. . . the Yang–Baxter equation, and Hopf–Galois structures

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Introduction

Notation and convention

Let $(G, \cdot)$ be a group, and $g \in G$.

- If $\psi \in \mathrm{End}(G, \cdot)$, we write ${}^{\psi}g$ for the image of g under ψ.
- We denote by λ the left regular representation, and by ρ the right regular representation.
- We write $\iota \colon (G, \cdot) \to \mathrm{Aut}(G, \cdot)$ for the homomorphism that sends $g \in G$ to the conjugation-by-g automorphism.
- If $\psi \in \mathrm{End}(G, \cdot)$, we write $[g, \psi] := g \cdot {}^{\psi}g^{-1}$, and $[G, \psi] = \langle [g, \psi] : g \in G \rangle$.
- If $(G, \cdot, \circ)$ is a skew brace, we denote by g^{-1} the inverse of g with respect to $\cdot$, and by $\overline{g}$ the inverse of g with respect to $\circ$.

Gamma functions

Theorem ([Guarnieri and Vendramin, 2017])

Let $(G, \cdot)$ be a group. The following data are equivalent.

- *An operation $\circ$ such that $(G, \cdot, \circ)$ is a skew brace.*
- *A regular subgroup $N \leq \mathrm{Perm}(G)$ which normalises $\lambda(G)$.*
- *A function $\gamma \colon G \to \mathrm{Aut}(G, \cdot)$ such that, for every $g, h \in G$,*

$$\gamma(g \cdot {}^{\gamma(g)}h) = \gamma(g)\gamma(h).$$

The function γ is called *gamma function*. Explicitly,

$$\begin{aligned} {}^{\gamma(g)}h &= g^{-1} \cdot (g \circ h), \\ N &= \{\lambda(g)\gamma(g) : g \in G\}. \end{aligned}$$

Bi-gamma functions

Theorem ([Childs, 2019], [Caranti, 2020])

Let $(G,\cdot)$ be a group. The following data are equivalent.

- *An operation $\circ$ such that $(G,\cdot,\circ)$ is a bi-skew brace.*
- *A regular subgroup $N \leq \mathrm{Perm}(G)$ which normalises, and is normalised by, $\lambda(G)$.*
- *An antihomomorphism $\gamma\colon (G,\cdot) \to \mathrm{Aut}(G,\cdot)$ such that, for every $g, h \in G$,*

$$\gamma(g \cdot {}^{\gamma(g)}h) = \gamma(g)\gamma(h).$$

The function γ is called *bi-gamma function*.

Our setting: $\varepsilon = -1$

Let $(G, \cdot)$ be a group, and $\psi \in \mathrm{End}(G, \cdot)$.

Theorem ([Caranti and LS, 2021])

The following are equivalent.

- ψ *satisfies* ${}^{\psi}[[G, \psi], G] \leq Z(G, \cdot)$.
- $(G, \cdot, \circ)$ *is a bi-skew brace, for* $g \circ h = g \cdot {}^{\psi}g^{-1} \cdot h \cdot {}^{\psi}g$.
- *The function* γ *define by* $\gamma(g) = \iota({}^{\psi}g^{-1})$ *is a bi-gamma function for* $(G, \cdot)$.

If any of these holds, then $N = \{\lambda(g)\iota({}^{\psi}g^{-1}) : g \in G\}$ is a regular subgroup of $\mathrm{Perm}(G)$ which normalises, and is normalised by, $\lambda(G)$.

This result generalises [Koch, 2021], where the map ψ is abelian.

The Yang–Baxter equation

Main definitions

Definition ([Drinfel'd, 1992])

A *set-theoretic solution* of the Yang–Baxter equation is a couple (X, r), where $X \neq \emptyset$ is a set, and

$$\begin{aligned} r\colon X \times X &\to X \times X \\ (x, y) &\mapsto (\sigma_x(y), \tau_y(x)) \end{aligned}$$

is a bijective map satisfying

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

We say that (X, r) is *non-degenerate* if, for every $x \in X$, σ_x and τ_x are bijective, and *involutive* if $r^2 = \mathrm{id}_{X \times X}$. For us, a *solution* is a non-degenerate set-theoretic solution of the Yang–Baxter equation.

