An introduction to the theory of Mean Field Games

Valeria De Mattei

Università di Pisa

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The philosophy

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$\to$ Analogy with mean field models

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$\to$ Analogy with mean field models

Mean field games

Study of strategic decision making in very large populations of small interacting individuals with symmetric payoffs.

Outline

1. A stochastic optimal control problem
   - mean field game (MFG)
   - the concept of ϵ-Nash Equilibrium
2. From game to PDEs: the mean field equations (MFE)
   - main hypothesis
   - existence theorem
   - uniqueness theorem
3. The link between MFG and MFE
   - an abstract control problem
   - asymptotic resolution of MFG

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Mean field game (MFG)

We have N players. For $i = 1, \cdots, N$, the player i has a dynamic described by the following SDE:

$$dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i.$$

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We suppose:
H1. X_0^i has a fixed law m_0 and are independent;
H2. (B_t^i) are independent d-dimensional Brownian motions.

The player i can choose his control α^i adapted to the filtration

$$(\mathcal{F}_t = \sigma(X_0^j, B_s^j : s \leq t, j = 1, \cdots, N))$$

Mean field game (MFG)

Player i's payoff is given by

$$J_i^N(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t^i|^2 + F\left(X_t^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{X_t^j}\right) dt\right].$$

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The problem

Minimize J_j^N conditioned to

$$dX_t^j = \alpha_t^j dt + \sqrt{2} dB_t^j$$

for all j.

The notion of ϵ - Nash equilibrium

We say that $(\alpha^{*,1}, \cdots, \alpha^{*,N})$ is a **Nash equilibrium** for $(J_i^N)_{i=1}^N$ if for all i and for all α

$$J_i^N(\alpha^{*,1}, \cdots, \alpha^{*,N}) \leq J_i^N((\alpha^{*,j})_{j \neq i}, \alpha)$$

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   - uniqueness theorem
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The Mean Field Equations (MFE)

In some sense, the MFG evolves to:

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = F(x,m) \\ \partial_t m - \Delta m - \mathrm{div}(m\nabla u) = 0 \\ u(x,T) = 0 \\ m(0) = m_0, \end{cases}$$

1. the first is an Hamilton Jacobi Bellman
2. the second is a Fokker Planck
3. they are coupled by F (the coupling term)
4. the system is forward - backward

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$$|F(x, m)| \leq C_0,$$
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$$\int_{\mathbb{R}^d} |x|^2 m_0(dx) < +\infty.$$

3. for all $m, m' \in \mathcal{P}^1(\mathbb{R}^d), m \neq m'$,
$$\int_{\mathbb{R}^d} (F(x,m) - F(x,m'))d(m-m')(x) > 0.$$

The Mean Field Equations (MFE): existence theorem

We say that a pair (u, m) is a *classical solutions* to MFE if

- $u, m : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ are continuous
- u, m are $\mathcal{C}^2$ in space and $\mathcal{C}^1$ in time

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Existence theorem

Under the above assumptions, there is at least one classical solution to MFE.

The Mean Field Equations (MFE): existence theorem

We denote with $C^{s+\alpha}$ ($s \in \mathbb{N}$, $\alpha \in (0,1]$)the maps $z : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ such that

- the derivatives $\partial_t^k D_x^l z$ exist if $2k + l \leq s$
- the derivatives are bounded and α- Holder in space and $\alpha/2$-Holder in time

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Existence and uniqueness result for the heat equation

If $a, b, f, w_0 \in \mathcal{C}^{\alpha}$, the there exists a unique weak solution to

$$\begin{cases} \partial_t w - \Delta w + \langle a(x,t), \nabla w \rangle + b(x,t) w = f(x,t) \\ w(x,0) = w_0(x). \end{cases}$$

Moreover $w \in \mathcal{C}^{2+\alpha}$.

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3. Apply a fixed point to $\Psi:\mu\mapsto m$.

