Simpson's shapes of schemes and stacks

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ABSTRACT. This is a compendium of Simpson's theory about shapes of schemes and stacks.

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To be written.

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2.1. Overview

Let X be a derived stack. We are often interested in knowing when the derived stack

$$\mathbf{Perf}(X) := \mathbf{Map}(X, \mathbf{Perf})$$

is representable by a geometric derived stack locally almost of finite presentation. This can be achieved by checking the assumptions of Artin-Lurie's representability theorem, which we recall below:

THEOREM 2.1.1 (Artin-Lurie Theorem, [Lur18, Theorem 18.3.0.1]). *A derived stack $F \in \mathsf{dSt}$ is geometric and locally almost of finite presentation if and only if it satisfies the following conditions:*

(1) (Truncatedness) There exists $n \geq 0$ such that for every affine underived scheme S, $F(S)$ is n-truncated.

(2) (Locally almost finitely presented) The functor F is locally almost of finite presentation, that is for every $n \geq 0$ and every cofiltered diagram $S \colon I \to \mathsf{dAff}^{\leqslant n}$ of n-truncated affine derived schemes, the canonical map

$$\operatorname*{colim}_{i \in I} F(S_i) \longrightarrow F\Big(\lim_{i \in I} S_i\Big)$$

is an equivalence.

(3) (Deformation theory) The functor F is infinitesimally cohesive, nilcomplete and admits an eventually connective global cotangent complex.

(4) (Integrability) The functor F is integrable, that is for every underived local ring A which is complete with respect to its maximal ideal $\mathfrak{m}$, the canonical map

$$F(\mathsf{Spec}(A)) \longrightarrow F(\mathsf{Spf}(A))$$

is an equivalence.

Our goal in this section is to spell out some useful criteria intrinsic on X that guarantee the assumptions of Artin-Lurie's theorem are satisfied for $\mathbf{Perf}(X)$. The most well known and well documented example is the case where X is a smooth and proper scheme. For our purposes, this is not enough, as we are often interested in the case of Simpson's shapes X_{B}, X_{dR}, X_{Dol} and X_{Del}.

A number of these conditions are easily checked:

- infinitesimal cohesiveness and nilcompleteness are essentially always satisfied;
- integrability enjoys some stability properties that make it easy to check it in the examples of our interests;
- truncatedness is relatively easy to verify in terms of flat presentations.

The hardest work is required to check that $\mathbf{Perf}(X)$ is locally almost finitely presented and admits a global cotangent complex. Combining ideas from [HLP14] and [PTVV13] we relate the problem of the existence of a global cotangent complex to two properties that are easily verified in practice: *categorical quasi-compactness* and *finite cohomological dimension*. We refer to a slight strengthening of the combination of these two properties as *categorical properness*.

2.2. Deformation theory I. Infinitesimal cohesiveness and nilcompleteness

We start discussing the easiest deformation theoretic properties of derived stacks. For later use, we consider things in the more general setting of derived stacks with values in Cat_∞.

DEFINITION 2.2.1. A *categorical derived stack* is a functor $F\colon \mathsf{dAff}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ which is a hypercomplete sheaf for the étale topology. We let $\mathsf{dSt}^{\mathsf{cat}}$ denote the ∞-category of categorical derived stacks. ⊘

The natural inclusion $i\colon \mathcal{S} \hookrightarrow \mathsf{Cat}_\infty$ of spaces inside ∞-categories is fully faithful and commutes with both limits and colimits. In particular, there is an induced fully faithful inclusion

$$i\colon \mathsf{dSt} \hookrightarrow \mathsf{dSt}^{\mathsf{cat}} .$$

If $X \in \mathsf{dSt}$ is a derived stack, we typically abuse of notation and see it, if needed, as a Cat_∞-valued derived stack implicitly using the above embedding. The functor i admits both a left adjoint L and a right adjoint R, which can be characterized as follows. If $F \in \mathsf{dSt}^{\mathsf{cat}}$ is a categorical derived stack, then $L(F)$ is the sheafification of the presheaf defined by

$$L(F)(S) \coloneqq \mathsf{Env}(F(S)) ,$$

where $\mathsf{Env}\colon \mathsf{Cat}_\infty \to \mathcal{S}$ denotes the *enveloping groupoid*. The right adjoint instead satisfies the relation

$$R(F)(S) \simeq F(S)^{\simeq} ,$$

where $(-)^{\simeq}\colon \mathsf{Cat}_\infty \to \mathcal{S}$ denotes the *maximal ∞-groupoid functor*. Observe that $R(F)$ is automatically a sheaf, with no need to sheafify. We refer to $R(F)$ as the *underlying derived stack of F*.

Since Cat_∞ is cartesian closed, the same goes for $\mathsf{dSt}^{\mathsf{cat}}$. In particular, given two categorical derived stacks F, G we have an internal hom $\mathbf{Map}(F,G) \in \mathsf{dSt}^{\mathsf{cat}}$. For every $S \in \mathsf{dAff}$ we have, tautologically:

$$\mathbf{Map}(F,G)(S) \coloneqq \mathrm{Map}_{\mathsf{dSt}^{\mathsf{cat}}}(F \times S, G) .$$

2.2.1. Infinitesimal cohesiveness. We start by discussing the notion of infinitesimal cohesiveness for categorical derived stacks.

Recall that for any $S = \mathrm{Spec}(A) \in \mathsf{dAff}$ an affine derived scheme and any $M \in \mathsf{QCoh}^{\geqslant 1}(S)$ be a quasi-coherent complex, we set $S[M] \coloneqq \mathrm{Spec}(A \oplus M)$ and we denote by d the derivation $S[M[-1]] \to S$. Finally, let $S_d[M]$ be the pushout

$$\begin{array}{ccc} S[M] & \xrightarrow{\ d\ } & S \\ \big\downarrow{\scriptstyle d_0} & & \big\downarrow{\scriptstyle f_0} \\ S & \xrightarrow{\ f\ } & S_d[M[-1]] \end{array} ,$$

where d_0 denotes the zero derivation.

DEFINITION 2.2.2. We say that a categorical derived stack $F \in \mathsf{dSt}^{\mathsf{cat}}$ is *infinitesimal cohesive* if for every $S \in \mathsf{dAff}$, every $M \in \mathsf{QCoh}^{\geqslant 1}(S)$ and every derivation $d\colon S[M] \to S$, the canonical square

$$\begin{array}{ccc} F(S_d[M[-1]]) & \longrightarrow & F(S) \\ \big\downarrow & & \big\downarrow \\ F(S) & \longrightarrow & F(S[M]) \end{array} \tag{2.2.1}$$

is a pullback. ⊘

Infinitesimally cohesive categorical derived stacks are closed under a certain number of operations:

PROPOSITION 2.2.3.

(1) If a categorical derived stack F is infinitesimally cohesive, then its underlying derived stack $R(F)$ is infinitesimally cohesive.

(2) The class of infinitesimally cohesive categorical derived stacks is closed under limits.

(3) Let $F \in \mathsf{dSt}$ be a derived stack and let $G \in \mathsf{dSt}^{\mathsf{cat}}$ be a categorical derived stack. If G is infinitesimal cohesive, the same goes for $\mathbf{Map}(F, G)$.

PROOF. The first statement follows from the fact that $(-)^{\simeq}$ commutes with limits. Moreover, since limits in $\mathsf{dSt}^{\mathsf{cat}}$ are computed objectwise, the second statement is obvious. For (3), we first observe that since F is a derived stack, we can write

$$F \simeq \operatorname*{colim}_{S \in \mathsf{dAff}_{/F}} S .$$

The inclusion $i\colon \mathsf{dSt} \to \mathsf{dSt}^{\mathsf{cat}}$ commutes with both limits and colimits. Therefore, we have

$$\mathbf{Map}(F, G) \simeq \lim_{S \in \mathsf{dAff}_{/F}} \mathbf{Map}(S, G) .$$

Thanks to point (2), we can therefore reduce to the case where F itself is an affine derived scheme. Let $S \in \mathsf{dAff}$ and choose $M \in \mathsf{QCoh}^{\geqslant 1}(S)$ and a derivation $d\colon S[M] \to S$. Then $F \times S$ is again an affine derived scheme. Let $p\colon F \times S \to S$ be the natural projection. Then there is a canonical equivalence

$$F \times S[M] \simeq (F \times S)[p^* M] ,$$

which induces a derivation $p^*(d)\colon (F \times S)[p^* M] \to F \times S$ and another equivalence

$$F \times S_d[M[-1]] \simeq (F \times S)_{p^*(d)}[p^* M[-1]] .$$

We can therefore rewrite the diagram (2.2.1) for $\mathbf{Map}(F, G)$ as the square

$$\begin{array}{ccc} G\big((F \times S)_{p^*(d)}[p^* M[-1]]\big) & \longrightarrow & G(F \times S) \\ \downarrow & & \downarrow \\ G(F \times S) & \longrightarrow & G\big((F \times S)[p^* M[-1]]\big) \end{array} ,$$

which is a pullback by the assumption on G. □

The following is a simple consequence of infinitesimal cohesiveness, which nevertheless is very often useful in practice:

PROPOSITION 2.2.4. *Let $F \in \mathsf{dSt}$ be an infinitesimally cohesive derived stack. Let $S \in \mathsf{dAff}$ be an affine derived scheme and let $x\colon S \to F$ be a morphism. Let $\Omega_x F := S \times_F S$ be the loop stack at x and let $\delta_x\colon S \to \Omega_x F$ the diagonal morphism. Then the following statements are equivalent:*

(1) the derived stack F admits a cotangent complex $x^* \mathbb{L}_F$ at the point x;

(2) the derived stack $\Omega_x F$ admits a cotangent complex $\delta_x^* \mathbb{L}_F$ at the point δ_x.

Furthermore, if these conditions are met, there is a canonical equivalence

$$x^* \mathbb{L}_F \simeq \delta_x^* \mathbb{L}_F[-1]$$

in $\mathsf{QCoh}(S)$.

PROOF. To be written. □

2.2.2. Nilcompleteness. We now turn to nilcompleteness.

DEFINITION 2.2.5. Let $n \in \mathbb{N} \cup \{\infty\}$ be a possibly infinite integer. We say that an affine derived scheme $X = \mathsf{Spec}(A)$ is *n-truncated* (or *n-coconnective*) if:

(1) when $n < \infty$, the groups $\pi_m(A)$ are zero for $m \geq n+1$;

(2) when $n = \infty$, there exists n_0 such that $\pi_m(A) = 0$ for $m \geq n_0$.

We refer to ∞-truncated affine derived schemes as *eventually coconnective* affine derived schemes. ⊘

For $n \in \mathbb{N} \cup \{\infty\}$ we let ${}^{<n}\mathsf{dAff}$ be the full subcategory of dAff spanned by $(n-1)$-truncated affine derived schemes. The étale topology on dAff induces a topology on ${}^{<n}\mathsf{dAff}$. We write

$$ {}^{<n}\mathsf{dSt} := \mathsf{Sh}({}^{<n}\mathsf{dAff}, \tau_{\text{ét}})^\wedge $$

for the category of hypercomplete étale sheaves on ${}^{<n}\mathsf{dAff}$. The natural inclusion functor

$$ i_n \colon {}^{<n}\mathsf{dAff} \hookrightarrow \mathsf{dAff} $$

is both continuous and cocontinuous, and therefore restriction along i_n induces a functor

$$ {}^{<n}(-) \colon \mathsf{dSt} \longrightarrow {}^{<n}\mathsf{dSt} , $$

which has both a left adjoint $i_{n!}$ and a right adjoint i_{n*}.

DEFINITION 2.2.6. Let $F \in \mathsf{dSt}^{\mathsf{cat}}$ be a categorical derived stack.

(1) The *n-truncation* of F is the derived stack

$$ \mathsf{t}_{\leq n} F := i_{n+1!}({}^{<n+1}F) . $$

(2) The *associated convergent derived stack* of F is the derived stack

$$ {}^{\mathsf{conv}}F := i_{\infty *}({}^{<\infty}F) . $$

⊘

EXAMPLE 2.2.7. If $F = \mathsf{Spec}(A)$ is an affine derived scheme, then $\mathsf{t}_{\leq n}F \simeq \mathsf{Spec}(\tau_{\leq n}A)$, while the associated convergent derived stack is F itself, ${}^{\mathsf{conv}}F \simeq F$. △

The terminology "associated convergent derived stack" is clarified by the following definition:

DEFINITION 2.2.8. We say that a categorical derived stack $F \in \mathsf{dSt}^{\mathsf{cat}}$ is *convergent* (or *nilcomplete*) if for every $S \in \mathsf{dAff}$ the canonical map

$$ F(S) \longrightarrow \lim_{n \geq 0} F(\mathsf{t}_{\leq n}(S)) $$

is an equivalence. ⊘

We summarize the basic properties of convergent derived stacks as follows:

PROPOSITION 2.2.9.

(1) A categorical derived stack F is nilcomplete if and only if the canonical map

$$ F \longrightarrow {}^{\mathsf{conv}}F $$

is an equivalence.

(2) If a categorical derived stack F is nilcomplete, then its underlying derived stack $R(F)$ is nilcomplete as well.

(3) The class of nilcomplete categorical derived stacks is closed under limits.

(4) Let $F \in \mathsf{dSt}$ be a derived stack and let $G \in \mathsf{dSt}^{\mathsf{cat}}$ be a categorical derived stack. If G is nilcomplete, the same goes for $\mathbf{Map}(F, G)$.

PROOF. We start by point (1). Consider the commutative triangle

$$\begin{array}{ccc} F & \longrightarrow & {}^{\mathrm{conv}}F \\ & \searrow \quad \swarrow & \\ & \lim_n i_{n*}({}^{<n}F) & \end{array}$$

The transitivity of right Kan extensions shows that the diagonal map on the right is an equivalence. Unraveling the definitions, we see that F is convergent if and only if the diagonal map on the left is an equivalence. Therefore, the conclusion follows from the 2-out-of-3 property of equivalences.

The second statement follows from the fact that $(-)^{\simeq}$ commutes with limits. Moreover, since limits in $\mathsf{dSt}^{\mathsf{cat}}$ are computed objectwise, the statement (3) is obvious. For (4), we first observe that since F is a derived stack, we can write

$$F \simeq \operatorname*{colim}_{S \in \mathsf{dAff}_{/F}} S .$$

The inclusion $i \colon \mathsf{dSt} \to \mathsf{dSt}^{\mathsf{cat}}$ commutes with both limits and colimits. Therefore, we have

$$\mathbf{Map}(F, G) \simeq \lim_{S \in \mathsf{dAff}_{/F}} \mathbf{Map}(S, G) .$$

Thanks to point (3), we can therefore reduce to the case where F itself is an affine derived scheme.

Consider the square

$$\begin{array}{ccc} G(\lim_n F \times \mathrm{t}_{\leq n} S) & \longrightarrow & \lim_n G(F \times \mathrm{t}_{\leq n} S) \\ \downarrow & & \downarrow \\ G(\lim_n \lim_m \mathrm{t}_{\leq m}(F \times \mathrm{t}_{\leq n} S)) & \longrightarrow & \lim_n \lim_m G(\mathrm{t}_{\leq m}(F \times \mathrm{t}_{\leq n} S)) \end{array} .$$

The left vertical map is obviously an equivalence. Since G is nilcomplete, the right vertical map and the bottom horizontal one are both equivalences. Therefore, the top horizontal map is an equivalence as well. This completes the proof. □

The following is the example of fundamental interest for us:

THEOREM 2.2.10 (Lurie). *The categorical stack*

$$\mathsf{QCoh}^{<-\infty} \colon \mathsf{dAff}^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty$$

of eventually connective quasi-coherent sheaves is infinitesimally cohesive and nilcomplete. The same goes for the categorical substacks APerf *and* Perf *of almost perfect complexes and perfect complexes, as well as for the underlying derived stacks* $\mathbf{APerf} := R(\mathsf{APerf})$ *and* $\mathbf{Perf} := R(\mathsf{Perf})$.

PROOF. Infinitesimally cohesiveness follows from [Lur18, Theorem 16.2.0.1 and Proposition 16.2.3.1-(6)]. Nilcompleteness follows from [Lur18, Propositions 19.2.1.5 and 2.7.3.2-(c)]. The last statement is a direct consequence of Propositions 2.2.3-(1) and 2.2.9-(2). □

2.3. Deformation theory II. Categorical properness and cotangent complex

Let $f \colon X \to S \in \mathsf{dSt}$ be a morphism of derived stacks. When the map is not representable by geometric stacks, it is not obvious to formulate a notion of properness for f. The following are some of the possible requirements one can put on f:

(1) the functor $f_* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(S)$ preserves filtered colimits, universally in S;

(2) for every pullback diagram

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

where g is representable by affine derived schemes, the Beck-Chevalley transformation

$$g^* \circ f_* \longrightarrow f'_* \circ g'^*$$

is an equivalence.