Theorem ([Rump, 2007], [Guarnieri and Vendramin, 2017])

Let $(G, \cdot, \circ)$ be a skew brace. Then

$$r\colon (g,h) \mapsto (g^{-1}\cdot(g\circ h), \overline{g^{-1}\cdot(g\circ h)}\circ g\circ h)$$

is a solution for G.

The solution (G, r) is involutive if and only if $(G, \cdot, \circ)$ is a brace, that is, if $(G, \cdot)$ is abelian.

Opposite skew brace, bi-skew braces and solutions

Definition ([Rump, 2019], [Koch and Truman, 2020a])

Let $(G, \cdot, \circ)$ be a skew brace. The *opposite skew brace* is $(G, \cdot', \circ)$, where, for every $g, h \in G$, $g \cdot' h = h \cdot g$.

Given a bi-skew brace $(G, \cdot, \circ)$, we find (up to) four solutions for G:

$$(G, \cdot, \circ) \rightsquigarrow (g, h) \mapsto (g^{-1} \cdot (g \circ h), \overline{g^{-1} \cdot (g \circ h)} \circ g \circ h),$$
$$(G, \cdot', \circ) \rightsquigarrow (g, h) \mapsto ((g \circ h) \cdot g^{-1}, \overline{(g \circ h) \cdot g^{-1}} \circ g \circ h),$$
$$(G, \circ, \cdot) \rightsquigarrow (g, h) \mapsto (\overline{g} \circ (g \cdot h), (\overline{g} \circ (g \cdot h))^{-1} \cdot g \cdot h),$$
$$(G, \circ', \cdot) \rightsquigarrow (g, h) \mapsto ((g \cdot h) \circ \overline{g}, ((g \cdot h) \circ \overline{g})^{-1} \cdot g \cdot h).$$

We may rewrite the solutions so that they depend only on $\cdot$ and γ.

Theorem ([Caranti and LS, 2021])

Let $(G, \cdot)$ be a group, and γ be a gamma function.
The we get (up to) two solutions:

$$(g, h) \mapsto ({}^{\gamma(g)}h, {}^{\gamma({}^{\gamma(g)}h)^{-1}}({}^{\gamma(g)}h^{-1} \cdot g \cdot {}^{\gamma(g)}h)),$$
$$(g, h) \mapsto ({}^{\iota(g)\gamma(g)}h, {}^{\gamma({}^{\iota(g)\gamma(g)}h)^{-1}}g).$$

If in addition γ is a bi-gamma function, then we get (up to) other two solutions:

$$(g, h) \mapsto ({}^{\gamma(g^{-1})}h, {}^{\gamma(g^{-1})}h^{-1} \cdot g \cdot h),$$
$$(g, h) \mapsto (g \cdot h \cdot {}^{\gamma(h)}g^{-1}, {}^{\gamma(h)}g^{-1}).$$

Yang–Baxter in our setting

Theorem ([Caranti and LS, 2021])

Let $(G, \cdot)$ be a group, and $\psi \in \mathrm{End}(G, \cdot)$. If ${}^{\psi}[[G, \psi], G] \le Z(G, \cdot)$, then we get (up to) four solutions:

$$(g, h) \mapsto ({}^{\psi}g^{-1} \cdot h \cdot {}^{\psi}g, {}^{\psi}(g^{-1} \cdot h) \cdot h^{-1} \cdot {}^{\psi}g \cdot g \cdot {}^{\psi}g^{-1} \cdot h \cdot {}^{\psi}(h^{-1} \cdot g)),$$
$$(g, h) \mapsto (g \cdot {}^{\psi}g^{-1} \cdot h \cdot {}^{\psi}g \cdot g^{-1}, {}^{\psi}h \cdot g \cdot {}^{\psi}h^{-1}),$$
$$(g, h) \mapsto ({}^{\psi}g \cdot h \cdot {}^{\psi}g^{-1}, {}^{\psi}g \cdot h^{-1} \cdot {}^{\psi}g^{-1} \cdot g \cdot h),$$
$$(g, h) \mapsto (g \cdot h \cdot {}^{\psi}h^{-1} \cdot g^{-1} \cdot {}^{\psi}h, {}^{\psi}h^{-1} \cdot g \cdot {}^{\psi}h).$$