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$$\sup_{s \neq t} \frac{d_1(\mu(s), \mu(t))}{|t-s|^{1/2}} \leq C$$

and

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Properties of $\mathcal{C}$:

- $\mathcal{C}$ is convex;
- $\mathcal{C}$ is compact in the topology of

$$d(\mu, \nu) = \sup_{t \in [0,T]} d_1(\mu(t), \nu(t)).$$

Step 2. The map Ψ

Associate to some $\mu \in \mathcal{C}$ the solution u of

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To see that a solution exists and is unique, we use Cole Hopf transform:

$$w = e^{-u/2};$$

then w has to satisfy

$$\begin{cases} -\partial_t w - \Delta w = wF(x,\mu) \\ w(x,T) = 1 \end{cases}$$

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Moreover $w \in \mathcal{C}^{2+\alpha}$.

The Mean Field Equations (MFE): existence theorem

In our situations:

- $a = 0$;
- $f = 0$;
- $b = F$;
- $w_0 = 1$;

It is sufficient to control that $(x,t) \mapsto F(x,\mu(t)) \in \mathcal{C}^\alpha$:

$$\begin{aligned} |F(x,\mu(t)) - F(x',\mu(t'))| &\leq C(|x-x'| + d_1(\mu(t),\mu(t'))) \\ &\leq C(|x-x'| + |t-t'|^{1/2}) \end{aligned}$$

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$\rightarrow$ **Existence and uniqueness of w (and u)**, $w \in C^{2+\alpha}$ (and $u \in C^{2+\alpha}$.)

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Properties of u:

- u is bounded;
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which can be written as

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Properties of u:

- u is bounded;
- u is Lipschitzian;
- $u \in C^{2+\alpha} \Rightarrow \nabla u, \Delta u \in C^{\alpha}$

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- u is bounded;
- u is Lipschitzian;
- $u \in C^{2+\alpha} \Rightarrow \nabla u, \Delta u \in C^{\alpha} \Rightarrow$ there exists $m \in C^{2+\alpha}, m \in \mathcal{C}$

Associate to u the solution m of

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Step 3. The properties of Ψ and fixed point theorem

Properties of Ψ:

- well defined ($m \in \mathcal{C}$)
- continuous

Schauder fixed point Theorem

Let X be a locally convex topological vector space. Let $K \subset X$ be a non-empty, convex and compact set. For any continuous function

$$f : K \to K,$$

there exists $x \in K$ such that $f(x) = x$.

The Mean Field Equations (MFE): uniqueness theorem

As to uniqueness we suppose: for all $m, m' \in \mathcal{P}^1(\mathbb{R}^d), m \neq m'$,

$$\int_{\mathbb{R}^d} (F(x,m) - F(x,m'))d(m-m')(x) > 0$$

Uniqueness Theorem

Under the above assumption, there exists a unique classical solution to MFE.

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The link between MFG and MFE: abstract control problem

Consider the following abstract control problem:

Abstract problem

We have

- a functional: $J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t|^2 + F(X_t, m_t)dt\right]$,
- a state: $dX_t = \alpha_t dt + \sqrt{2} dB_t$.

Find

$$\inf_{\alpha} J(\alpha)$$

The link between MFG and MFE: abstract control problem

Resolution of abstract control problem

- fix (u, m) solution to the MFE;

The link between MFG and MFE: abstract control problem

Resolution of abstract control problem

- fix (u, m) solution to the MFE;
- let $\bar{X}$ solves

$$d\bar{X}_t = -\nabla u(\bar{X}_t, t)dt + \sqrt{2}dB_t;$$

- put

$$\bar{\alpha}_t = -\nabla u(\bar{X}_t, t);$$

The link between MFG and MFE: abstract control problem

Resolution of abstract control problem

- fix (u, m) solution to the MFE;
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$$d\bar{X}_t = -\nabla u(\bar{X}_t, t)dt + \sqrt{2}dB_t;$$

- put

$$\bar{\alpha}_t = -\nabla u(\bar{X}_t, t);$$

Then

$$J(\bar{\alpha}) = \inf_{\alpha} J(\alpha).$$

The link between MFG and MFE: abstract control problem

$$
\begin{aligned}
0 &= \mathbb{E}[u(X_T, T)] \\
&= \mathbb{E}\left[u(X_0, 0) + \int_0^T \partial_t u(X_s, s) + \langle \alpha_s, \nabla u(X_s, s)\rangle + \Delta u(X_s, s) ds\right] \\
&= \mathbb{E}\left[u(X_0, 0) + \int_0^T \frac{1}{2}|\nabla u(X_s, s)|^2 + \langle \alpha_s, \nabla u(X_s, s)\rangle - F(X_s, m_s) ds\right] \\
&\geq \mathbb{E}\left[u(X_0, 0) + \int_0^T -\frac{1}{2}|\alpha_s|^2 - F(X_s, m_s) ds\right] \\
&= \mathbb{E}\left[u(X_0, 0)\right] - J(\alpha).
\end{aligned}
$$