(3) the functor $f_* \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ preserves almost perfect complexes, universally in S.

Typically, each of these conditions requires non-trivial arguments to be verified, as the following example shows:

EXAMPLE 2.3.1.

(1) Let S be an affine scheme and let $f \colon X \to S$ be a quasi-compact morphism of schemes. Then the functor

$$f_* \colon \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(S)$$

commutes with filtered colimits, universally in S. Indeed, if $T \to S$ is a map in dAff, it is in particular quasi-compact, and therefore the base-change $X_T \to T$ is again quasi-compact. Thus, it is enough to deal with the case $S = T$. Denote by X_{Zar} the small Zariski site of X. There is a canonical commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{i_X} & \mathrm{PSh}_{\mathrm{Mod}_{\mathbb{C}}}(X_{\mathrm{Zar}}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(S) & \xrightarrow{i_S} & \mathrm{PSh}_{\mathrm{Mod}_{\mathbb{C}}}(S_{\mathrm{Zar}}) \end{array} .$$

The functors i_X and i_S are conservative. Since S and f are quasi-compact, the same goes for X. Hence the small Zariski sites S_{Zar} and X_{Zar} are quasi-compact. This implies that the functors i_S and i_X commutes with filtered colimits (see for example [PY18, Lemma 5.5]).

(2) Assume now that $f \colon X \to S$ is a quasi-compact and quasi-separated morphism of schemes. Then [Toë12, Proposition 1.4] implies that for any pullback diagram

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

the Beck-Chevalley transformation

$$g^* f_*(\mathcal{F}) \longrightarrow f'_* g'^*(\mathcal{F})$$

is an equivalence for every $\mathcal{F} \in \mathrm{QCoh}(X)$. When X is itself affine, the statement is trivial. In general, one needs quasi-compactness and quasi-separatedness of f to write X as a finite colimit of affine schemes and open immersions between them. This allows to compute f_* as a finite limit (using for example [PY16, §8.2]). The conclusion follows because g^* commutes with finite limits, being a functor between stable ∞-categories.

(3) Finally, assume that $f \colon X \to S$ is a proper morphism of schemes. Using [Lur18, Theorem 5.6.0.2], we see that the functor $f_* \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ restricts to a functor

$$f_* \colon \mathrm{APerf}(X) \longrightarrow \mathrm{APerf}(S) .$$

In the case of schemes, one can see this as a combination of Grothendieck direct image theorem plus the fact that f has finite cohomological dimension. Assume furthermore that f has finite tor-amplitude (see [Lur18, Definition 6.1.1.1]). Then f_* preserves objects of finite tor-amplitude (see [Lur18, Proposition 6.1.3.1]), and therefore the characterization of perfect complexes via tor-amplitude provided in [Lur17, Proposition 7.2.4.23] shows that f_* restricts to

$$f_* \colon \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}(S) .$$

See also [Lur18, Theorem 6.1.3.2].

△

In the above example, we used in a significant way the fact that X was a derived *scheme*. It is not difficult to relax this condition a little, for example to derived algebraic spaces. Nevertheless, to remove altogether every geometricity condition on X is challenging.

2.3.1. Categorical quasi-compactness. We start by exploring the formal consequences of commuting with filtered colimits. The following definition is motivated by Example (1):

DEFINITION 2.3.2. Let $f \colon X \to S \in \mathsf{dSt}$ be a morphism of derived stacks. We say that f is *categorically quasi-compact* if the functor

$$f_* \colon \mathsf{QCoh}(X) \longrightarrow \mathsf{QCoh}(S)$$

commutes with filtered colimits. ⊘

REMARK 2.3.3. In [PTVV13, Definition 2.1], the authors introduce the notion of *(strict) $\mathcal{O}$-compact morphism*. A morphism of derived stacks $f \colon X \to S$ is said to be strictly $\mathcal{O}$-compact if

$$f_* \colon \mathsf{QCoh}(X) \longrightarrow \mathsf{QCoh}(S)$$

commutes with filtered colimits and it preserves perfect complexes. As Example (3) shows, in the geometric case preservation of perfect complexes is a consequence of properness (which guarantees that almost perfect complexes are preserved) and of finite tor-amplitude (which further guarantees that perfectness is preserved). △

PROPOSITION 2.3.4. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pullback square in dSt. *Assume that f is representable by affine derived schemes. Then:*

(1) *the functor*

$$f_* \colon \mathsf{QCoh}(X) \longrightarrow \mathsf{QCoh}(Y)$$

is conservative and commutes with filtered colimits.

(2) *For every $\mathcal{F} \in \mathsf{QCoh}(X)$ the Beck-Chevalley transformation*

$$g^* f_*(\mathcal{F}) \longrightarrow f'_* g'^*(\mathcal{F})$$

is an equivalence.

REMARK 2.3.5. One can deduce the above Proposition as a special case of [HLP14, Proposition A.1.5]. In our setting, the proof can be simplified, so we include it for the convenience of the reader. △

PROOF. We start by proving (1). Since f is representable by affine derived schemes, pullback along f induces a morphism

$$- \times_Y X \colon \mathsf{dAff}_{/Y} \longrightarrow \mathsf{dAff}_{/X} .$$

Since dSt is an ∞-topos, colimits are universal and therefore we see that the canonical map

$$\operatorname*{colim}_{U \in \mathsf{dAff}_{/Y}} U \times_Y X \longrightarrow X$$

is an equivalence. Let now

$$\begin{array}{ccc} V & \xrightarrow{g} & U \\ & {\scriptstyle v}\searrow \quad \swarrow{\scriptstyle u} & \\ & Y & \end{array}$$

be a morphism in $\mathsf{dAff}_{/Y}$. It gives rise to a pullback square

$$\begin{array}{ccc} V \times_Y X & \xrightarrow{p} & V \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ U \times_Y X & \xrightarrow{q} & U \end{array}$$

in dAff. In particular, the induced diagram

$$\begin{array}{ccc} \mathsf{QCoh}(U) & \xrightarrow{q^*} & \mathsf{QCoh}(U \times_Y X) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathsf{QCoh}(V) & \xrightarrow{p^*} & \mathsf{QCoh}(V \times_Y X) \end{array}$$

is horizontally right adjointable. This implies that for every pullback diagram

$$\begin{array}{ccc} U \times_Y X & \xrightarrow{f'} & U \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ X & \xrightarrow{f} & Y \end{array}$$

where $U \in \mathsf{dAff}$ and for every $\mathcal{F} \in \mathsf{QCoh}(X)$, the canonical map

$$u^*(f_*(\mathcal{F})) \longrightarrow f'_*(u'^*(\mathcal{F})) \tag{2.3.1}$$

is an equivalence. Since f' is a map between affine derived schemes, f'_* is conservative and commutes with filtered colimits. Since the functors $u^* \colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(U)$ are jointly conservative for $u \colon U \to Y$ in $\mathsf{dAff}_{/Y}$ and they commute with colimits, the conclusion follows.

As for statement (2), we first observe that we can replace Y' by a affine derived scheme. In this case, the conclusion follows directly from the fact that (2.3.1) is an equivalence. □

We can now collect the fundamental properties of categorically quasi-compact morphisms:

PROPOSITION 2.3.6. *Let*

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

be a pullback square in dSt. *Assume that f is categorically quasi-compact and g is representable by affine derived schemes. Then:*

(1) the map f' is categorically quasi-compact.

Assume furthermore that S is an affine derived scheme. Then:

(2) *for every* $\mathcal{F} \in \mathsf{QCoh}(X)$ *and* $\mathcal{G} \in \mathsf{QCoh}(S)$, *the canonical map*

$$f_*(\mathcal{F}) \otimes \mathcal{G} \longrightarrow f_*(\mathcal{F} \otimes f^*(\mathcal{G})) \tag{2.3.2}$$

is an equivalence.

(3) *For every* $\mathcal{F} \in \mathsf{QCoh}(X)$ *the Beck-Chevalley transformation*

$$g^* f_*(\mathcal{F}) \longrightarrow f'_* g'^*(\mathcal{F})$$

is an equivalence.

PROOF. We start proving (1). Let $F \colon I \to \mathsf{QCoh}(X_T)$ be a filtered diagram. For every $\alpha \in I$, set $\mathcal{F}_\alpha := F(\alpha)$ and let

$$\mathcal{F} := \operatorname*{colim}_{\alpha \in I} \mathcal{F}_\alpha .$$

Consider the natural map

$$\operatorname*{colim}_{\alpha \in I} f'_*(\mathcal{F}_\alpha) \longrightarrow f'_*(\mathcal{F}) .$$

Thanks to Proposition 2.3.4-(1) the functor g_* is conservative. It is therefore enough to check that the above map is an equivalence after applying g_*. Since by the same result g_* commutes with filtered colimits, we reduce ourselves to check that the map

$$\operatorname*{colim}_{\alpha \in I} g_*(f'_*(\mathcal{F}_\alpha)) \longrightarrow g_*(f'_*(\mathcal{F}_\alpha))$$

is an equivalence. Using the natural equivalence $g_* \circ f'_* \simeq f_* \circ g'_*$, plus the fact that g'_* commutes with filtered colimits (since it is again representable by affine derived schemes), we finally reduce ourselves to the assumption that f_* commutes with filtered colimits. This proves (1).

We now turn to statement (2). Let $\mathcal{C}$ be the full subcategory of $\mathsf{QCoh}(S)$ spanned by the objects $\mathcal{G}$ for which the morphism (2.3.2) is an equivalence. Since f is strictly categorically quasi-compact, f_* commutes with filtered colimits. Since $\mathsf{QCoh}(X)$ and $\mathsf{QCoh}(S)$ are stable ∞-categories, it follows that f_* commutes with arbitrary colimits. As tensor products and the functor f^* commute with arbitrary colimits as well, it follows $\mathcal{C}$ is closed under arbitrary colimits. Since S is affine, it is therefore enough to observe that $\mathcal{O}_S$ belongs to $\mathcal{C}$.

We finally prove point (3). Since g is representable by affine derived schemes, Proposition 2.3.4-(1) guarantees that g_* is conservative. It is therefore enough to prove that the induced map

$$g_*(g^*(f_*(\mathcal{F}))) \longrightarrow g_*(f'_*(g'^*(\mathcal{F}))) \simeq f_*(g'_*(g'^*(\mathcal{F})))$$

is an equivalence. Combining Proposition 2.3.4-(1) and statement (2), we have a canonical equivalences

$$g_*(g^*(f_*(\mathcal{F}))) \simeq f_*(\mathcal{F}) \otimes g_*(\mathcal{O}_T) , \quad g'_* g'^*(\mathcal{F}) \simeq g'_*(\mathcal{O}_{X_T}) \otimes \mathcal{F} .$$

On the other hand, since g is representable by affine derived schemes. Proposition 2.3.4-(2) shows that the canonical map

$$f^*(g_*(\mathcal{O}_T)) \longrightarrow g'_*(f'^*(\mathcal{O}_T)) \simeq g'_*(\mathcal{O}_{X_T})$$

is an equivalence. Applying statement (2) once more, we obtain the equivalence

$$f_*(g'_*(\mathcal{O}_{X_T}) \otimes \mathcal{F}) \simeq f_*(\mathcal{F}) \otimes g_*(\mathcal{O}_T) .$$

The conclusion follows. □

2.3.2. Finite cohomological dimension. In practice, it is useful to have a criterion allowing to check whether a morphism $f\colon X \to S$ is categorically quasi-compact. If one has a bound on its cohomological dimension, the verification is often simpler, as we are going to discuss in this section.

DEFINITION 2.3.7. Let $f\colon X \to S$ be a morphism in dSt and let $n \geq 0$ be an integer. We say that f has *cohomological dimension* $\leq n$ if for every $\mathcal{F} \in \mathsf{QCoh}^{\heartsuit}(X)$, the quasi-coherent sheaf $f_*(\mathcal{F})$ belongs to $\mathsf{QCoh}^{\geqslant -n}(S)$. We say that f has *finite cohomological dimension* if there exists an integer $n \geq 0$ such that f has cohomological dimension $\leq n$. ⊘

PROPOSITION 2.3.8. *Let*

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

be a pullback square in dSt. *Assume that f has cohomological dimension $\leq n$ and g is representable by affine derived schemes. Then f' has cohomological dimension $\leq n$.*

PROOF. Let now $\mathcal{F} \in \mathsf{QCoh}^{\heartsuit}(X_T)$. Proposition 2.3.4-(1) implies that the functor g_* is conservative, while point (2) implies that g_* is t-exact. It follows that $f'_*(\mathcal{F})$ belongs to $\mathsf{QCoh}^{\geqslant -n}(T)$ if and only if

$$g_*(f'_*(\mathcal{F})) \simeq f_*(g'_*(\mathcal{F}))$$

belongs to $\mathsf{QCoh}^{\geqslant -n}(S)$. Since g is representable by affine derived schemes, the same goes for g'. Using once again the fact that g'_* is t-exact, the conclusion follows. □

PROPOSITION 2.3.9. *Let $S \in \mathsf{dAff}$ be an affine derived scheme and let $f\colon X \to S$ be a morphism in* dSt. *If f has finite cohomological dimension, the following statements are equivalent:*

(1) The morphism f is categorically quasi-compact.

(2) The functor

$$f_*\colon \mathsf{QCoh}^{\heartsuit}(X) \longrightarrow \mathsf{QCoh}(S)$$

commutes with filtered colimits.

PROOF. Since the t-structure on $\mathsf{QCoh}(X)$ is compatible with filtered colimits, etc. □

The following two propositions relate the property of having finite cohomological dimension to categorical quasi-compactness.

PROPOSITION 2.3.10. *Let $S \in \mathsf{dAff}$ be an affine derived scheme and let $f\colon X \to S$ be a morphism in* dSt. *Assume that there exists a simplicial stack*

$$U_\bullet\colon \mathbf{\Delta} \longrightarrow \mathsf{dSt}_{/X}$$

such that:

(1) the canonical morphism $|U_\bullet| \to X$ is an equivalence;

(2) for every $[n] \in \mathbf{\Delta}$, the canonical morphism $u_n\colon U_n \to X$ is flat;

(3) there exists an integer $m \geq 0$ such that for every $[n] \in \mathbf{\Delta}^n$, the canonical morphism $f_n\colon U_n \to S$ has cohomological dimension $\leq n$ and is categorically quasi-compact.

Then f is categorically quasi-compact. If furthermore the maps $u_n\colon U_n \to X$ are universally flat relative to S and the maps $f_n\colon U_n \to S$ have universal finite cohomological dimension $\leq m$ and are categorically quasi-compact, then f is categorically quasi-compact as well.

REMARK 2.3.11. Observe that we do not require the U_n to be geometric stacks. Moreover, X is not required to have finite cohomological dimension in itself. △

PROOF. Let I be a filtered category and let $F\colon I \to \mathrm{QCoh}(X)$ be a diagram. Write $\mathcal{F}_\alpha := F(\alpha)$ for $\alpha \in I$ and set

$$\mathcal{F} := \operatorname*{colim}_{\alpha \in I} \mathcal{F}_\alpha .$$

Using [PY16, §8.2], we have a natural identification

$$f_*(\mathcal{F}) \simeq \lim_{[n]\in \mathbf{\Delta}} f_{n*} u_n^*(\mathcal{F}) , \tag{2.3.3}$$

and similarly for $f_*(\mathcal{F}_\alpha)$. Consider the canonical comparison map

$$\phi\colon \operatorname*{colim}_{\alpha} f_*(\mathcal{F}_\alpha) \longrightarrow f_*(\mathcal{F}) .$$

Since S is affine, it is enough to check that $\pi_i(\phi)$ is an isomorphism for every $i \in \mathbb{Z}$. Replacing the diagram F by $F[i]$, we see it is enough to prove that $\pi_0(\phi)$ is an equivalence. Since the t-structure on $\mathrm{QCoh}(S)$ is compatible with filtered colimits, we see that the canonical map

$$\operatorname*{colim}_{\alpha} \pi_0(f_*(\mathcal{F}_\alpha)) \longrightarrow \pi_0(\operatorname*{colim}_{\alpha} f_*(\mathcal{F}_\alpha))$$

is an equivalence. Using (2.3.3) and the fact that each f_n has cohomological dimension $\leq m$, we can replace the diagram F by $\tau_{\leq m} F$. In other words, we can assume that each $\mathcal{F}_\alpha$ and $\mathcal{F}$ are m-coconnective. Since each u_n is flat, $f_{n*}u_n^*(\mathcal{F})$ and $f_{n*}u_n^*(\mathcal{F}_\alpha)$ are again m-coconnective. Using [PY18, Corollary 9.4], we see that

$$\pi_0\left(\lim_{[n]\in\mathbf{\Delta}} f_{n*}u_n^*(\mathcal{F})\right) \longrightarrow \pi_0\left(\lim_{[n]\in\mathbf{\Delta}_{\leq m+2}} f_{n*}u_n^*(\mathcal{F})\right)$$

is an equivalence, and similarly for $\mathcal{F}_\alpha$ in place of $\mathcal{F}$. As each f_{n*} commutes with filtered colimits, the conclusion follows because $\mathbf{\Delta}_{\leq m+2}$ is a finite category and filtered colimits commute with finite limits. □

2.3.3. Morphisms of finite tor-amplitude. We start by defining the notion of finite tor-amplitude for quasi-coherent sheaves on derived stacks:

DEFINITION 2.3.12. Let $S \in \mathrm{dSt}$ be a derived stack and let $a \leq b$ be integers. We say that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(S)$ has *tor-amplitude contained in* $[a,b]$ if for every map $f\colon T \to S$ where $T \in \mathrm{dAff}$ is an affine derived scheme, $f^*(\mathcal{G})$ has tor-amplitude[1] contained in $[a,b]$. We say that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(S)$ has *globally finite tor-amplitude* if there exist integers $a \leq b$ such that $\mathcal{F}$ has tor-amplitude contained in $[a,b]$. ⊘

Obviously, we have:

LEMMA 2.3.13. *Let $f\colon X \to S$ be a morphism of derived stacks. The functor*

$$f^*\colon \mathrm{QCoh}(S) \longrightarrow \mathrm{QCoh}(X)$$

preserves objects of tor-amplitude contained in $[a,b]$.

