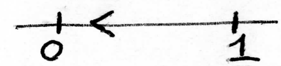
Calcolare (se esistono) massimo, minimo, estremo superiore ed inferiore per gli insiemi, le successioni e le funzioni che seguono.

- $A = \left\{ x \in \mathbb{R} : \operatorname{sen} \frac{1}{x} = 0 \right\}$

$$\frac{1}{x} = k\pi \quad (k \in \mathbb{Z} - 0) \longrightarrow x = \frac{1}{k\pi} \quad (k \in \mathbb{Z} - 0)$$

$k = n : \quad x = \frac{1}{n\pi}$

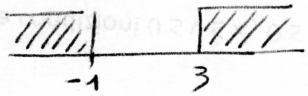


$k = -n : \quad x = -\frac{1}{n\pi}$

$M = L = \frac{1}{\pi}, \quad m = l = -\frac{1}{\pi}$

$$x_n = |n^2 - 2n - 3|$$

$x^2 - 2x - 3 \geq 0$



$x_1 = 4$

$x_2 = 3$

per $n \geq 3$ $x_n = n^2 - 2n - 3$

- è crescente $x_{n+1} > x_n \iff \cancel{n^2} + 2n + 1 - \cancel{2n} - 2 - \cancel{3} \geq \cancel{n^2} - \cancel{2n} - \cancel{3}$

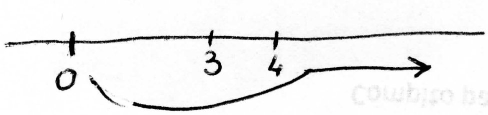
$\iff n \geq 1/2$ sempre vera

- $x_3 = 0$
- x_n diventa arbitrariamente grande

Fissato $M > 0$ arbitrario, proviamo che $n^2 - 2n - 3 > M$ per almeno un $n \geq 3$ (in realtà, come vedremo, è vero definitivamente, cioè per tutti gli n maggiori di un certo valore)

$n^2 - 2n - 3 - M > 0 \iff n < 1 - \sqrt{4+M} \quad \vee \quad n > 1 + \sqrt{4+M}$

$n = 1 \pm \sqrt{4+M}$ non ci interessa OK.



min = inf = 0
sup = $+\infty$
max non $\exists$

$$1,\overline{20} = \sup \{ 1{,}20 \quad 1{,}2020 \quad 1{,}202020 \ \dots \}$$

$$1 + 20\left(10^{-2} + 10^{-4} + \dots + 10^{-2n}\right)$$ al passo n-esimo

$$1 + 20 \cdot 10^{-2}\left(1 + 10^{-2} + \dots + 10^{-2(n-1)}\right)$$

$$1 + \frac{20}{100} \sum_{k=0}^{n-1} \left(\frac{1}{100}\right)^k =$$

NB $\displaystyle\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ se $x \neq 1$

$$1 + \frac{20}{100} \frac{1-\left(\frac{1}{100}\right)^n}{1-\frac{1}{100}} =$$

$$1 + \frac{20}{100} \frac{1-\frac{1}{100^n}}{\frac{99}{100}} = 1 + \frac{20}{100}\frac{100}{99} - \frac{20}{100}\frac{100}{99}\frac{1}{100^n}$$

$$\frac{119}{99} - \boxed{\frac{20}{99}\frac{1}{100^n}} \qquad \longrightarrow \frac{119}{99}$$

diventa arbitrariamente piccola al crescere di n

il sup. è $\frac{119}{99}$

- $x_n < \frac{119}{99} \quad \forall n$
- $x_n > \frac{119}{99} - \varepsilon$ per almeno un n (in realtà, per tutti gli n da un certo punto in poi)

$$\frac{119}{99} - \frac{20}{99}\frac{1}{100^n} > \frac{119}{99} - \varepsilon$$

$$\frac{20}{99}\frac{1}{100^n} < \varepsilon$$

$$100^n > \frac{20}{99\varepsilon}$$ vero per $n > \log_{100} \frac{20}{99\varepsilon}$

cioè definitivamente

- $x_n = \lg(1+e^{-n})$

  - è decrescente
  - $\max = \sup = x_1 = \lg(1+1/e)$
  - al crescere di n si avvicina a 0, che è inf (ma non min)

