\mathrm{Stk}/k) \to L_0(\mathrm{Ab})$, where the RHS is generated by classes of finitely generated abelian groups modulo $\{A \oplus B\} = \{A\} + \{B\}$. Applied to the class of BG for a finite group G they provide obstructions to the expected class formula. When $i = 2$ they recover the unramified Brauer group.

We plan to try and understand the invariants $\mathrm{H}^i(\{\mathrm{B}\Delta_n\})$ for $i \geq 3$ (for $i = 1, 2$ we know them to be zero). Finding a non-zero invariant would give a counterexample to the expected class formula for connected algebraic groups[6].

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[1] That is, every G-torsor is Zariski-locally trivial.

[2] In general a vector bundle over an algebraic stack will be smooth-locally trivial.

[3] Given G algebraic group, is there a faithful representation V of G such that V/G is rational? Finding a connected counterexample is a big open problem in the theory of algebraic groups.

[4] Otherwise we would have $\mathbb{Z}/47\mathbb{Z} = \mu_{47}$ and the class would be trivial by the computation above

[5] The minimum number of independent parameters needed to define a G-torsor.

[6] For technical reasons, a direct computation on the class of $\mathrm{B}\Delta_n$ has no chance to successfully discover a counterexample.