PROOF. Let $g\colon T \to X$ be a morphism where $T \in \mathrm{dAff}$ is an affine derived scheme. Then $g^* \circ f^* \simeq (f \circ g)^*$, whence the conclusion. □

Our goal is to formulate the notion of finite tor-amplitude for a morphism of derived stacks, not necessarily representable. We start by considering the following example:

EXAMPLE 2.3.14. Let $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ be two affine derived schemes and let $f\colon X \to S$ be a morphism of derived schemes. The following statements are equivalent:

[1] We refer to [Lur17, Definition 7.2.4.21] for the notion of tor-amplitude of A-modules, where $T = \mathrm{Spec}(A)$.

(1) For every discrete A-module $N \in A\text{-}\mathsf{Mod}^{\heartsuit}$, the pullback $f^*(N)$ has cohomological amplitude contained in $[a,b]$.

(2) The A-module $f_*(B)$ has tor-amplitude $[a,b]$.

(3) If $M \in A\text{-}\mathsf{Mod}$ has tor-amplitude $[a',b']$, then $f_*(M)$ has tor-amplitude $[a+a',b+b']$.

The projection formula

$$N \otimes f_*(M) \simeq f_*(f^*(N) \otimes_B M)$$

plus the fact that f_* is t-exact and conservative immediately implies the equivalence (1) $\Leftrightarrow$ (3). The same formula applied with $M = B$ shows that (1) $\Leftrightarrow$ (2) holds as well. △

When moving to the non-affine setting, the equivalence between (1) and (3) no longer holds. This produces two different ways of generalizing the notion of finite tor-amplitude for a non-representable morphism. We will refer to the one corresponding to point formulation (1) as *local tor-amplitude*, and the one corresponding to the formulation (3) as *global tor-amplitude*. We will see that under suitable finiteness assumptions, these two notions still agree.

2.3.3.1. *Local tor-amplitude.* The following is the immediate generalization of statement (1) in Example 2.3.14:

DEFINITION 2.3.15. Let $a \leq b$ be integers. We say that a morphism $f \colon X \to S$ of derived stacks *has local tor-amplitude contained in* $[a,b]$ if for every $\mathcal{F} \in \mathsf{QCoh}^{\heartsuit}(S)$, the pullback $f^*(\mathcal{F})$ belongs to $\mathsf{QCoh}^{\leqslant a \cap \geqslant b}(X)$. We say that a morphism $f \colon X \to S$ *has finite local tor-amplitude* if there exist integers $a \leq b$ such that f has local tor-amplitude contained in $[a,b]$. ⊘

PROPOSITION 2.3.16.

(1) Morphisms of finite local tor-amplitude are stable under compositions.

(2) Let

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

be a pullback square in dSt. *If g is representable by affine derived schemes and f has local tor-amplitude contained in $[a,b]$, then the same goes for f'.*

(3) A morphism $f \colon X \to S$ has local tor-amplitude contained in $[0,0]$ if and only if it is flat. In other words, if and only if the pullback functor

$$f^* \colon \mathsf{QCoh}(S) \longrightarrow \mathsf{QCoh}(X)$$

is t-exact.

PROOF. Points (1) and (3) just follow from the definitions. As for point (2), let $\mathcal{F} \in \mathsf{QCoh}^{\heartsuit}(T)$. We have to check that $f'^*(\mathcal{F})$ belongs to $\mathsf{QCoh}^{\leqslant a \cap \geqslant b}(X_T)$. Since g is representable by affine derived schemes, Proposition 2.3.4 shows that the canonical map

$$f^* g_*(\mathcal{F}) \longrightarrow g'_* f'^*(\mathcal{F})$$

is an equivalence. Applying the same proposition again, we see that g_* and g'_* are both t-exact. Therefore $g_*(\mathcal{F}) \in \mathsf{QCoh}^{\heartsuit}(S)$ and so $f^* g_*(\mathcal{F}) \in \mathsf{QCoh}^{\leqslant a \cap \geqslant b}(X)$. As g'_* is also conservative, the conclusion follows. □

The property of finite local tor-amplitude comes in handy to check the truncatedness assumption of Artin-Lurie representability theorem (cf. (1)):

PROPOSITION 2.3.17. *Let $S \in \mathsf{Aff}$ be an affine underived scheme and let $f \colon X \to S$ be a morphism in* dSt. *Assume that:*

(1) *the morphism f has finite local tor-amplitude;*

(2) *there exists a flat effective epimorphism*[2] $u\colon U \to X$*, where U is a quasi-compact derived scheme.*

Then for every pair of integers $a \le b$, there exist integers $a' \le b'$ such that if $\mathcal{F} \in \mathsf{Perf}(X)$ has tor-amplitude contained in $[a,b]$, then $\mathcal{F}$ is contained in cohomological amplitude $[a',b']$. In particular, $\mathsf{Perf}^{[a,b]}(X)$ is a $(b'-a')$-category[3].

PROOF. Using Proposition 2.3.16-(1), we deduce that the composition $U \to S$ has finite tor-amplitude. In particular, the structure sheaf of U is locally bounded, and therefore every perfect complex on U is locally bounded. Since U is quasi-compact, we deduce that every perfect complex on U is bounded, uniformly in its tor-amplitude. Since u is a flat effective epimorphism, the pullback functor

$$u^*\colon \mathsf{QCoh}(U) \longrightarrow \mathsf{QCoh}(X)$$

is conservative and t-exact. This implies that every perfect complex on X is bounded, uniformly in its tor-amplitude. The second statement is a trivial consequence of the first one. □

2.3.3.2. *Global tor-amplitude.* We now consider the natural generalization of statement (3) in Example 2.3.14:

DEFINITION 2.3.18. We say that a morphism $f\colon X \to S$ in dSt *has finite global tor-amplitude* if the pushforward functor

$$f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(S)$$

preserves objects of globally finite tor-amplitude. We say that f *has universally finite global tor-amplitude* if for every map $T \to S$ representable by affine derived schemes, the map $X \times_S T \to T$ has finite global tor-amplitude.

We saw in Example 2.3.14 that for a map of affine derived schemes $f\colon X \to S$, being of finite local tor-amplitude is equivalent to being finite global tor-amplitude. In the non-affine setting the equivalence typically does not hold. Nevertheless, under suitable finiteness conditions, local finite tor-amplitude still implies

PROPOSITION 2.3.19. *Let S be an affine derived scheme and let $f\colon X \to S$ be a morphism in dSt. Assume that f is categorically quasi-compact, of finite cohomological dimension and of finite local tor-amplitude. Then f has universally finite global tor-amplitude.*

PROOF. Let $T \to S$ be a morphism in dAff. Combining Propositions 2.3.6-(1), 2.3.8 and 2.3.16-(2), we deduce that the projection $X \times_S T \to T$ is again categorically quasi-compact, of finite cohomological dimension and of finite local tor-amplitude. It is therefore enough to prove the proposition when $T = S$.

Let $\mathcal{F} \in \mathsf{QCoh}(X)$ be an object of globally finite tor-amplitude. We have to prove that there are integers $a \le b$ such that for every $\mathcal{G} \in \mathsf{QCoh}^{\heartsuit}(S)$, the tensor product $f_*(\mathcal{F}) \otimes \mathcal{G}$ is contained in cohomological amplitude $[a,b]$. Since f is categorically proper, Proposition 2.3.6-(2) implies that the canonical map

$$f_*(\mathcal{F}) \otimes \mathcal{G} \longrightarrow f_*(\mathcal{F} \otimes f^*(\mathcal{G}))$$

is an equivalence. Since f has finite local tor-amplitude, we see that $f^*(\mathcal{G})$ is contained in cohomological amplitude $[a_0, b_0]$ (where a_0 and b_0 are independent of $\mathcal{G}$). Say that $\mathcal{F}$ has tor-amplitude contained in $[a_1, b_1]$. Then $\mathcal{F} \otimes f^*(\mathcal{G})$ is contained in cohomological amplitude $[a_0 - a_1, b_0 + b_1]$. Let n be an upper bound for the cohomological dimension of f. Then $f_*(\mathcal{F} \otimes f^*(\mathcal{G}))$ is contained in cohomological amplitude $[a_0 - a_1, b_0 + b_1 + n]$. The conclusion follows. □

[2] See [Lur09, § 6.2.3] for the definition of the effective epimorphism and [Lur09, Proposition 7.2.1.14] for a characterization of it that we use freely in the paper.

[3] Cf. [Lur09, §2.3.4] for the notion of n-categories. The relation between mapping spaces and truncatedness of categories is addressed in [Lur09, Proposition 2.3.4.18].

2.3.4. Categorical properness. We start by introducing the following definition which resembles [HLP14, Definition 2.4.1]:

DEFINITION 2.3.20. Let $f\colon X \to S$ be a morphism in dSt. We say that f has the *strict coherent pushforward property* if the functor

$$f_*\colon \mathsf{QCoh}(X) \longrightarrow \mathsf{QCoh}(S)$$

takes $\mathsf{APerf}(X)$ to $\mathsf{APerf}(S)$. We say that f has the *coherent pushforward property* if for every morphism $T \to S$ representable by affine derived schemes, the morphism $f'\colon X \times_S T \to T$ has the strictly coherent pushforward property. Here, $f'\colon X \times_S T \to T$ is the base change of f. ⊘

We are finally ready to introduce the notion of categorical properness:

DEFINITION 2.3.21. Let $f\colon X \to S$ be a morphism in dSt. We say that f is *strictly categorically proper* if it is categorically quasi-compact, has finite cohomological dimension and the strictly coherent pushforward property. We say that f is *categorically proper* if for every morphism $T \to S$ representable by affine derived schemes, the morphism $f'\colon X \times_S T \to T$ is *strictly categorically proper*. ⊘

REMARK 2.3.22.

(1) Combining Propositions 2.3.6-(1) and 2.3.8 we see that $f\colon X \to S$ is categorically proper if and only if it is strictly categorically proper and for every morphism $T \to S$ representable by affine derived schemes, pushforward along $f'\colon X \times_S T \to T$ preserves almost perfect complexes.

(2) In [HLP14], the authors developed a theory of *formally proper morphisms* between geometric derived stacks (cf. Definition 1.1.3 of *loc.cit.*). It depends on another geometric notion, which is the *completion along closed immersions* (cf. Definition 1.1.1 in *loc.cit.*). In Theorem 2.4.3 of *loc.cit.*, they prove that formally proper morphisms have the coherent pushforward property. It is not clear to us how to modify Definition 1.1.1. in *loc.cit.* in the non-geometric setting (we see already a problem for the de Rham shape of a derived stack). Thus, it is not clear to us how to extend the notion of formally properness to the non-geometric setting.

△

If a morphism is categorically proper and has finite local tor-amplitude, then it automatically preserves perfect complexes:

PROPOSITION 2.3.23. *Let $S \in \mathsf{dAff}$ be an affine derived scheme and let $f\colon X \to S$ be a morphism in dSt. Assume that f is categorically proper and has finite local tor-amplitude. If $\mathcal{F} \in \mathsf{Perf}(X)$ has tor-amplitude contained in $[a,b]$, then $f_*(\mathcal{F})$ belongs to $\mathsf{Perf}(S)$. In particular, if for every object $\mathcal{F} \in \mathsf{Perf}(X)$ there are integers $a \leq b$ such that $\mathcal{F}$ is contained in tor-amplitude $[a,b]$, then*

$$f_*\colon \mathsf{QCoh}(X) \longrightarrow \mathsf{QCoh}(S)$$

restricts to a functor

$$f_*\colon \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}(S)\ .$$

PROOF. The second half of the proposition is a trivial consequence of the first one. Let therefore $\mathcal{F} \in \mathsf{Perf}(X)$ and assume that there are integers $a \leq b$ such that $\mathcal{F}$ has tor-amplitude contained in $[a,b]$. Since f is categorically proper, $f_*(\mathcal{F})$ belongs to $\mathsf{APerf}(S)$ by definition. On the other hand, Proposition 2.3.19 implies that $f_*(\mathcal{F})$ has finite tor-amplitude. Therefore, [Lur17, Proposition 7.2.4.23-(4)] shows that $f_*(\mathcal{F})$ is perfect. The conclusion follows. □

The following lemma helps in checking that every perfect complex on X has globally finite tor-amplitude:

LEMMA 2.3.24. *Let $X \in \mathsf{dSt}$ be a derived stack. Assume that there exists an effective epimorphism $u \colon U \to X$, where U is a quasi-compact derived scheme. Then every $\mathcal{F} \in \mathsf{Perf}(X)$ has globally finite tor-amplitude.*

PROOF. Since U is quasi-compact, there are integers $a \leq b$ such that $u^*(\mathcal{F})$ has tor-amplitude contained in $[a,b]$. We claim that $\mathcal{F}$ has tor-amplitude contained in $[a,b]$ as well. Let $S \in \mathsf{dAff}$ be an affine derived scheme and let $f \colon S \to X$ be a morphism. We have to check that $f^*(\mathcal{F})$ has tor-amplitude contained in $[a,b]$. This can be checked étale locally on S. Since u is an effective epimorphism, we can choose an étale cover $\{s_\alpha \colon S_\alpha \to S\}_{\alpha \in I}$ such that for each $\alpha \in I$ there is a factorization

$$\begin{array}{ccccc} & & & \overset{f_\alpha}{\nearrow} & U \\ & & & & \downarrow{\scriptstyle u} \\ S_\alpha & \xrightarrow{s_\alpha} & S & \xrightarrow{f} & X \end{array} .$$

At this point, isomorphism

$$s_\alpha^*(f^*(\mathcal{F})) \simeq f_\alpha^*(u^*(\mathcal{F}))$$

implies that $f^*(\mathcal{F})$ has tor-amplitude contained in $[a,b]$. The conclusion follows. □

REMARK 2.3.25. Let S be an affine derived scheme and let $f \colon X \to S$ be a morphism in dSt. Assume that f is categorically proper and has finite local tor-amplitude, and that there exists an effective epimorphism $u \colon U \to X$, where U is a quasi-compact derived scheme. Then f is $\mathcal{O}$-compact in the sense of [PTVV13, Definition 2.1]. △

2.3.5. Plus pushforward and cotangent complex. Let $S \in \mathsf{dAff}$ be an affine derived scheme and let $f \colon X \to S$ be a morphism in dSt. Throughout this section we make the following assumptions:

(1) the morphism f is categorically proper and has finite local tor-amplitude;

(2) there exists an effective epimorphism $u \colon U \to X$, where U is a quasi-compact derived scheme.

We start with the following construction:

CONSTRUCTION 2.3.26. Under the above assumptions (1) and (2), Lemma 2.3.24 and Proposition 2.3.23 imply that $f_* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(S)$ restricts to a functor

$$f_* \colon \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}(S) .$$

Since perfect complexes are dualizable, we are authorized to set

$$f_+(\mathcal{F}) \coloneqq (f_*(\mathcal{F}^\vee))^\vee .$$

The natural morphism

$$\mathcal{F} \otimes f^* f_*(\mathcal{F}^\vee) \longrightarrow \mathcal{F} \otimes \mathcal{F}^\vee \longrightarrow \mathcal{O}_X$$

induces a canonical morphism

$$\eta_{\mathcal{F}} \colon \mathcal{F} \longrightarrow f^* f_+(\mathcal{F}) ,$$

which is easily seen to be functorial in $\mathcal{F} \in \mathsf{Perf}(X)$.