These coincide with the solutions found in [Koch, 2021], where ψ is abelian.

Hopf–Galois structures

Main definition and results

Fix a finite Galois extension L/K with Galois group $(G, \cdot)$.

Definition

A Hopf–Galois structure on L/K consists of a cocommutative K-Hopf algebra H, together with an action of H on L satisfying certain technical properties.

Theorem ([Greither and Pareigis, 1987])

The Hopf–Galois structures on L/K are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(G)$ *normalised by* $\lambda(G)$.

The K-Hopf algebra $L[N]^G$ corresponds to the subgroup N.

Moreover, the K-sub-Hopf algebras of $L[N]^G$ are in bijective correspondence with the subgroups of N normalised by $\lambda(G)$.

We would like to use gamma functions to analyse Hopf–Galois structures. Notice that a gamma function for $(G,\cdot)$ yields a regular subgroup which normalises $\lambda(G)$, while we need a regular subgroup normalised by $\lambda(G)$.

Fact

If γ is a bi-gamma function for $(G,\cdot)$, then

$$N = \{\lambda(g)\gamma(g) : g \in G\}$$

is a regular subgroup of $\mathrm{Perm}(G)$ *which normalises, and is normalised by, $\lambda(G)$. In particular, $L[N]^G$ gives a Hopf–Galois structure on L/K.*

Hopf–Galois structures in our setting

Let L/K be a finite Galois extension with Galois group $(G, \cdot)$, and $\psi \in \mathrm{End}(G, \cdot)$ such that ${}^{\psi}[[G, \psi], G] \leq Z(G, \cdot)$.
Then γ, defined by $\gamma(g) = \iota({}^{\psi}g^{-1})$, is a bi-gamma function, and so $L[N]^G$ gives a Hopf–Galois structure on L/K, where

$$N = \{\lambda(g)\iota({}^{\psi}g^{-1}) : g \in G\}.$$

Question

Can we determine the type of N?

Five subgroups of N

As in [Koch, 2021], we can always find (up to) five subgroups of N normalised by $\lambda(G)$, and these correspond to five K-sub-Hopf algebras of $L[N]^G$.

For example, the λ-points and ρ-points, introduced in [Koch and Truman, 2020b]:

$$\begin{aligned}\Lambda_N &= N \cap \lambda(G) = \{\lambda(g) : g \in \ker(\gamma)\} \\ &= \{\lambda(g) : g \text{ satisfies } {}^{\psi}g \in Z(G,\cdot)\}, \\ P_N &= N \cap \rho(G) = \{\rho(g) : g \text{ satisfies } \gamma(g) = \iota(g^{-1})\} \\ &= \{\rho(g) : g \text{ satisfies } g \cdot {}^{\psi}g^{-1} \in Z(G,\cdot)\}.\end{aligned}$$

Some of the five subgroups may coincide, but we can find examples in which they are all distinct.

When we can be explicit

- If ψ is a fixed point free abelian endomorphism, then $N \cong (G, \cdot)$ ([Childs, 2013], [Koch, 2021]).
- If ψ is different from zero and idempotent, then for every $n \geq 1$, $\psi^n = \psi$, and ${}^{\psi}G = \{g \in G : {}^{\psi}g = g\}$. We can use a version of the Fitting's Lemma for groups ([Caranti, 1985]) to deduce that $N \cong (\ker(\psi), \cdot) \times ({}^{\psi}G, \cdot)$.

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