The link between MFG and MFE: abstract control problem

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&= \mathbb{E}\left[u(X_0, 0) + \int_0^T \frac{1}{2}|\nabla u(X_s, s)|^2 + \langle \alpha_s, \nabla u(X_s, s)\rangle - F(X_s, m_s) ds\right] \\
&\geq \mathbb{E}\left[u(X_0, 0) + \int_0^T -\frac{1}{2}|\alpha_s|^2 - F(X_s, m_s) ds\right] \\
&= \mathbb{E}\left[u(X_0, 0)\right] - J(\alpha).
\end{aligned}
$$

Then

$$
J(\alpha) \geq \mathbb{E}\left[u(X_0, 0)\right].
$$

The link between MFG and MFE: asymptotic resolution of MFG

We have the following situation:

- a payoff for each player:

$$J_i^N(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t^i|^2 + F\left(X_t^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{X_t^j}\right) dt\right]$$

- a state for each player:

$$dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i$$

The link between MFG and MFE: asymptotic resolution of MFG

Main theorem

Fix (u, m) solution to MFE. For all i, put

$$d\bar{X}_t^i = -\nabla u(\bar{X}_t^i, t)dt + \sqrt{2}dB_t^i$$

$$\bar{\alpha}_t^i = -\nabla u(\bar{X}_t^i, t)$$

The link between MFG and MFE: asymptotic resolution of MFG

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Then $(\bar{\alpha}^1, \cdots, \bar{\alpha}^N)$ is a ϵ_N- Nash equilibrium for $(J_1^N, \cdots, J_N^N)$ with $\epsilon_N \to 0$ as $N \to \infty$.

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Proof. We have to evaluate:

$$J_i^N(\bar{\alpha}^1, \cdots, \bar{\alpha}^N) - J_i^N((\bar{\alpha}^j)_{j\neq i}, \alpha)$$

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which is dominated by

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It is sufficient to show that

$$J_i^N(\bar{\alpha}^1, \cdots, \bar{\alpha}^N) - J(\bar{\alpha}^i) \to 0$$

and

$$J(\alpha) - J_i^N((\bar{\alpha}^j)_{j\neq i}, \alpha) \to 0.$$

The link between MFG and MFE: asymptotic resolution of MFG

Consider the first:

$$J_i^N(\bar{\alpha}^1, \cdots, \bar{\alpha}^N) - J(\bar{\alpha}^i) \leq \mathbb{E}\left[\int_0^T d_1\left(m(t), \frac{1}{N-1}\sum_{j\neq i}\delta_{\bar{X}_t^j}\right) dt\right],$$

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which goes to zero, since $\bar{X}^j$ are independent and identically distributed with law m :

$$d\bar{X}_t^i = -\nabla u(\bar{X}_t^i, t)dt + \sqrt{2}dB_t^j$$

$$\partial_t m - \Delta m - div(m\nabla u) = 0$$

The link between MFG and MFE: asymptotic resolution of MFG

So we get

$$\epsilon_N^1 = J_i^N(\bar{\alpha}^1, \cdots, \bar{\alpha}^N) - J(\bar{\alpha}^i) \to 0$$

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then

$$J_i^N(\bar{\alpha}^1, \cdots, \bar{\alpha}^N) - J_i^N((\bar{\alpha}^j)_{j \neq i}, \alpha) \leq \epsilon_N$$

with

$$\epsilon_N = \epsilon_N^1 + \epsilon_N^2 \to 0.$$

The link between MFG and MFE: asymptotic resolution of MFG

The MFG evolves to:

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = F(x,m) \\ \partial_t m - \Delta m - \mathrm{div}(m\nabla u) = 0 \\ u(x,T) = 0 \\ m(0) = m_0, \end{cases}$$

- optimality is given by the notion of $\epsilon-$ Nash equilibrium;
- the optimal control is $-\nabla u$;
- the law of the optimal state is m.

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