    - $\lg(1+e^{-n}) > 0 \quad \forall n$
    - $\lg(1+e^{-n}) < \varepsilon$ (deve esserlo per almeno un n)

$$\Updownarrow$$

$$1+e^{-n} < e^{\varepsilon} \leftrightarrow e^{-n} < \underbrace{e^{\varepsilon}-1}_{\oplus} \leftrightarrow -n < \lg(e^{\varepsilon}-1)$$

$$\leftrightarrow n > -\lg(e^{\varepsilon}-1) \text{, cioè definitivamente}$$

- $x_n = \dfrac{n^2+4}{2n^2+3}$

$$x_n = \frac{1}{2}\,\frac{2n^2+8}{2n^3+3} = \frac{1}{2}\,\frac{2n^2+3+5}{2n^2+3} = \frac{1}{2}\left(1+\frac{5}{2n^2+3}\right)$$

  - decrescente
  - si avvicina arbitrariamente a $\frac{1}{2}$

    - $\dfrac{1}{2} + \dfrac{5}{2(2n^2+3)} > \dfrac{1}{2}$
    - $\dfrac{1}{2} + \dfrac{5}{2(2n^2+3)} < \dfrac{1}{2} + \varepsilon \leftrightarrow$

$$\frac{5}{2(2n^2+3)} < \varepsilon \leftrightarrow 2(2n^2+3) > \frac{5}{\varepsilon} \leftrightarrow \dots$$

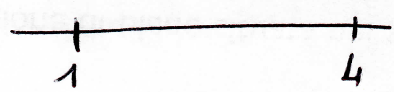
$$n^2 > \frac{5/\varepsilon - 6}{4}$$

Se $\frac{5}{\varepsilon} - 6 < 0$, è sempre vero;
altrimenti è vero definitivamente $\left(n > \dfrac{\sqrt{5/\varepsilon - 6}}{2}\right)$

- $$\begin{cases} x_1 = 4 \\ x_{n+1} = \sqrt{\dfrac{x_n^2+1}{x_n+1}} \end{cases}$$

- ben definita e positiva
- $x_{n+1} < x_n \iff \sqrt{\dfrac{x_n^2+1}{x_n+1}} < x_n \iff x_n^2+1 < x_n^2 + x_n$

  $\iff x_n > 1$



- $x_n > 1$

  Per induzione
  - per $n=1$ è vera
  - $x_n > 1 \Rightarrow \sqrt{\dfrac{x_n^2+1}{x_n+1}} > 1$?

    $\Updownarrow$

    $x_n^2 + 1 > x_n + 1$

    $\Updownarrow$

    $x_n > 1$ OK

  - 1 è l'unico pto fisso

  Conclusioni:

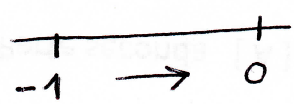
  $\max = \sup = 4$, $\inf = 1$, min non $\exists$

$A = \{ x : n x^2 + 2nx + 1 = 0 \}$

$x_n = -1 \pm \sqrt{1 - \frac{1}{n}}$

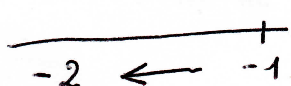
A è l'unione dei termini di queste due successioni

$-1 + \sqrt{1 - \frac{1}{n}}$



$\sup = 0$
max non $\exists$

$-1 - \sqrt{1 - \frac{1}{n}}$



$\inf = -2$
min non $\exists$

$$x_n = \frac{9 + (1 + (-1)^n)\, 2^n}{3^n}$$

n pari $\quad \dfrac{9 + 2^{n+1}}{3^n} = \dfrac{9}{3^n} + 2\left(\dfrac{2}{3}\right)^n$