PROPOSITION 2.3.27. *Under the assumptions (1) and (2) on $f \colon X \to S$, we have:*

(1) For every $\mathcal{F} \in \mathsf{Perf}(X)$ and every $\mathcal{G} \in \mathsf{QCoh}(S)$, the morphism $\eta_{\mathcal{F}} \colon \mathcal{F} \to f^* f_+(\mathcal{F})$ induces an equivalence

$$\mathrm{Map}_{\mathsf{QCoh}(S)}(f_+(\mathcal{F}), \mathcal{G}) \longrightarrow \mathrm{Map}_{\mathsf{QCoh}(X)}(\mathcal{F}, f^*\mathcal{G}) . \tag{2.3.4}$$

(2) *For every pullback diagram*

$$\begin{array}{ccc} X_T & \xrightarrow{f'} & X \\ \downarrow{g'} & & \downarrow{f} \\ T & \xrightarrow{g} & S \end{array},$$

where T is an affine derived scheme, the diagram

$$\begin{array}{ccc} \mathsf{Perf}(X_T) & \xleftarrow{g'^*} & \mathsf{Perf}(X) \\ \uparrow{f'^*} & & \uparrow{f^*} \\ \mathsf{Perf}(T) & \xleftarrow{g^*} & \mathsf{Perf}(S) \end{array}$$

is vertically left adjointable.

PROOF. Proposition 2.3.6-(2) implies that the canonical map

$$f_*(\mathcal{F}^\vee) \otimes \mathcal{G} \longrightarrow f_*(\mathcal{F}^\vee \otimes f^*\mathcal{G})$$

is an equivalence. Applying $\mathsf{Map}_{\mathsf{QCoh}(S)}(\mathcal{O}_S, -)$ and using the adjunction $f^* \dashv f_*$, we see that

$$\mathsf{Map}_{\mathsf{QCoh}(S)}(\mathcal{O}_S, f_*(\mathcal{F}^\vee) \otimes \mathcal{G}) \longrightarrow \mathsf{Map}_{\mathsf{QCoh}(X)}(\mathcal{O}_X, \mathcal{F}^\vee \otimes f^*\mathcal{G})$$

is an equivalence. Using the fact that $f_*(\mathcal{F}^\vee)$ is perfect and hence dualizable, we see that the above morphism coincides with (2.3.4). This proves point (1).

We now turn to point (2). Since f is categorically proper and $T \to S$ is a map between affine derived schemes, we see that f' is again categorically proper. In particular, the previous point shows that f^* and f'^* admit left adjoints when restricted to the ∞-categories of perfect complexes. Let $\mathcal{F} \in \mathsf{Perf}(X)$ and consider the induced Beck-Chevalley transformation

$$f'_+ g'^*(\mathcal{F}) \longrightarrow g^* f_+(\mathcal{F}) .$$

Unraveling the definitions, we reduce to check that the map

$$g^* f_*(\mathcal{F}^\vee) \longrightarrow f'_* g'^*(\mathcal{F})$$

is an equivalence, which is true thanks to Proposition 2.3.6-(3). □

Combining all the results obtained so far, we obtain the following:

COROLLARY 2.3.28. *Let S be an affine derived scheme and let $f: X \to S$ be a morphism in* dSt. *Assume that:*

(1) *the morphism f is categorically proper and has finite local tor-amplitude;*

(2) *there exists an effective epimorphism $u: U \to X$, where U is a quasi-compact derived scheme.*

Then the mapping stack

$$\mathbf{Perf}_S(X) := \mathbf{Map}_S(X, \mathbf{Perf} \times S)$$

admits a global cotangent complex.

PROOF. Let $T \in \mathsf{dAff}_{/S}$ be an affine derived scheme over S and let

$$x: T \to \mathbf{Map}_S(X, \mathbf{Perf} \times S)$$

be a morphism. Consider the diagram

$$\begin{array}{ccccc} T \times_S X & \longrightarrow & X \times_S \mathbf{Map}_S(X, \mathbf{Perf} \times S) & \xrightarrow{\mathsf{ev}} & \mathbf{Perf} \\ \downarrow{p} & & \downarrow{q} & & \\ T & \xrightarrow{x} & \mathbf{Map}_S(X, \mathbf{Perf} \times S) & & \end{array},$$

where the vertical maps are the canonical projection maps. Let $\mathcal{E} \in \mathsf{Perf}(T \times_S X)$ be the perfect complex classified by x. Write

$$G := \Omega_x \mathbf{Map}_S(X, \mathbf{Perf} \times S)$$

for the loop stack of $\mathbf{Map}_S(X, \mathbf{Perf} \times S)$ at the point x. Combining Theorem 2.2.10 with Proposition 2.2.3-(3), we see that $\mathbf{Map}_S(X, \mathbf{Perf} \times S)$ is infinitesimally cohesive. Therefore, Proposition 2.2.4 shows that it is enough to prove that G admits a cotangent complex at the diagonal morphism $\delta_x \colon T \to G$.

Let $\mathcal{F} \in \mathsf{QCoh}(T)$. Unraveling the definitions, we see that

$$\mathsf{Der}_{G,\delta_x}(T; \mathcal{F}) \simeq \mathsf{Map}_{\mathsf{QCoh}(T \times_S X)}(\mathcal{E}, \mathcal{E} \otimes p^*(\mathcal{F})) \simeq \mathsf{Map}_{\mathsf{QCoh}(T \times_S X)}(\mathcal{E} \otimes \mathcal{E}^\vee, p^*(\mathcal{F})) ,$$

where we used the fact that $\mathcal{E}$ is dualizable. Using Proposition 2.3.16-(2), we see that $p \colon T \times_S X \to T$ is again categorically proper and has finite tor-amplitude. Furthermore, $T \times_S U \to T \times_S X$ is again an effective epirmorphism, and $T \times_S U$ is a quasi-compact derived scheme. In conclusion, the assumptions of Proposition 2.3.27 are satisfied. This supplies us with a natural equivalence

$$\mathsf{Map}_{\mathsf{QCoh}(T \times_S X)}(\mathcal{E} \otimes \mathcal{E}^\vee, p^*(\mathcal{F})) \simeq \mathsf{Map}_{\mathsf{QCoh}(T)}(p_+(\mathcal{E} \otimes \mathcal{E}^\vee), \mathcal{F}) ,$$

which implies that the cotangent complex of G at δ_x exists and it is given by $p_+(\mathcal{E} \otimes \mathcal{E}^\vee)$. In turn, this implies that the cotangent complex of $\mathbf{Map}_S(X, \mathbf{Perf} \times S)$ at x exists and it is given by

$$x^* \mathbb{L}_{\mathbf{Map}_S(X, \mathbf{Perf} \times S)} \simeq p_+(\mathcal{E} \otimes \mathcal{E}^\vee)[1] .$$

The fact that it is a global cotangent complex simply follows from Proposition 2.3.27-(2). $\square$

We conclude this section stating the following obvious but useful consequence of Proposition 2.3.27:

COROLLARY 2.3.29. *Let $f \colon X \to S$ be a morphism satisfying assumptions (1) and (2). Let*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ F & \longrightarrow & S \end{array}$$

be a pullback square in dSt. *Then:*

(1) the functor

$$g^* \colon \mathsf{Perf}(F) \longrightarrow \mathsf{Perf}(Y)$$

has a left adjoint $g_+ \colon \mathsf{Perf}(Y) \to \mathsf{Perf}(F)$.

(2) For every $T \in \mathsf{dAff}_{/F}$, *let*

$$\begin{array}{ccc} Y_T & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle g_T} & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{q} & F \end{array}$$

be induced pullback square. Then for every $\mathcal{F} \in \mathsf{Perf}(Y)$, *the Beck-Chevalley transformation*

$$g_{T+}(p^*(\mathcal{F})) \longrightarrow q^*(g_+(\mathcal{F}))$$

is an equivalence.

PROOF. Let $T \in \mathsf{dAff}_{/F}$ and let $f \colon S \to F$ be the structural morphism. Then the squares in the diagram

$$\begin{array}{ccccc} Y_T & \longrightarrow & Y & \longrightarrow & X \\ \downarrow{\scriptstyle g_T} & & \downarrow{\scriptstyle g} & & \downarrow \\ T & \longrightarrow & F & \longrightarrow & S \end{array}$$

are both pullbacks. Since the map $T \to S$ is representable by affine derived schemes, Proposition 2.3.16-(2) shows that g_T is of finite local tor-amplitude. It is furthermore categorically proper, and $U \times_S T \to Y_T$ is again an effective epimorphism, with $U \times_S T$ a derived quasi-compact scheme. Therefore Proposition 2.3.27-(1) shows that the functor

$$g_T^* \colon \mathsf{Perf}(T) \longrightarrow \mathsf{Perf}(Y_T)$$

admits a left adjoint g_{T+}, and point (2) of the same proposition shows that it is compatible with base change along morphisms $T' \to T$ in $\mathsf{dAff}_{/F}$. It therefore induces a well defined functor

$$g_+ \colon \mathsf{Perf}(Y \times F) \longrightarrow \mathsf{Perf}(F)\ ,$$

which is left adjoint to g^* and satisfies base-change against maps $S \to F$ from an affine derived scheme $S \in \mathsf{dAff}$ by construction. □

2.4. Morphisms locally almost of finite presentation

We start by recalling what it classically means for a morphism $X \to Y$ of derived stacks to be locally almost of finite presentation. To simplify later discussions, we introduce it for general derived prestacks. Recall that dPreSt simply denotes the category of presheaves $\mathsf{PSh}(\mathsf{dAff})$. We have:

DEFINITION 2.4.1 (cf. [Lur18, Definition 17.4.1.1]). A morphism $X \to Y$ in dPreSt is said to be *locally almost of finite presentation* if for every integer $n \geq 0$ and every cofiltered diagram $\{S_\alpha\}_{\alpha \in I}$ of n-truncated affine derived schemes, the square

$$\begin{array}{ccc} \operatorname{colim}_\alpha X(S_\alpha) & \longrightarrow & X(\lim_\alpha S_\alpha) \\ \downarrow & & \downarrow \\ \operatorname{colim}_\alpha Y(S_\alpha) & \longrightarrow & Y(\lim_\alpha S_\alpha) \end{array}$$

is a pullback. We say that a morphism of derived stacks $X \to Y \in \mathsf{dSt}$ is *locally almost of finite presentation* if its image in $\mathsf{PSh}(\mathsf{dAff})$ is. We say that a derived stack X is *locally almost of finite presentation* if the map $X \to \mathsf{Spec}(k)$ is locally almost of finite presentation. ⊘

The following are the basic properties of morphisms locally almost of finite presentation:

LEMMA 2.4.2.

(1) *Morphisms locally almost of finite presentation are closed under pullbacks.*

(2) *If $g \colon Y \to Z$ is locally almost of finite presentation, then a morphism $f \colon X \to Y$ is locally almost of finite presentation if and only if the composite $g \circ f$ is.*

(3) *Morphisms locally almost of finite presentation are closed under finite limits in $\mathsf{Fun}(\Delta^1, \mathsf{dSt})$ and under arbitrary colimits in $\mathsf{Fun}(\Delta^1, \mathsf{PSh}(\mathsf{dAff}))$.*

(4) *An affine derived scheme $X = \mathsf{Spec}(A)$ is locally almost of finite presentation if and only if $\pi_0(A)$ is of finite presentation as k-algebra and the homotopy groups $\pi_i(A)$ are finitely generated as $\pi_0(A)$-module.*

PROOF. The first two statements follow at once unraveling the definitions.For the third one, we first observe that a derived stack X is locally almost of finite presentation if and only if for every $n \geq 0$ and every cofiltered diagram $\{S_\alpha\}_{\alpha \in I}$ of n-truncated affine derived schemes, the natural map

$$\operatorname*{colim}_\alpha X(S_\alpha) \longrightarrow X(\lim_\alpha S_\alpha)$$

is an equivalence. As limits in dSt are computed objectwise and the above colimit is filtered, this shows that the first half of (3) is satisfied. For the second half, it is enough to recall that colimits in $\mathsf{PSh}(\mathsf{dAff})$ are computed objectwise. Finally, since the base ring k is noetherian, the last statement is a consequence of the derived Hilbert's basis theorem, see [Lur17, Proposition 7.2.4.31]. □

Let us record the following useful consequence of point (3) of the above lemma:

COROLLARY 2.4.3. *Let* $X, Y \in \mathsf{dSt}$ *be derived stacks. Assume that:*

(1) *X can be written as a finite colimit of affine schemes of finite tor-amplitude.*

(2) *Y is locally almost of finite presentation.*

Then $\mathbf{Map}(X, Y)$ *is locally almost of finite presentation.*

PROOF. Thanks to Lemma 2.4.2-(3) it is enough to assume that X itself is an affine scheme locally almost of finite presentation and of finite tor-amplitude. Let $\{S_\alpha\}_{\alpha \in I}$ be a cofiltered diagram of n-truncated affine derived schemes and let

$$S := \lim_{\alpha \in I} S_\alpha$$

be its limit. Since X has finite tor-amplitude, there exists $m \geq n$ such that $S_\alpha \times X$ is m-truncated for every $\alpha \in I$. Furthermore,

$$S \times X \simeq \lim_{\alpha \in I} S_\alpha \times X .$$

Since Y is locally almost of finite presentation, we see that the canonical map

$$\operatorname*{colim}_{\alpha \in I} \mathsf{Map}(S_\alpha \times X, Y) \longrightarrow \mathsf{Map}(S \times X, Y)$$

is an equivalence. This completes the proof. □

In virtue of the colimit part of Lemma 2.4.2-(3), it is relatively simple to guarantee that a derived prestack is locally almost of finite presentation. Combining it with point (4), we see that the functor

$$j_! \colon \mathsf{PSh}(\mathsf{dAff}^{\mathsf{aft}}) \longrightarrow \mathsf{dPreSt} ,$$

given by left Kan extension along the inclusion $j \colon \mathsf{dAff}^{\mathsf{aft}} \hookrightarrow \mathsf{dAff}$ lands in the (non full) subcategory of locally almost of finite presentation derived prestacks and morphisms pf locally almost of finite presentation between them. Unfortunately, colimits in dPreSt are not particularly useful in situations of geometric relevance.[4] For this reason, we investigate under which conditions the hypersheafification functor preserves the locally almost of finite presentation condition.

2.4.1. Truncated derived (pre)stacks. Recall from Section 2.2.2 the canonical inclusion

$$i_n \colon {}^{<n}\mathsf{dAff} \longrightarrow \mathsf{dAff} .$$

Equipping both sides with the étale topology, i_n becomes both continuous and cocontinuous. Write

$${}^{<n}\mathsf{dPreSt} := \mathsf{PSh}({}^{<n}\mathsf{dAff}) .$$

Since i_n is continuous, the commutative diagram

$$\begin{array}{ccc} {}^{<n}\mathsf{dPreSt} & \xrightarrow{i_{n!}} & \mathsf{dPreSt} \\ \uparrow & & \uparrow \\ {}^{<n}\mathsf{dSt} & \xrightarrow{i_{n!}} & \mathsf{dSt} \end{array}$$

is vertically left adjointable, see [PY16, Lemma 2.14]. On the other hand, since i_n is cocontinuous, we see that the above diagram is also horizontally right adjointable, see [PY16, Lemma 2.18]. In particular, if $F \in \mathsf{dPreSt}$ is a derived prestack and

$$F \longrightarrow L(F)$$

exhibit $L(F)$ as the hypersheafification of F, then the induced morphism

$${}^{<n}F \longrightarrow {}^{<n}L(F)$$

[4]For instance, the Yoneda embedding $\mathsf{dAff} \to \mathsf{dPreSt}$ does not even respect disjoint unions.

exhibit ${}^{<n}L(F)$ as the associated hypersheafification of ${}^{<n}F$.

