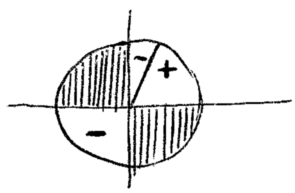
Soluzioni

1. 2π-periodica: si studia in $[0, 2\pi]$.

$f'(x) = 2\cos x + 2\,\mathrm{sgn}(\mathrm{sen}\,2x)\cdot\cos 2x$

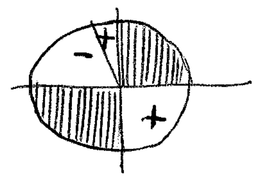
(i) $\mathrm{sen}\,x\cos x > 0$

$f'(x) > 0$ se $2\cos^2 x + \cos x - 1 > 0$
cioè se $\cos x > 1/2$.



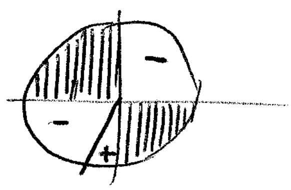
(ii) $\mathrm{sen}\,x\cos x < 0$

$f'(x) > 0$ se $2\cos^2 x - \cos x - 1 < 0$
cioè se $\cos x > -1/2$.



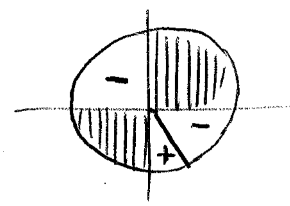
$f''(x) = -2\,\mathrm{sen}\,x - 4\,\mathrm{sgn}(\mathrm{sen}\,2x)\,\mathrm{sen}\,2x$

(i) $\mathrm{sen}\,x\cos x > 0$

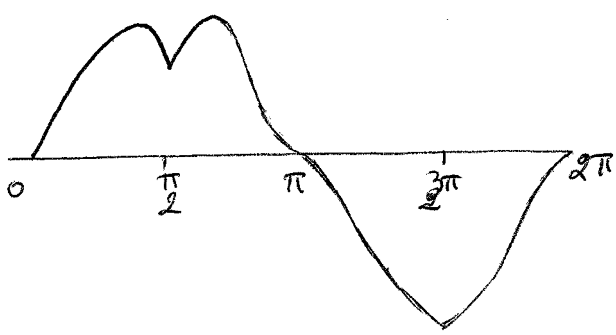


$-2\,\mathrm{sen}\,x(1+4\cos x) > 0$

(ii) $\mathrm{sen}\,x\cos x < 0$



$2\,\mathrm{sen}\,x(4\cos x - 1) > 0$



2. Per $x \to 0$ $f(x) \sim 1/\sqrt{x}$: infinito di ordine $1/2$

$(x - \frac{1}{2})^2 = x^2 - x + \frac{1}{4}$

$x - x^2 = \frac{1}{4} - (x - \frac{1}{2})^2$.

Posto $x - \frac{1}{2} = \frac{1}{2}\,\mathrm{sen}\,t$, $dx = \frac{1}{2}\cos t\,dt$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos t}{1 + \mathrm{sen}\,t + \cos t}\,dt.$$

Con la consueta posizione $\mathrm{tg}\,\frac{t}{2} = z$:

$$\int_{-1}^{1} \frac{1-z}{1+z^2}\,dz = \left[\mathrm{arctg}\,z - \frac{1}{2}\log(1+z^2)\right]_{-1}^{1} = \frac{\pi}{2}.$$

In alternativa:

$$\sqrt{x-x^2} = x\sqrt{\frac{1-x}{x}}$$

$$\sqrt{\frac{1-x}{x}} = t, \quad x = \frac{1}{1+t^2}, \quad dx = -\frac{2t}{(1+t^2)^2}\,dt.$$

$$y = \int_0^\infty \frac{2t}{(1+t)(1+t^2)}\,dt \underset{\text{Hermite}}{=} \int_0^\infty \left(-\frac{1}{1+t} + \frac{t}{1+t^2} + \frac{1}{1+t^2}\right) dt =$$

$$= \left[\operatorname{arctg} t + \lg\frac{\sqrt{t^2+1}}{|t+1|}\right]_0^\infty = \frac{\pi}{2}.$$

3. $\lg a_n = n^2 \lg\cos\frac{2}{n} \sim n^2\left(\cos\frac{2}{n} - 1\right) \sim n^2\left(-\frac{2}{n^2}\right) \to -2.$

$a_n \to e^{-2}.$

$$a_n - e^{-2} = e^{n^2 \lg\cos\frac{2}{n}} - e^{-2} \cong e^{n^2\log\left(1-\frac{2}{n^2}+\frac{2}{3n^4}\right)} - e^{-2} =$$

$$\cong e^{n^2\left(-\frac{2}{n^2}+\frac{2}{3n^4}-\frac{2}{n^4}\right)} - e^{-2} =$$

$$= e^{-2}\left(e^{\frac{2}{3n^2}} - 1\right) \sim \frac{2e^{-2}}{3n^2}$$ Serie convergente.

4. $\forall \varepsilon > 0\ \exists \bar{n}: \quad \forall n > \bar{n} \quad -\varepsilon < \operatorname{arcsen}\frac{n}{n+1} < \varepsilon.$ basta studiare la seconda condizione.

Poiché $n/(n^2+1) \in (0,1)$,

$\operatorname{arcsen}\frac{n}{n^2+1} < \varepsilon \to \frac{n}{n^2+1} < \operatorname{sen}\varepsilon \to (\operatorname{sen}\varepsilon)n^2 - n + (\operatorname{sen}\varepsilon) > 0.$

Scegliamo $\varepsilon \in (0, \pi$[redacted]$)$ in modo che sia $\operatorname{sen}\varepsilon > 0$. Scegliendo ulteriormente $\varepsilon \in (0, \pi/6)$, si ottiene $n > \frac{1+\sqrt{1-4\operatorname{sen}^2\varepsilon}}{2}$,

La successione è decrescente; dunque $\max = \sup = x_1 = \frac{\pi}{3}$

$\inf = 0$, min non $\exists$.

5. $a(x) = 1, \quad A(x) = x \to (e^x y)' = 4x^3 \to e^x y = x^4 + c$

$\to y = (x^4 + c)e^{-x}.$

$\cancel{(e^x y' + e^x y)}$

La C.I. è soddisfatta per $c=1$.