decresce ; si avvicina arbitr. a 0

$0 \longleftarrow 17/9$

n dispari $\quad \dfrac{9}{3^n}$

decresce, si avvicina a 0

$0 \longleftarrow 3$

max = sup = $\frac{17}{9}$, inf = 3, min non $\exists$

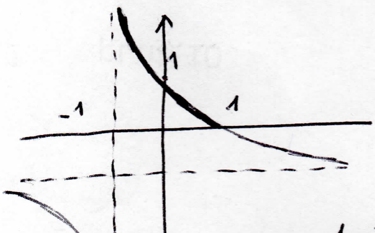
$$x_n = \sqrt{n^2-1} - n$$

$$x_n = \frac{-1}{\sqrt{n^2-1}+n}$$

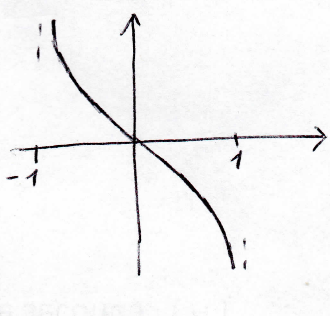
…cresce, si avvicina a 0

min = inf = -1 sup = 0 inf $\nexists$

$$f(x) = \lg \frac{1-x}{1+x}$$

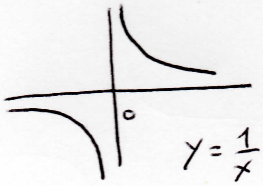


$y = \dfrac{1-x}{1+x}$

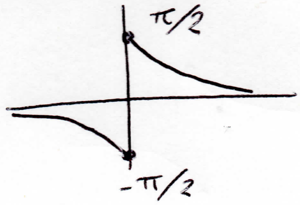


sup = $+\infty$
inf = $-\infty$
max, min $\nexists$

• $f(x) = \text{arctg}\,\frac{1}{x}$



$y = \frac{1}{x}$

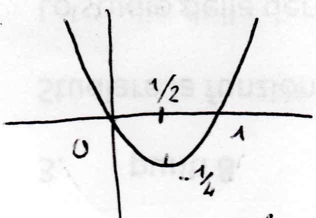


$\sup = \pi/2$

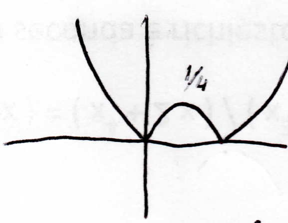
$\inf = -\pi/2$

max, min $\nexists$

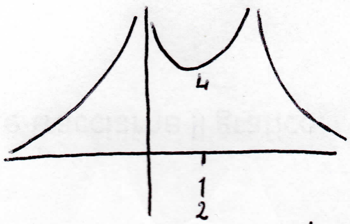
• $f(x) = \lg \frac{1}{|x^2 - x|}$



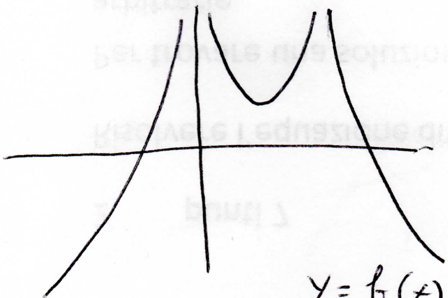
$y = x^2 - x$



$y = |x^2 - x|$



$y = \frac{1}{|x^2 - x|}$



$y = f(x)$

$\sup = +\infty$

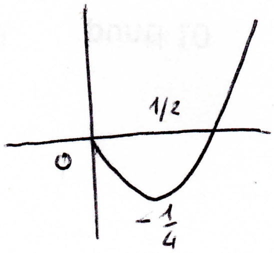
$\inf = -\infty$

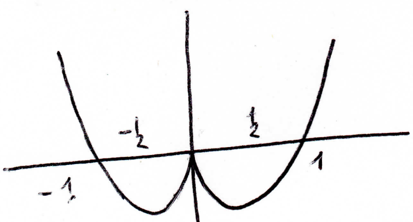
max, min non $\exists$

• $f(x) = x^2 - |x|$

fz. pari: il suo grafico è simmetrico rispetto all'asse delle y.

Possiamo limitarci a studiare il grafico per $x \geq 0$; in questo modo la fz. diventa $x^2 - x$ (come sopra)



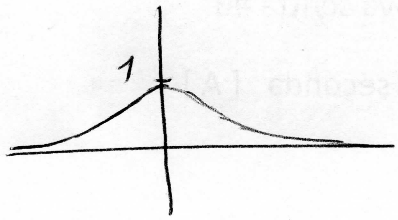


$\min = \inf = -\frac{1}{4}$

$\sup = +\infty$

max $\nexists$

• $f(x) = e^{-x^2}$



max = sup = 1
inf = 0
min $\nexists$

• $f(x) = \text{arcsen}\ \frac{1}{2x}$