We give the following definition:

DEFINITION 2.4.4. A collection of derived prestacks $\{F_\alpha\}_{\alpha\in I}$ is said to be *uniformly truncated* if for every $n \geq 0$ there exists an integer $m = m_n$ such that the derived prestack ${}^{<n}F_\alpha \in {}^{<n}\mathsf{dPreSt}$ takes values in $\mathcal{S}_{\leq m}$ for every $\alpha \in I$. We say that a derived prestack $F \in \mathsf{dPreSt}$ is *truncated* if the family $\{F\}$ is uniformly truncated. ⊘

EXAMPLE 2.4.5. Every geometric derived stack is truncated. △

LEMMA 2.4.6. *Let $f\colon X \to Y$ be a morphism locally almost of finite presentation between truncated derived prestacks. Then the induced morphism $\overline{f}\colon L(X) \to L(Y)$ between the associated derived stacks is locally almost of finite presentation.*

PROOF. Let $\{S_\alpha\}_{\alpha\in I}$ be a cofiltered diagram of n-truncated affine derived schemes and let

$$S \coloneqq \lim_{\alpha\in I} S_\alpha$$

be its limit. We have

$$\mathsf{Map}(S_\alpha, L(X)) \simeq \operatorname*{colim}_{U_\bullet \in \mathsf{HCov}(S_\alpha)} \lim_{[\ell]\in\mathbf{\Delta}} \mathsf{Map}(U_\ell, X) .$$

Since X is truncated, we can find an integer $m = m_n$ that only depends on n such that the canonical map

$$\lim_{[\ell]\in\mathbf{\Delta}} \mathsf{Map}(U_\ell, X) \longrightarrow \lim_{[\ell]\in\mathbf{\Delta}_{\leq m}} \mathsf{Map}(U_\ell, X)$$

is an equivalence (cf. [PY18, Corollary 9.4]). The latter limit being finite, we can therefore commute it with the filtered colimit over the hypercovers of S_α and with the filtered colimit over α. The conclusion follows. □

COROLLARY 2.4.7. *Let $X \in \mathsf{dSt}$ be a derived stack. Assume that:*

(1) it can be written as a colimit

$$X \simeq \operatorname*{colim}_i X_i$$

in dSt, where each X_i is a affine derived scheme locally almost of finite presentation.

(2) It is truncated.

Then X is locally almost of finite presentation.

PROOF. Combine Lemmas 2.4.2-(3) and 2.4.6. □

We now offer a variation on Corollary 2.4.3:

COROLLARY 2.4.8. *Let $X_\bullet\colon \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{dSt}$ be a simplicial derived stack and let*

$$X \coloneqq |X_\bullet|$$

be its geometric realization. Let $Y \in \mathsf{dSt}$ be a derived stack such that:

(1) the family $\{\mathbf{Map}(X_n, Y)\}_{[n]\in\mathbf{\Delta}}$ of derived (pre)stacks is uniformly truncated.

(2) For every $[n] \in \mathbf{\Delta}$, the mapping stack $\mathbf{Map}(X_n, Y)$ is locally almost of finite presentation.

Then $\mathbf{Map}(X, Y)$ is locally almost of finite presentation.

PROOF. Let $n \geq 0$ be an integer and let $\{S_\alpha\}_{\alpha\in I}$ be a cofiltered family of n-truncated affine derived schemes. Let

$$S \coloneqq \lim_{\alpha\in I} S_\alpha$$

be its limit. Let $m \geq 0$ be an integer such that $\mathsf{Map}(T \times X_\ell, Y) \in \mathcal{S}_{\leq m}$ for every n-truncated affine derived scheme T and every $[\ell] \in \mathbf{\Delta}$. Consider the commutative square

$$\begin{array}{ccc} \mathrm{colim}_{\alpha \in I} \lim_{[\ell]\in\mathbf{\Delta}} \mathsf{Map}(S_\alpha \times X_\ell, Y) & \longrightarrow & \lim_{[\ell]\in\mathbf{\Delta}} \mathsf{Map}(S \times X_\ell, Y) \\ \downarrow & & \downarrow \\ \mathrm{colim}_{\alpha \in I} \lim_{[\ell]\in\mathbf{\Delta}_{\leq m+2}} \mathsf{Map}(S_\alpha \times X_\ell, Y) & \longrightarrow & \lim_{[\ell]\in\mathbf{\Delta}_{\leq m+2}} \mathsf{Map}(S \times X_\ell, Y) \end{array} .$$

Using [PY18, Corollary 9.4], we see that the vertical maps are equivalences. As the category $\Delta_{\leq m+2}$ is finite and I is filtered, we can commute the limit and the colimit in the bottom left vertex of the above diagram. At that point, the conclusion follows from assumption (2). □

2.5. Integrability and formal GAGA

Recall the notion of integrable stack:

DEFINITION 2.5.1 (cf. [Lur18, Definition 17.3.4.1]). We say that a derived stack $X \in \mathsf{dSt}$ is *integrable* if for every local Noetherian derived ring A which is complete with respect to its maximal ideal $\mathfrak{m} \subset \pi_0(A)$, the inclusion $\mathsf{Spf}(A) \hookrightarrow \mathsf{Spec}(A)$ induces an equivalence

$$\mathsf{Map}_{\mathsf{dSt}}(\mathsf{Spec}(A), X) \longrightarrow \mathsf{Map}_{\mathsf{dSt}}(\mathsf{Spf}(A), X) .$$

Here $\mathsf{Spf}(A) \in \mathsf{dSt}$ denotes the formal spectrum of A with respect to $\mathfrak{m}$ (see [Lur18, Construction 8.1.1.10 and Theorem 8.1.5.1]). ⊘

DEFINITION 2.5.2. We say that a derived stack $X \in \mathsf{dSt}$ satisfies the *universal formal GAGA property* if the derived stack

$$\mathbf{Perf}(X) := \mathbf{Map}(X, \mathbf{Perf})$$

is integrable. ⊘

We have the following formal properties:

PROPOSITION 2.5.3.

(1) *The full subcategory of* dSt *spanned by integrable stacks is closed under (small) limits.*

(2) *The full subcategory of* dSt *spanned by derived stacks satisfying the universal formal GAGA property is closed under (small) colimits.*

PROOF. Point (2) is a direct consequence of point (1). For point (1), let A be a local Noetherian derived ring which is complete with respect to its maximal ideal $\mathfrak{m} \subset \pi_0(A)$. Let $X = \lim_\alpha X_\alpha$ be a derived stack and assume that every X_α is integrable. Consider the square

$$\begin{array}{ccc} \mathsf{Map}_{\mathsf{dSt}}(\mathsf{Spec}(A), X) & \longrightarrow & \lim_\alpha \mathsf{Map}_{\mathsf{dSt}}(\mathsf{Spec}(A), X_\alpha) \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathsf{dSt}}(\mathsf{Spf}(A), X) & \longrightarrow & \lim_\alpha \mathsf{Map}_{\mathsf{dSt}}(\mathsf{Spf}(A), X_\alpha) \end{array} .$$

The horizontal maps are equivalences by the Yoneda lemma and the right vertical map is an equivalence because every X_α is integrable. Therefore the left vertical map is an equivalence as well. □

2.6. Representability

Artin-Lurie's representability theorem implies[5]:

THEOREM 2.6.1. *Let $X \in \mathsf{dSt}$ be a derived stack. Assume that:*

[5] See also [HLP14, Theorem 5.1.1].

(1) X is categorically proper;

(2) X is of local finite tor-amplitude;

(3) there exists a flat effective epimorphism $U \to X$, where U is a quasi-compact derived scheme;

(4) X satisfies universal formal GAGA property;

(5) The derived stack $\mathbf{Perf}(X)$ is locally almost of finite presentation.

Then $\mathbf{Perf}(X)$ is locally geometric.

PROOF. We check that the assumptions of Artin-Lurie's representability theorem 2.1.1 are satisfied.

Since **Perf** is infinitesimally cohesive and nilcomplete by Theorem 2.2.10, Proposition 2.2.3-(1) and (3), and Proposition 2.2.9-(2) and (4) imply, respectively, that $\mathbf{Perf}(X)$ is infinitesimally cohesive and nilcomplete. Thanks to the assumptions (1), (2) and (3), Corollary 2.3.28 implies that $\mathbf{Perf}(X)$ admits a global cotangent complex. Thus, the point (3) of the Artin-Lurie theorem is satisfied.

Thanks to the assumptions (2) and (3), Proposition 2.3.17 guarantees that $\mathbf{Perf}(X)$ is truncated. Thus, the point (1) of the Artin-Lurie theorem is satisfied. Assumption (4) translates to say that $\mathbf{Perf}(X)$ is integrable, thus (4) of the Artin-Lurie theorem is satisfied. Since by hypothesis, $\mathbf{Perf}(X)$ is locally almost of finite presentation, the conclusion follows. □

REMARK 2.6.2. Using Corollaries 2.4.3 and 2.4.8 one can often verify in practice that $\mathbf{Perf}(X)$ is locally almost of finite presentation. This method works particularly well when X is a geometric stack. In later chapters, we will need an additional effort to check that this assumption is met for X_{dR} and X_{Dol}. △

CHAPTER 3

Betti shape

3.1. Definition and geometrical properties

The Betti shape encodes the theory of local systems, and it can more generally be attached to any space $K \in \mathcal{S}$. The obvious functor $\pi \colon \mathsf{dAff} \to *$ induces an adjunction

$$\pi^* \colon \mathcal{S} \leftrightarrows \mathsf{dSt} \colon \pi_* ,$$

where π_* sends $F \in \mathsf{dSt}$ to $F(\mathbb{C})$ and π^* takes $K \in \mathcal{S}$ to the (étale) sheafification of the constant presheaf associated to K. We write

$$K_{\mathsf{B}} \coloneqq \pi^*(K) \in \mathsf{dSt} ,$$

and we refer to K_{B} as the *Betti shape of K* (or *Betti stack* of K).

Working over the complex numbers, there is a natural functor

$$(-)^{\mathsf{htop}} \colon \mathsf{dSch}^{\mathsf{laft}} \longrightarrow \mathcal{S}$$

sending a (derived) $\mathbb{C}$-scheme locally almost of finite presentation to the underlying homotopy type of its analytification (cf. [Por15, §4] for the analytification in the derived setting). It is insensitive to the derived structure, in the sense that the canonical map ${}^{\mathsf{cl}}(X) \to X$ induces an equivalence $({}^{\mathsf{cl}}(X))^{\mathsf{htop}} \simeq X^{\mathsf{htop}}$. We commit abuse of notation and define

$$X_{\mathsf{B}} \coloneqq (X^{\mathsf{htop}})_{\mathsf{B}} \in \mathsf{dSt} .$$

We refer to this stack as the *Betti shape* of X (or *Betti stack of* X). It is well known that if X is quasi-compact and locally almost of finite presentation, then X^{htop} is a finite space. Therefore, all the results in what follows apply to this case. As working with Betti stacks arising from schemes leads to no simplification, we offer a general treatment.

The basic properties of K_{B} can be summarized as follows:

PROPOSITION 3.1.1.

(1) *There is a canonical equivalence* $*_{\mathsf{B}} \simeq \mathsf{Spec}(\mathbb{C})$.

(2) *For any* $\mathsf{Spec}(A) \in \mathsf{dAff}$, *there is a symmetric monoidal equivalence of stable ∞-categories*

$$\mathsf{QCoh}(K_{\mathsf{B}} \times \mathsf{Spec}(A)) \simeq \mathsf{Fun}(K, A\text{-}\mathsf{Mod}) .$$

This equivalence restricts to an equivalence

$$\mathsf{Vect}_n(K_{\mathsf{B}} \times \mathsf{Spec}(A)) \simeq \mathsf{Fun}(K, \mathsf{Vect}_n(A)) .$$

(3) *Let $K \in \mathcal{S}$ be a homotopy type and let $x \colon * \to K$ be a point. The induced map*

$$\mathsf{Spec}(\mathbb{C}) \simeq *_{\mathsf{B}} \longrightarrow X_{\mathsf{B}}$$

is universally flat. If moreover K is connected, then this map is an effective epimorphism as well.

(4) *Let $K \in \mathcal{S}^{\mathsf{fin}}$ be a finite homotopy type. Then K_{B} is universally categorically proper and universally of global tor-amplitude ≤ 0.*

PROOF. The functor π^* is part of a geometric morphism of ∞-topoi and therefore commutes with finite limits. In particular, it preserves the initial object. This proves (1). To prove point (2), we first observe that if $K = \emptyset$ there is nothing to say. Otherwise, we write K as colimit of a tower

$$K_0 \longrightarrow K_1 \longrightarrow \cdots K_n \longrightarrow K_{n+1} \longrightarrow \cdots ,$$

where K_0 is discrete and for each $n \geq 0$ the map $K_n \to K_{n+1}$ fits in a pushout square

$$\begin{array}{ccc} \coprod_{I_n} S^n & \longrightarrow & \coprod_{I_n} * \\ \downarrow & & \downarrow \\ K_n & \longrightarrow & K_{n+1} \end{array} ,$$

Since π^* commutes with arbitrary colimits and $\mathsf{QCoh}(-)$ and $\mathsf{Fun}(-, A\text{-}\mathsf{Mod})$ commute with colimits in their variables (compatibly with symmetric monoidal structures), we are readily reduced to prove the statement when $K = S^n$ is the n-sphere or $K = *$. In the latter case, the statement follows from point (1). As for the spheres, using the fact that $S^n \simeq \Sigma(S^{n-1})$ for $n \geq 0$, we further reduce to prove the statement for $K = \emptyset$, in which case it is obvious.

We now prove (3). If K is connected, then [Lur09, Proposition 7.2.1.14] immediately shows that the induced map $\mathsf{Spec}(\mathbb{C}) \to K_{\mathrm{B}}$ is an effective epimorphism. As for the universal flatness, first we observe that since the n-sphere S^n is connected for $n > 0$, we can always factor (up to homotopy) the map $x \colon * \to K$ through the map $K_0 \to K$. Denote the induced map $x_0 \colon * \to K_0$. Since K_0 is discrete, we identify it to a set and we observe that the map

$$\mathsf{Fun}(K_0, A\text{-}\mathsf{Mod}) \simeq \prod_{y \in K_0} A\text{-}\mathsf{Mod} \longrightarrow A\text{-}\mathsf{Mod}$$

induced by x_0 corresponds to the projection on the x_0-factor. In particular, it is t-exact. Proceeding by induction, it is enough to prove that for every $n \geq 1$ the map

$$\mathsf{Fun}(K_{n+1}, A\text{-}\mathsf{Mod}) \longrightarrow \mathsf{Fun}(K_n, A\text{-}\mathsf{Mod})$$

is t-exact. Using once again the t-exactness of the projections from the product, we reduce ourselves to the case where $K_n \to K_{n+1}$ fits in a pushout

$$\begin{array}{ccc} S^n & \longrightarrow & * \\ \downarrow & & \downarrow \\ K_n & \longrightarrow & K_{n+1} \end{array} .$$

This induces to the pullback

$$\begin{array}{ccc} \mathsf{Fun}(K_{n+1}, A\text{-}\mathsf{Mod}) & \longrightarrow & \mathsf{Fun}(K_n, A\text{-}\mathsf{Mod}) \\ \downarrow & & \downarrow \\ A\text{-}\mathsf{Mod} & \longrightarrow & \mathsf{Fun}(S^n, A\text{-}\mathsf{Mod}) \end{array} .$$

Using [HPV16, Lemma 3.20], it is enough to prove that the bottom and the right vertical maps are t-exact. Choose a map $* \to S^n$. As $n \geq 1$, this map is an effective epimorphism, and in particular the induced map

$$\mathsf{Fun}(S^n, A\text{-}\mathsf{Mod}) \longrightarrow A\text{-}\mathsf{Mod}$$

is conservative. It is therefore enough to prove that it is t-exact as well, for the composite

$$\mathsf{Fun}(K_n, A\text{-}\mathsf{Mod}) \longrightarrow \mathsf{Fun}(S^n, A\text{-}\mathsf{Mod}) \longrightarrow A\text{-}\mathsf{Mod}$$

is flat by inductive hypothesis. Using the relation $S^n \simeq \Sigma(S^{n-1})$, we can therefore reduce to the case of S^0, which is a particular case of the discussion for K_0 we already gave.

We finally prove (4). Let $A \in \mathrm{CAlg}$ be a derived ring and let $S = \mathrm{Spec}(A)$. Let $p \colon K_{\mathrm{B}} \times S \to S$ be the natural projection. We have to prove that

$$p_* \colon \mathrm{QCoh}(K_{\mathrm{B}} \times S) \longrightarrow \mathrm{QCoh}(S)$$

has finite cohomological dimension and tor-amplitude ≤ 0. Since K is finite, we have $K = K_n$ and each map $K_i \to K_{i+1}$ is obtained by attaching a finite number of cells. We proceed once again by induction on n. When $n = 0$, $K = K_0$ is (equivalent to) a finite set. Let $m = |K_0|$ be its cardinality, so we have

$$\mathrm{Fun}(K_0, A\text{-}\mathrm{Mod}) \simeq \prod_{K_0} A\text{-}\mathrm{Mod} \simeq A^m\text{-}\mathrm{Mod} \, .$$

The pushforward coincides with the forgetful functor along the natural ring map $A \to A^m$, which is finite and flat. Therefore, p_* has cohomological dimension 0 and tor-amplitude ≤ 0. Assume now that the statement holds true for K_n and that K_{n+1} is obtained by K_n by attaching a single cell:

$$\begin{array}{ccc} S^\ell & \longrightarrow & * \\ \downarrow & & \downarrow \\ K_n & \longrightarrow & K_{n+1} \end{array} \, .$$

Using [PY16, § 8.2] and the fact that bounded A-modules and A-modules of tor-amplitude ≤ 0 are closed under finite limits, we reduce ourselves to prove the same statement for S^n. Using the relation $S^\ell \simeq \Sigma(S^{\ell-1})$, we further reduce ourselves to the case $\ell = 0$, where the statement follows from what we already discussed in the case $K = K_0$. □

REMARK 3.1.2. Let A be an underived commutative ring and let $K \in \mathcal{S}$ be a space. Then Proposition 3.1.1-(2) provides a canonical equivalence

$$\mathrm{Vect}_n(K_{\mathrm{B}} \times \mathrm{Spec}(A)) \simeq \mathrm{Fun}(K, \mathrm{Vect}_n(A)) \, .$$

As A is underived, $\mathrm{Vect}_n(A)$ is a 1-category. Therefore, we obtain

$$\mathrm{Vect}_n(K_{\mathrm{B}} \times \mathrm{Spec}(A)) \simeq \mathrm{Fun}(\tau_{\leq 1} K, \mathrm{Vect}_n(A)) \, .$$

In other words, we can identify rank n vector bundles on $K_{\mathrm{B}} \times \mathrm{Spec}(A)$ with representations of the fundamental groupoid of K. △

The above remark shows that K_{B} encodes the theory of local systems in a rather combinatorial way. We now make the link with the theory of locally constant sheaves. For this we suppose that K is the homotopy type of a topological space X. Let $\mathrm{Opens}(X)$ be the poset of open subsets of X. Consider the functor

$$(-)^{\mathrm{htop}} \colon \mathrm{Opens}(X) \longrightarrow \mathcal{S}$$

sending $U \in \mathrm{Opens}(X)$ to the homotopy type U^{htop}. The colimit of this functor is X^{htop}. Let $X^{\mathrm{htop}}_{\mathrm{lax}}$ be the lax colimit of the same functor. We have a canonical zig-zag

$$\begin{array}{ccc} X^{\mathrm{htop}}_{\mathrm{lax}} & \xrightarrow{\phi} & X^{\mathrm{htop}} \\ \downarrow{\scriptstyle \psi} & & \\ \mathrm{Opens}(X)^{\mathrm{op}} & & \end{array} \, ,$$

where ϕ is an ∞-categorical localization and ψ is a right fibration[1]. The following is a simplified version of [Lur17, Theorems A.1.15 and A.4.19]:

[1] It is the right fibration classifying the functor $(-)^{\mathrm{htop}}$.

PROPOSITION 3.1.3. *Let $\mathcal{E}$ be a presentable ∞-category. Assume that X is locally contractible. Then the functor*

$$\psi_! \circ \phi^* \colon \mathsf{Fun}(X^{\mathsf{htop}}, \mathcal{E}) \longrightarrow \mathsf{Fun}(\mathsf{Opens}(X)^{\mathsf{op}}, \mathcal{E})$$

is fully faithful and its essential image consists of locally constant sheaves. Here ϕ^* denotes the precomposition with ϕ and $\psi_!$ denotes the left Kan extension along ψ.

PROOF. We first observe that an object in $X^{\mathsf{htop}}_{\mathsf{lax}}$ corresponds to a pair (U, x), where U is an open in X and $x \in U$ is a point. The space of morphisms from (U, x) to (V, y) is empty if $V \not\subset U$ and it coincides with $\mathsf{Paths}_V(x, y)$ otherwise. Every morphism in $X^{\mathsf{htop}}_{\mathsf{lax}}$ is ψ-cartesian.

Let $F \colon \mathsf{Opens}(X)^{\mathsf{op}} \to \mathcal{E}$ be a functor. Let U be an open in X and set

$$X_{U/} := X^{\mathsf{htop}}_{\mathsf{lax}} \times_{\mathsf{Opens}(X)^{\mathsf{op}}} \mathsf{Opens}(X)^{\mathsf{op}}_{U/} \ .$$

By definition, $X_{U/}$ is the full subcategory of $X^{\mathsf{htop}}_{\mathsf{lax}}$ spanned by objects (V, x) such that $U \subseteq V$. We have

$$\psi_! \psi^*(F)(U) \simeq \operatorname*{colim}_{(V,x) \in X_{U/}} F(V) \ .$$

Assume that $U \subset V$ is the inclusion of two contractible open subsets of X. Then the induced map

$$X_{U/} \longrightarrow X_{V/}$$

is final, and in particular the canonical map

$$\psi_! \psi^*(F)(V) \longrightarrow \psi_* \psi^*(F)(U)$$

is an equivalence. In other words, $\psi_! \psi^*(F)$ is locally constant.

This shows that if the map

$$\psi_! \psi^*(F) \longrightarrow F$$

is an equivalence, then F is locally constant. Assume vice-versa that F is locally constant and let U be a contractible open. Let $x \in U$ be a point and consider the diagram

$$\begin{array}{ccc} \psi_! \psi^*(F)(U) & \longrightarrow & \operatorname{colim}_{x \in V \subseteq U} \psi_! \psi^*(F)(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & \operatorname{colim}_{x \in V \subseteq U} F(V) \end{array} \ ,$$

where the colimits are taken over the categories of contractible open neighborhoods of x inside U. The horizontal arrows are equivalences because F and $\psi_! \psi(F)$ are locally constant. Furthermore, it is easily checked that the right vertical map is an equivalence. It follows that the left vertical map is an equivalence as well. Since X is locally contractible, it has a basis of contractible open neighborhoods, and therefore it follows that $\psi_! \psi^*(F) \to F$ is an equivalence.

This shows that the functor

$$\psi^* \colon \mathsf{Fun}(\mathsf{Opens}(X)^{\mathsf{op}}, \mathcal{E}) \longrightarrow \mathsf{Fun}(X^{\mathsf{htop}}_{\mathsf{lax}}, \mathcal{E})$$

is fully faithful once restricted to the full subcategory of locally constant sheaves. Furthermore, its essential image consists of those functors $F \colon X^{\mathsf{htop}}_{\mathsf{lax}} \to \mathcal{E}$ that take every morphism in $X^{\mathsf{htop}}_{\mathsf{lax}}$ in equivalences of $\mathcal{E}$. Indeed, if F is locally constant, then for every morphism $(U, x) \to (V, y)$ such that $V \subset U$ is an inclusion of contractible opens, then the induced morphism

$$\psi^*(F)(U, x) \longrightarrow \psi^*(F)(V, y)$$

is equivalent to $F(U) \to F(V)$, which is an equivalence. As X is locally contractible, every morphism $(U, x) \to (V, y)$ can be written as composition of morphisms of the previous form, whence the conclusion.

On the other hand ϕ is the ∞-categorical localization of $X^{\mathsf{htop}}_{\mathsf{lax}}$ at cartesian morphisms of $X^{\mathsf{htop}}_{\mathsf{lax}}$. Since every morphism in $X^{\mathsf{htop}}_{\mathsf{lax}}$ is cartesian, the proof is achieved. □

In particular, this allows us to show that coherent cohomology of K_{B} recovers singular cohomology of K:

COROLLARY 3.1.4. *Let X be a locally contractible topological space and let $K := X^{\mathsf{htop}}$ be its homotopy type. Then:*

(1) *via the equivalence supplied by Proposition 3.1.3, the structure sheaf*
$$\mathcal{O}_{K_{\mathrm{B}}} \in \mathsf{QCoh}(K_{\mathrm{B}}) \simeq \mathsf{Fun}(K, \mathsf{Mod}_{\mathbb{C}})$$
corresponds to the constant sheaf $\underline{\mathbb{C}}_X$ on X.

(2) *Let $\mathcal{F} \in \mathsf{QCoh}(K_{\mathrm{B}})$ and let F be the locally constant sheaf corresponding to $\mathcal{F}$ via Proposition 3.1.3. Then there is a canonical equivalence*
$$\mathrm{R}\Gamma(K_{\mathrm{B}}; \mathcal{F}) \simeq \mathrm{R}\Gamma(X; F) .$$
In particular, there is a canonical equivalence
$$\mathrm{R}\Gamma(K_{\mathrm{B}}; \mathcal{O}_{K_{\mathrm{B}}}) \simeq \mathrm{R}\Gamma(X; \underline{\mathbb{C}}_X) .$$
In other words, the derived global sections of $\mathcal{O}_{K_{\mathrm{B}}}$ compute singular cohomology of X.

PROOF. When X is contractible, statement (1) is a consequence of Proposition 3.1.1-(1). The general case follows by descent along a covering of X made by contractible open subsets. As for the second statement, let
$$\mathfrak{F} := \phi^*(\mathcal{F}) .$$
Then Proposition 3.1.3 provides canonical equivalences
$$\psi_!(\mathfrak{F}) \simeq F , \quad \phi_!(\mathfrak{F}) \simeq \mathcal{F} .$$
Consider the canonically commutative diagram
$$\begin{array}{ccc} X^{\mathsf{htop}}_{\mathsf{lax}} & \xrightarrow{\phi} & X^{\mathsf{htop}} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle q} \\ \mathsf{Opens}(X)^{\mathsf{op}} & \xrightarrow{p} & * \end{array} .$$
Then
$$\mathrm{R}\Gamma(K_{\mathrm{B}}, \mathcal{F}) \simeq q_!(\mathcal{F}) , \quad \mathrm{R}\Gamma(X; F) \simeq p_!(F) .$$
The conclusion therefore follows from the functoriality of left Kan extensions, $p_! \circ \psi_! \simeq q_! \circ \phi_!$. □

3.2. Representability of the stack of perfect complexes

PROPOSITION 3.2.1. *Let $K \in \mathcal{S}^{\mathsf{fin}}$ be a finite space. Then* $\mathbf{Perf}(K_{\mathrm{B}})$ *is a locally geometric stack, locally of finite presentation.*

PROOF. When K is discrete, Proposition 3.1.1-(1) shows that $K_{\mathrm{B}} \simeq \mathsf{Spec}(\mathbb{C})^{\amalg n}$, where $n = |\pi_0(K)|$. Therefore $\mathbf{Perf}(K_{\mathrm{B}}) \simeq \mathbf{Perf}^n$. In the general case, one describes K starting from a finite discrete set attaching a finite number of cells. Since locally geometric stacks, locally of finite presentation are closed under finite limits, the conclusion follows. □

REMARK 3.2.2. Remark 3.1.2 implies that if K is connected, then the truncation of the derived stack $\mathbf{Vect}_n(K_{\mathrm{B}})$ coincides with the usual stack of n-dimensional representations of $\pi_1(K)$. △

CHAPTER 4

De Rham shape

4.1. Definition and geometrical properties

Let $i\colon \mathsf{Aff}^{\mathsf{red}} \hookrightarrow \mathsf{dAff}$ be the inclusion of the full subcategory of dAff spanned by reduced affine schemes. This is a cocontinuous morphism of sites (for the étale topology) and therefore it induces an adjunction

$$i^*\colon \mathsf{dSt} \leftrightarrows \mathsf{Sh}(\mathsf{Aff}^{\mathsf{red}}, \tau_{\text{ét}})\colon i_* ,$$

where i_* denotes the right Kan extension along i. The *de Rham functor* is the endofunctor

$$(-)_{\mathsf{dR}} \coloneqq i_* \circ i^* \colon \mathsf{dSt} \longrightarrow \mathsf{dSt} .$$

For a derived stack F, we refer to F_{dR} as its *de Rham shape*. Unraveling the definitions, we see that

$$F_{\mathsf{dR}}(\mathsf{Spec}(A)) \simeq F(\mathsf{Spec}(\pi_0(A)^{\mathsf{red}})) .$$

We denote the unit transformation of the adjunction $i^* \dashv i_*$ by

$$\lambda_F\colon F \longrightarrow F_{\mathsf{dR}} .$$

We collect in the following proposition the most basic properties of F_{dR}:

PROPOSITION 4.1.1.

(1) *For every derived stack $F \in \mathsf{dSt}$, the cotangent complex of F_{dR} exists and it is zero. Moreover, if F is a derived Artin stack, the canonical map $\lambda_F\colon F \to F_{\mathsf{dR}}$ is formally étale.*

(2) *Let $f\colon X \to Y$ be a morphism of derived schemes locally almost of finite presentation over $\mathbb{C}$. Then $X \times_{X_{\mathsf{dR}}} Y_{\mathsf{dR}}$ is canonically equivalent to the derived formal completion of the graph $\Gamma_f\colon X \to X \times Y$. In particular, if f is an lci closed immersion of classical schemes, then $X \times_{X_{\mathsf{dR}}} Y_{\mathsf{dR}}$ is equivalent to the usual formal completion of X along Y[1].*

(3) *If X is a scheme locally almost of finite presentation over $\mathbb{C}$. Then the map $\lambda_X\colon X \to X_{\mathsf{dR}}$ is an effective epimorphism if and only if X is smooth.*

(4) *Let X be a smooth scheme over $\mathbb{C}$. Then the map $\lambda_X\colon X \to X_{\mathsf{dR}}$ is universally flat.*

(5) *Let X be an underived scheme almost of finite presentation over $\mathbb{C}$. Then there is a canonical equivalence of $\mathsf{QCoh}(X_{\mathsf{dR}})$ with the derived ∞-category of $\mathcal{D}_X$-modules.*

(6) *Let X be a smooth and proper scheme over $\mathbb{C}$. Then X_{dR} is universally categorically proper and universally of global tor-amplitude ≤ 0.*

PROOF. Statement (1) just follows from the definitions (see [CPT+17, Lemma 2.1.10]). The first part of statement (2) is [CPT+17, Proposition 2.1.8]. The second part follows from the fact that for lci closed embeddings there is no difference between derived completion and the usual completion. This can be for instance deduced from [Bha12, Example 4.5 and Proposition 4.16].

[1] The lci assumption is necessary.

Statement (3) is equivalent to say that for every affine derived scheme $S \in \mathsf{dAff}$ the lifting problem

$$\begin{array}{ccc} S^{\mathrm{red}} & \longrightarrow & X \\ \downarrow & \nearrow & \\ S & & \end{array}$$

admits a solution étale locally on S. This is equivalent to the infinitesimal lifting property of X, i.e. to its formal smoothness. The conclusion follows because X is almost of finite presentation.

For statement (4), consider the Čech nerve

$$\widehat{X}^\bullet_\Delta \coloneqq \check{\mathcal{C}}(X \xrightarrow{\lambda_X} X_{\mathrm{dR}}) .$$

As the notation suggests, one can identify the components of this Čech nerve with the formal completions of X^n along its main diagonal. Moreover, [GR14, Proposition 3.4.3] guarantees that for every $S \in \mathsf{dAff}$ the canonical map

$$\mathsf{QCoh}(X_{\mathrm{dR}} \times S) \longrightarrow \lim \mathsf{QCoh}(\widehat{X}^\bullet_\Delta \times S)$$

is an equivalence. Since X is noetherian, the transition maps in this limit are flat. Therefore, [HPV16, Lemma 3.20] shows that $\mathsf{QCoh}(X_{\mathrm{dR}} \times S)$ inherits a t-structure characterized by the fact that the pullback to $X \times S$ via $\lambda_X \times \mathrm{id}_S$ is t-exact. As X is smooth, point (3) guarantees that this is the canonical t-structure on $\mathsf{QCoh}(X_{\mathrm{dR}} \times S)$.

Statement (5) follows from [GR17, §4.4.1.4]. Now let $S \in \mathsf{dAff}$ be an affine derived scheme and let $q_S \colon X_{\mathrm{dR}} \times S \to S$ be the canonical projection. Passing to right adjoints in [GR17, Lemma 4.4.1.6][2], we obtain a canonical equivalence

$$q_{S*}(\mathcal{F}) \simeq p_*((\lambda_X \times \mathrm{id}_S)^*\mathcal{F} \otimes_{\mathcal{O}_{X\times S}} |\mathrm{DR}(X \times S/S)|) .$$

Here $|\mathrm{DR}(X \times S/S)|$ is the realization of the mixed de Rham algebra of $X \times S$ relative to S. Since X is smooth, we can simply identify it with the complex

$$\mathcal{O}_{X\times S} \xrightarrow{\mathrm{d_{dR}}} \Omega^1_{X\times S/S} \xrightarrow{\mathrm{d_{dR}}} \cdots \xrightarrow{\mathrm{d_{dR}}} \Omega^n_{X\times S/S} \quad .$$

Since X is of dimension n, the spectral sequence for descent implies that X_{dR} has finite cohomological dimension. Moreover, since X is proper, we see that $p_*(\mathcal{F} \otimes_{\mathcal{O}_{X\times S}} \Omega^i_{X\times S/S})$ has coherent cohomology. Therefore, the spectral sequence for descent implies once again that X_{dR} is universally categorically proper. Observe now that since X is underived, $(\lambda_X \times \mathrm{id}_S)^*\mathcal{F}$ belongs to $\mathsf{Coh}^{\mathrm{b}}(X \times S)$. This together with the derived base change immediately implies that X_{dR} has universally global tor-amplitude ≤ 0. This proves statement (6). □

In the middle of the proof of statement (6), we used the following result, which we extract:

COROLLARY 4.1.2. *Let X be a smooth and proper scheme over $\mathbb{C}$. Then $\mathrm{R}\Gamma(X_{\mathrm{dR}}; \mathcal{O}_{X_{\mathrm{dR}}})$ is canonically equivalent to the hypercohomology of the algebraic de Rham complex of X.*

4.2. Representability of the stack of perfect complexes

COROLLARY 4.2.1. *Let X be a smooth and proper scheme over $\mathbb{C}$. Then $\mathbf{Perf}(X_{\mathrm{dR}})$ is a locally geometric stack.*

PROOF. This is a consequence of Theorem 2.6.1 and Proposition 4.1.1-(6). □

REMARK 4.2.2. An alternative way of proving the geometricity of $\mathbf{Perf}(X_{\mathrm{dR}})$ is to combine Simpson's proof of the geometricity of the corresponding underived stack [Sim09], Proposition 4.1.1-(6) which implies the existence of the cotangent complex for $\mathbf{Perf}(X_{\mathrm{dR}})$, and the easier version of Lurie's representability theorem [Lur18, Theorem 18.1.0.2].

[2] See also [Bha12, Corollary 4.30] and [CPT+17, Proposition 2.2.3].

CHAPTER 5

Dolbeault shape

5.1. Definition

Let X be a geometric derived stack. The Dolbeault stack of X is defined as follows: let

$$\mathsf{T}X := \mathsf{Spec}_X(\mathsf{Sym}_{\mathcal{O}_X}(\mathbb{L}_X))$$

be the derived tangent bundle to X. It is a linear stack (cf. Definition A.2.1).

Let $\widehat{\mathsf{T}X} := X_{\mathsf{dR}} \times_{(\mathsf{T}X)_{\mathsf{dR}}} \mathsf{T}X$ be the formal completion of $\mathsf{T}X$ along the zero section. Using [Lur17, 4.2.2.9] we can convert the natural commutative group structure of $\mathsf{T}X$ relative to X (seen as an associative one) into a simplicial diagram $\mathsf{T}^\bullet X \colon \mathbf{\Delta}^{\mathsf{op}} \to (\mathsf{dSt})_{/X}$. Unwinding the definitions, we see that $\mathsf{T}^\bullet X$ can be identified with the n-fold product $\mathsf{T}X \times_X \cdots \times_X \mathsf{T}X$. The zero section $X \to \mathsf{T}X$ allows to promote $\mathsf{T}^\bullet X$ to a simplicial diagram

$$\mathsf{T}^\bullet X \colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow (\mathsf{dSt})_{X//X} .$$

Formal completion along the natural maps $X \to \mathsf{T}^n X$ provides us with a new simplicial object

$$\widehat{\mathsf{T}^\bullet X} \colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow (\mathsf{dSt})_{/X} .$$

The *Dolbeault shape of* X is the geometric realization

$$X_{\mathsf{Dol}} := \left| \widehat{\mathsf{T}^\bullet X} \right| \in (\mathsf{dSt})_{/X} ,$$

while the *nilpotent* Dolbeault shape of X is the geometric realization

$$X_{\mathsf{Dol}}^{\mathsf{nil}} := |\mathsf{T}^\bullet X| \in (\mathsf{dSt})_{/X} .$$

REMARK 5.1.1. Note that X_{Dol} coincides with the relative classifying stack $\mathcal{B}_X\widehat{\mathsf{T}X}$, while $X_{\mathsf{Dol}}^{\mathsf{nil}} \simeq \mathcal{B}_X\mathsf{T}X$. △

We let

$$\kappa_X \colon X \longrightarrow X_{\mathsf{Dol}} \quad \text{and} \quad \kappa_X^{\mathsf{nil}} \colon X \longrightarrow X_{\mathsf{Dol}}^{\mathsf{nil}}$$

be the natural maps. In addition, $\kappa_X^{\mathsf{nil}} = \imath_X \circ \kappa_X$, where $\imath_X \colon X_{\mathsf{Dol}} \to X_{\mathsf{Dol}}^{\mathsf{nil}}$ is the canonical map induced by $\widehat{\mathsf{T}^\bullet X} \to \mathsf{T}^\bullet X$.

PROPOSITION 5.1.2. *Let $X \in \mathsf{dSt}$ be a derived stack for which there exists a cotangent complex $\mathbb{L}_X$, which is dualizable, that is $\mathbb{L}_X \in \mathsf{Perf}(X)$. We denote by $\mathbb{T}_X$ the dual of $\mathbb{L}_X$. Then we have*

$$\mathsf{QCoh}(X_{\mathsf{Dol}}) \simeq \mathsf{Mod}_{\mathsf{Sym}_{\mathcal{O}_X}(\mathbb{T}_X)}(\mathsf{QCoh}(X)) .$$

In particular, $\mathsf{QCoh}(X_{\mathsf{Dol}})^\heartsuit$ is the category of Higgs sheaves on X.

PROOF. Note that by definition, we have

$$\mathsf{QCoh}(X_{\mathsf{Dol}}) \simeq \mathsf{Mod}_{\widehat{\mathsf{Sym}_{\mathcal{O}_X}}(\mathbb{L}_X)^\vee}(\mathsf{QCoh}(X)) .$$

Now, since $\widehat{\mathsf{Sym}_{\mathcal{O}_X}}(\mathbb{L}_X) \simeq \mathsf{Sym}_{\mathcal{O}_X}(\mathbb{T}_X)^\vee$, we obtain the assertion. □

5.2. The BNR correspondence for perfect complexes

In this section, we shall provide a version of the Beauville-Narasimhan-Ramanan correspondence, in the sense of Simpson [Sim94, Lemma 6.8], for perfect complexes.

DEFINITION 5.2.1. Let $p\colon X \to Y$ be a morphism of derived schemes. Let $\mathcal{F} \in \mathsf{Perf}(X)$ be a perfect complex. We say that $\mathcal{F}$ is *properly supported with respect to* p if there exists a closed subscheme $i\colon Z \to X$ such that:

(1) the composition $Z \xrightarrow{i} X \xrightarrow{p} Y$ is proper;

(2) let $j\colon X \smallsetminus Z \hookrightarrow X$ be inclusion of open complementary of Z. Then $j^*\mathcal{F} \simeq 0$.

We let $\mathsf{Perf}_{p\text{-prop}}(X)$ denote the full subcategory of $\mathsf{Perf}(X)$ spanned by perfect complexes properly supported with respect to p. ⊘

LEMMA 5.2.2. *Let $p\colon X \to Y$ be a quasi-compact and quasi-separated morphism of derived schemes of finite tor-amplitude. Let $\mathcal{F} \in \mathsf{Perf}(X)$ be a perfect complex which is properly supported with respect to p. Then $p_*(\mathcal{F})$ is perfect.*

PROOF. Since p has finite tor-amplitude and X and Y are schemes, a Čech cohomology argument shows that $p_*(\mathcal{F})$ has finite tor-amplitude. It is therefore enough to prove that $p_*(\mathcal{F})$ is almost perfect. Since $j^*\mathcal{F} \simeq 0$, we see that each $\pi_i(\mathcal{F})$ is set-theoretically supported on Z. Therefore, the cohomological descent spectral sequence

$$\mathsf{R}^i p_*(\pi_j(\mathcal{F})) \Rightarrow \mathsf{R}^{i+j} p_*(\mathcal{F})$$

implies that each $\mathsf{R}^i p_*(\mathcal{F})$ is coherent and that $\mathsf{R}^i p_*(\mathcal{F}) = 0$ for $i \gg 0$. The conclusion follows. □

PROPOSITION 5.2.3. *Let X be a smooth and proper scheme. Let $\mathsf{T}^*X := \mathsf{Spec}_X(\mathsf{Sym}_{\mathcal{O}_X}(\mathbb{T}_X))$ and let $p\colon \mathsf{T}^*X \to X$ be the natural projection. Then the functor $p_*\colon \mathsf{QCoh}(\mathsf{T}^*X) \to \mathsf{QCoh}(X)$ restricts to an equivalence*

$$\mathsf{Perf}_{p\text{-prop}}(\mathsf{T}^*X) \simeq \mathsf{Perf}(X_{\mathsf{Dol}}) .$$

PROOF. The functor $p_*\colon \mathsf{QCoh}(\mathsf{T}^*X) \simeq \mathsf{QCoh}(X)$ induces an equivalence

$$\mathsf{QCoh}(\mathsf{T}^*X) \simeq \mathsf{Mod}_{\mathsf{Sym}_{\mathcal{O}_X}(\mathbb{T}_X)}(\mathsf{QCoh}(X)) \simeq \mathsf{QCoh}(X_{\mathsf{Dol}}) .$$

Lemma 5.2.2 implies that the functor p_* restricts to a functor

$$\mathsf{Perf}_{p\text{-prop}}(\mathsf{T}^*X) \longrightarrow \mathsf{Perf}(X_{\mathsf{Dol}}) .$$

On the other hand, let $\mathcal{F} \in \mathsf{QCoh}(\mathsf{T}^*X)$ be such that $p_*(\mathcal{F}) \in \mathsf{Perf}(X_{\mathsf{Dol}})$. We want to prove that it is properly supported with respect to p. Since X is smooth, we have that $\pi_i(\mathcal{F}) \neq 0$ for only finitely many integers i. It is therefore enough to check that each $\pi_i(\mathcal{F})$ is properly supported with respect to p. This follows from the classical BNR correspondence (cf. [Sim94, Lemma 6.8]). □

By using the same arguments as above and [Sim94, Lemma 6.10], one can prove:

COROLLARY 5.2.4. *Let X be a smooth and proper scheme. Then the functor $p_*\colon \mathsf{QCoh}(\mathsf{T}^*X) \to \mathsf{QCoh}(X)$ restricts to an equivalence*

$$\mathsf{Perf}_X(\mathsf{T}^*X) \simeq \mathsf{Perf}(X_{\mathsf{Dol}}^{\mathsf{nil}}) ,$$

where $\mathsf{Perf}_X(\mathsf{T}^*X)$ is the full subcategory of $\mathsf{Perf}(\mathsf{T}^*X)$ of perfect complexes set-theoretically supported at X, seen as the zero-section of T^*X.

5.3. Geometrical properties of the Dolbeault shape

LEMMA 5.3.1. *Let X be a geometric derived stack over $\mathbb{C}$. The map κ_X is universally flat and an effective epimorphism. The same properties hold for κ_X^{nil}.*

PROOF. The map κ_X is an effective epimorphism by construction. The flatness follows from the fact that the transition maps in the diagram $\widehat{\mathrm{T}^\bullet X}$ are flat. By arguing similarly, one proves the same statement for κ_X^{nil}. □

LEMMA 5.3.2. *The derived stack X_{Dol} is categorically proper and universally of tor-amplitude ≤ 0.*

PROOF. Using the BNR correspondence for perfect complexes proven in Proposition 5.2.3 , we can identify $\mathrm{Perf}(X_{\mathrm{Dol}})$ with the ∞-category $\mathrm{Perf}_{p\text{-prop}}(X)$ of perfect complexes properly supported with respect to the projection $p \colon \mathrm{T}^*X \to X$. Under this equivalence, the functor

$$q_* \colon \mathrm{QCoh}(X_{\mathrm{Dol}}) \longrightarrow \mathrm{QCoh}(X)$$

is identified with the global section functor on T^*X. Since X is smooth, T^*X is smooth as well, and therefore we conclude that X_{Dol} has universally tor-amplitude ≤ 0. Finite cohomological dimension follows immediately. Finally, categorical properness is consequence of the properness of X, Lemma 5.2.2 and the BNR correspondence. □

By using similar arguments as above and Corollary 5.2.4, one can prove the following.

LEMMA 5.3.3. *The derived stack $X_{\mathrm{Dol}}^{\mathrm{nil}}$ is categorically proper and universally of tor-amplitude ≤ 0.*

Let X be a smooth scheme over $\mathbb{C}$. Then the category of (quasi) coherent sheaves on X_{Dol} is canonically equivalent to the category of (quasi) coherent Higgs sheaves on X (cf. [**Sim96**, **Sim97**, **Sim02**]).

5.4. The stack of perfect complexes

5.4.1. Relation with linear stacks. Let X be a smooth proper connected complex scheme. Define

$$\widetilde{\mathcal{E}} := q_+\left(q_X^*\mathbb{T}_X \otimes \mathrm{ev}^*\mathcal{E}nd(\mathcal{E}_{\mathrm{univ}})\right) ,$$

where the maps are

$$\begin{array}{ccc} \mathbf{Perf} & & \\ \uparrow{\scriptstyle \mathrm{ev}} & & \\ X \times \mathbf{Perf}(X) & \xrightarrow{q_X} & X \\ \downarrow{\scriptstyle q} & & \\ \mathbf{Perf}(X) & & \end{array} .$$

PROPOSITION 5.4.1. *Let X be a smooth proper connected complex scheme of dimension n. Then there exists a map*

$$\mathbf{Perf}(X_{\mathrm{Dol}}) \longrightarrow \mathbb{V}_{\mathbf{Perf}(X)}(\widetilde{\mathcal{E}}) ,$$

which is an equivalence when $n = 1$.

PROOF. Let us start by giving an explicit description of the points of $\mathbb{V}_{\mathbf{Perf}(X)}(\widetilde{\mathcal{E}})$.

Fix $A \in \mathrm{dAff}$ and let $x \colon \mathrm{Spec}(A) \to \mathbf{Perf}(X)$ be a point. By Formula A.2, a point $\mathrm{Spec}(A) \to \mathbb{V}_{\mathbf{Perf}(X)}(\widetilde{\mathcal{E}})$ corresponds to a morphism as A-Mod:

$$x^*\widetilde{\mathcal{E}} \to A . \tag{5.4.1}$$

Consider the diagram

$$\begin{array}{ccc} X_A & \xrightarrow{\mathrm{id}\times x} & X \times \mathbf{Perf}(X) \\ \big\downarrow{\scriptstyle p_A} & & \big\downarrow{\scriptstyle q} \\ \mathrm{Spec}(A) & \xrightarrow{x} & \mathbf{Perf}(X) \end{array} ,$$

where $X_A := X \times \mathrm{Spec}(A)$ and $p_A \colon X_A \to \mathrm{Spec}(A)$ is the projection to the second factor. We have

$$x^* q_+ \simeq p_{A+}(\mathrm{id} \times x)^* .$$

Thus

$$x^* \widetilde{\mathcal{E}} \simeq p_{A+}\Big(p_X^* \mathbb{T}_X \otimes \mathcal{E}nd(\mathcal{F}_A)\Big) ,$$

where $\mathcal{F}_A := (\mathrm{id} \times x)^* \mathrm{ev}^*(\mathcal{E}_{\mathrm{univ}})$ and $p_X \colon X_A \to X$ is the projection to the first factor. Therefore, the morphism (5.4.1) is equivalent, by adjunction, to the morphism of $\mathcal{O}_{X_A}$-modules:

$$p_X^* \mathbb{T}_X \otimes \mathcal{E}nd(\mathcal{F}_A) \longrightarrow \mathcal{O}_{X_A} . \tag{5.4.2}$$

Since $\mathcal{F}_A$ is perfect, we get a morphism $p_X^* \mathbb{T}_X \to \mathcal{E}nd(\mathcal{F}_A)$, which induces a morphism

$$\mathcal{T}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X) \longrightarrow \mathcal{E}nd(\mathcal{F}_A) \tag{5.4.3}$$

from the tensor algebra $\mathcal{T}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X)$ of $p_X^* \mathbb{T}_X$.

Recall now that

$$\mathrm{Perf}(X_{\mathrm{Dol}} \times \mathrm{Spec}(A)) \simeq \mathrm{Perf}(\mathcal{B}_{X_A} \widehat{\mathbb{V}_{X_A}(p_X^* \mathbb{L}_X)}) \simeq \mathrm{Mod}_{\mathrm{Sym}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X)}(\mathrm{Perf}(X_A)) .$$

By arguing in a way similar to above, a point $\mathrm{Spec}(A) \to \mathrm{Perf}(X_{\mathrm{Dol}} \times \mathrm{Spec}(A))$ corresponds to a morphism

$$\mathrm{Sym}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X) \longrightarrow \mathcal{E}nd(\mathcal{F}_A) .$$

By composing the natural projection $\mathcal{T}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X) \to \mathrm{Sym}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X)$ with the above morphism, we get a morphism of the form (5.4.3). Thus, we have a canonical morphism

$$\mathbf{Perf}(X_{\mathrm{Dol}}) \longrightarrow \mathbb{V}_{\mathbf{Perf}(X)}(\widetilde{\mathcal{E}}) . \tag{5.4.4}$$

Moreover, the canonical projection $\mathcal{T}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X) \to \mathrm{Sym}_{\mathcal{O}_{X_A}}(p_X^* \mathbb{T}_X)$ is an isomorphism when $n = 1$. Hence, the map (5.4.4) is an equivalence when $n = 1$. □

We now give a characterization of $\mathbb{V}_{\mathbf{Perf}(X)}(\widetilde{\mathcal{E}})$ when $n = 1$. Recall that, by Corollary 2.3.28, the global tangent complex of $\mathbf{Perf}(X)$ is

$$\mathbb{T}_{\mathbf{Perf}(X)} \simeq q_+\Big(\mathrm{ev}^* \mathcal{E}nd(\mathcal{E}_{\mathrm{univ}})[1]\Big) .$$

PROPOSITION 5.4.2. *Let X be a smooth proper connected complex scheme of dimension one. Then*

$$\mathbb{V}_{\mathbf{Perf}(X)}(\widetilde{\mathcal{E}}) \simeq \mathrm{T}^*[0]\mathbf{Perf}(X) := \mathbb{V}_{\mathbf{Perf}(X)}(\mathbb{T}_{\mathbf{Perf}(X)}) .$$

PROOF. Let us start by giving an explicit description of the points of $\mathbb{V}_{\mathbf{Perf}(X)}(\mathbb{T}_{\mathbf{Perf}(X)})$.

Fix $A \in \mathrm{dAff}$ and let $x \colon \mathrm{Spec}(A) \to \mathbf{Perf}(X)$ be a point. By Formula A.2, a point $\mathrm{Spec}(A) \to \mathbb{V}_{\mathbf{Perf}(X)}(\mathbb{T}_{\mathbf{Perf}(X)})$ corresponds to a morphism as A-Mod:

$$x^* \mathbb{T}_{\mathbf{Perf}(X)} \to A .$$

By using arguments similar to those in the proof of Proposition 5.4.1, the above morphism corresponds to the morphism of $\mathcal{O}_{X_A}$-modules:

$$\mathcal{E}nd(\mathcal{F}_A)[1] \longrightarrow \mathcal{O}_{X_A} . \tag{5.4.5}$$

By Grothendieck-Serre duality, we have

$$\mathcal{E}nd(\mathcal{F}_A)[1] \simeq p_X^* \mathbb{T}_X \otimes \mathcal{E}nd(\mathcal{F}_A) .$$

Thus morphism (5.4.5) is equivalent to (5.4.2). This proves the assertion.

□

By combining the above two propositions, we obtain:

COROLLARY 5.4.3. *Let X be a smooth proper connected complex scheme of dimension one. Then*

$$\mathbf{Perf}(X_{\mathsf{Dol}}) \simeq \mathsf{T}^*[0]\mathbf{Perf}(X) \ .$$

REMARK 5.4.4. A version of the above corollary for $\mathbf{Bun}(X_{\mathsf{Dol}})$ has been proved in [GR18, Lemma 4.3]. △

5.4.2. Representability. To be written.

CHAPTER 6

Deligne shape

6.1. Definition and geometrical properties

Following ideas of Deligne and Simpson,[1] we consider the cosimplicial affine scheme

$$\mathsf{Del}^\bullet \colon \mathbf{\Delta} \longrightarrow \mathsf{dAff}_{/\mathbb{A}^1} ,$$

given by

$$\mathsf{Del}^n := \mathsf{Spec}(\mathbb{C}[X,Y]/(X^n - Y^n)) ,$$

where the structural map to $\mathbb{A}^1 := \mathsf{Spec}(\mathbb{C}[T])$ is given by $T \mapsto Y$. Moreover $\mathbb{G}_m$ naturally acts on Del^n in an equivariant way with respect to $\mathbb{A}^1$. This gives rise to a cosimplicial stack

$$\mathsf{Del}^\bullet_{\mathbb{G}_m} := [\mathsf{Del}^\bullet / \mathbb{G}_m] \colon \mathbf{\Delta} \longrightarrow \mathsf{dSt}_{/[\mathbb{A}^1/\mathbb{G}_m]} .$$

Let now X be a smooth scheme over $\mathbb{C}$. Then

$$\mathbf{Map}_{/[\mathbb{A}^1, \mathbb{G}_m]}(\mathsf{Del}^\bullet_{\mathbb{G}_m}, X \times [\mathbb{A}^1/\mathbb{G}_m])$$

is a simplicial object over $[\mathbb{A}^1/\mathbb{G}_m]$. Pulling back along the atlas $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ (that is, forgetting the $\mathbb{G}_m$-action) we obtain the $\mathbb{A}^1$-cosimplicial object

$$\mathbf{Map}_{/\mathbb{A}^1}(\mathsf{Del}^\bullet, X \times \mathbb{A}^1) ,$$

which can explicitly be described as follows: over the open $\mathbb{A}^1 \smallsetminus \{0\}$, it is canonically equivalent to the simplicial object

$$[n] \mapsto X^n \times (\mathbb{A}^1 \smallsetminus \{0\}) ,$$

while over the point $0 \in \mathbb{A}^1$ it becomes the simplicial object

$$[n] \mapsto \mathsf{T}^n X$$

described in the previous section. The canonical map

$$\mathsf{Del}^\bullet_{\mathbb{G}_m} \longrightarrow [\mathbb{A}^1, \mathbb{G}_m]$$

gives rise to a map

$$\delta \colon X \times [\mathbb{A}^1, \mathbb{G}_m] \longrightarrow \mathbf{Map}_{/[\mathbb{A}^1/\mathbb{G}_m]}(\mathsf{Del}^\bullet_{\mathbb{G}_m}, X \times [\mathbb{A}^1/\mathbb{G}_m]) .$$

Given $[n] \in \mathbf{\Delta}$, the induced family of morphisms

$$\begin{array}{ccc} X \times \mathbb{A}^1 & \longrightarrow & \mathbf{Map}_{/\mathbb{A}^1}(\mathsf{Del}^n, X \times \mathbb{A}^1) \\ & \searrow \quad \swarrow & \\ & \mathbb{A}^1 & \end{array} ,$$

[1]The reader might want to compare with the general construction performed in [GR17, §9.1.6].

coincides with the deformation to the normal cone of the diagonal embedding $X \hookrightarrow X^n$. We now define the simpicial object $X^{\bullet}_{\mathsf{Del},\mathbb{G}_m}$ as the fiber product

$$\begin{array}{ccc} X^{\bullet}_{\mathsf{Del},\mathbb{G}_m} & \longrightarrow & \mathbf{Map}_{/[\mathbb{A}^1/\mathbb{G}_m]}(\mathsf{Del}^{\bullet}_{\mathbb{G}_m}, X \times [\mathbb{A}^1/\mathbb{G}_m]) \\ \downarrow & & \downarrow \\ (X \times [\mathbb{A}^1/\mathbb{G}_m])_{\mathsf{dR}} & \xrightarrow{\delta_{\mathsf{dR}}} & \mathbf{Map}_{/[\mathbb{A}^1/\mathbb{G}_m]}(\mathsf{Del}^{\bullet}_{\mathbb{G}_m}, X \times [\mathbb{A}^1/\mathbb{G}_m])_{\mathsf{dR}} \end{array} .$$

In other words, $X^{\bullet}_{\mathsf{Del},\mathbb{G}_m}$ is the formal completion of $\mathbf{Map}_{/[\mathbb{A}^1/\mathbb{G}_m]}(\mathsf{Del}^{\bullet}_{\mathbb{G}_m}, X \times [\mathbb{A}^1, \mathbb{G}_m])$ along the diagonal morphism δ. Finally, we let

$$X_{\mathsf{Del},\mathbb{G}_m} := \left| X_{\mathsf{Del},\mathbb{G}_m} \right| \in \mathsf{dSt}_{/[\mathbb{A}^1/\mathbb{G}_m]} .$$

We also let X_{Del} be the pullback of $X_{\mathsf{Del},\mathbb{G}_m}$ along the atlas $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$.

Let X be a smooth scheme over $\mathbb{C}$. Then for any (quasi)-coherent sheaf E on X_{Del}, its fiber $E|_{\lambda}$ at $\lambda \in \mathbb{A}^1$ is a (quasi) coherent λ-connection on X (see e.g. [**Sim09**, §7]).

6.1.1. Representability. To be written.

APPENDIX A

Linear stacks

A.1. Picard stacks

We follow and generalize [SGA73, Exposé XVIII, §1.4]. Recalling the (homotopy) equivalence between groupoids and 1-homotopy types, we can rephrase Definition 1.4.5 in *loc.cit.* as follows:

DEFINITION A.1.1. Let $\mathcal{X}$ be an ∞-topos. A *Picard stack* over X is a sheaf

$$\mathcal{F}\colon \mathcal{X}^{\mathsf{op}} \to \mathsf{sAb}^{\leqslant 1} ,$$

where $\mathsf{sAb}^{\leqslant 1}$ denotes the ∞-category of simplicial abelian groups whose underlying space is a 1-homotopy type. We let $\mathsf{Pic}(\mathcal{X})$ denote the ∞-category of Picard stacks on $\mathcal{X}$. ⊘

The main result of [SGA73, Exposé XVIII, §1.4] can then be summarized as follows:

PROPOSITION A.1.2 (Proposition 1.4.15 & Corollary 1.4.17 in *loc.cit.*). *Let $\mathcal{X}$ be an ∞-topos. There is an equivalence of ∞-categories*

$$\mathsf{Pic}(\mathcal{X}) \simeq \mathsf{Sh}_{\mathcal{D}^{[-1,0]}(\mathsf{Ab})}(\mathcal{X}) ,$$

where $\mathcal{D}^{[-1,0]}(\mathsf{Ab})$ denotes the full ∞-subcategory of $\mathcal{D}(\mathsf{Ab})$ (the ∞-derived category of abelian groups) spanned by objects in cohomological amplitude $[-1,0]$.

From a modern point of view, the proof is a direct consequence of the Dold-Kan equivalence

$$\mathsf{sAb} \simeq \mathcal{D}^{\leqslant 0}(\mathsf{Ab}) ,$$

combined with the remark that objects in cohomological degree $[-1,0]$ in $\mathcal{D}^{\leqslant 0}(\mathsf{Ab})$ correspond to 1-homotopy types. The language of higher stacks, allows us to generalize the above proposition:

PROPOSITION A.1.3. *Let $\mathcal{X}$ be an ∞-topos. There is an equivalence of ∞-categories*

$$\mathsf{Sh}_{\mathsf{sAb}}(\mathcal{X}) \simeq \mathsf{Sh}_{\mathcal{D}^{\leqslant 0}(\mathsf{Ab})}(\mathcal{X}) .$$

We will call the elements of $\mathsf{Sh}_{\mathcal{D}^{\leqslant 0}(\mathsf{Ab})}(\mathcal{X})$ *higher Picard stacks.*

We now consider a special case of interest: namely, we will suppose that $\mathcal{X}$ is the smooth-étale site of some geometric derived stack X. In this case, we have a forgetful functor

$$U\colon \mathsf{QCoh}(X) \to \mathsf{Sh}_{\mathcal{D}(\mathsf{Ab})}(\mathcal{X}) .$$

Composing with the truncation functor

$$\tau^{\leqslant 0}\colon \mathcal{D}(\mathsf{Ab}) \to \mathcal{D}^{\leqslant 0}(\mathsf{Ab})$$

we obtain a functor

$$U^{\leqslant 0}\colon \mathsf{QCoh}(X) \to \mathsf{Sh}_{\mathcal{D}(\mathsf{Ab})}(\mathcal{X}) \to \mathsf{Sh}_{\mathcal{D}^{\leqslant 0}(\mathsf{Ab})}(\mathcal{X})$$

that allows to see a quasi-coherent sheaf $\mathcal{F}$ on X as a higher Picard stack on X.

REMARK A.1.4. Let $\mathcal{F} \in \mathsf{QCoh}(X)$. Then the canonical map $\tau^{\leqslant 0}\mathcal{F} \to \mathcal{F}$ induces an equivalence

$$U^{\leqslant 0}(\tau^{\leqslant 0}(\mathcal{F})) \simeq U^{\leqslant 0}(\mathcal{F}) .$$

For this reason, we will more often consider the restriction of $U^{\leqslant 0}$ to $\mathsf{QCoh}(X)^{\leqslant 0}$. △

A.2. Definition of a linear stack

We now introduce an example of higher Picard stack.

DEFINITION A.2.1. Let $F \in \mathsf{dSt}$ be a derived stack. Let $\mathcal{E} \in \mathsf{QCoh}(F)$. The *linear stack over F associated to* $\mathcal{E}$ is the derived stack $\mathbb{V}(\mathcal{E}) \in \mathsf{dSt}_{/F}$:

$$\mathbb{V}(\mathcal{E}) := \mathsf{Spec}_X(\mathsf{Sym}_{\mathcal{O}_X}(\mathcal{E})) .$$

When F is clear from the context, we will simply write $\mathbb{V}(\mathcal{E})$ instead of $\mathbb{V}_F(\mathcal{E})$. ⊘

Unraveling this definition, we see that for any $x\colon \mathsf{Spec}(A) \to F$, one has

$$\begin{aligned} \mathsf{Map}_{/F}(\mathsf{Spec}(A), \mathbb{V}(\mathcal{E})) &\simeq \mathsf{Map}_{\mathcal{O}_X/}(\mathsf{Sym}_{\mathcal{O}_X}(\mathcal{E}), x_*\mathcal{O}_A) \\ &\simeq \mathsf{Map}_{\mathsf{QCoh}(X)}(\mathcal{E}, x_*\mathcal{O}_A) \simeq \mathsf{Map}_{A\text{-}\mathsf{Mod}}(x^*\mathcal{E}, A) . \end{aligned}$$

REMARK A.2.2. As A-Mod is naturally enriched in $\mathcal{D}(\mathsf{Ab})$ and since for any $\mathcal{F}, \mathcal{G} \in A$-Mod we have

$$\mathsf{Map}_{A\text{-}\mathsf{Mod}}(\mathcal{F}, \mathcal{G}) \simeq \tau^{\leqslant 0}\mathsf{Map}_{A\text{-}\mathsf{Mod}}^{\mathcal{D}(\mathsf{Ab})}(\mathcal{F}, \mathcal{G}) ,$$

it is then clear that $\mathbb{V}(E)$ defines a higher Picard stack on F. △

Note that, by definition, $\mathbb{V}(\mathcal{E}) \in \mathsf{dSt}_{/F}$ is a derived stack equipped with a canonical map

$$\pi\colon \mathbb{V}(\mathcal{E}) \to F .$$

The following result is evident:

PROPOSITION A.2.3. *Let F be a geometric derived stack and let* $\mathcal{E} \in \mathsf{QCoh}(F)^{\leqslant 0}$. *Then the map* $\pi\colon \mathbb{V}(\mathcal{E}) \to F$ *is representable by affine derived schemes. In particular,* $\mathbb{V}(\mathcal{E})$ *is a geometric derived stack.*

Let now F be a geometric derived stack X.

PROPOSITION A.2.4. *Let* $\mathsf{Perf}(X)^{\geqslant 0}$ *be the category of perfect complexes on X that are in positive cohomological amplitude. Let* $(-)^\vee\colon \mathsf{Perf}(X)^{\geqslant 0} \to A\text{-}\mathsf{Mod}^{\leqslant 0}$ *be the duality functor:*

$$\mathcal{E}^\vee := \mathsf{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) ,$$

for $\mathcal{E} \in \mathsf{Perf}(X)^{\geqslant 0}$. *Then the diagram*

$$\begin{array}{ccccc} & & & & \mathsf{QCoh}(X)^{\leqslant 0} \\ & & & \overset{(-)^\vee}{\nearrow} & \downarrow U \\ \mathsf{Perf}(X)^{\geqslant 0} & \hookrightarrow & \mathsf{QCoh}(X) & \xrightarrow{\mathbb{V}(-)} & \mathsf{Sh}_{\mathcal{D}^{\leq 0}(\mathsf{Ab})}(X) \end{array}$$

commutes.

PROOF. Let $x\colon \mathsf{Spec}(A) \to X$ be a fixed map and let $\mathcal{E} \in \mathsf{Perf}(X)^{\geqslant 0}$. Since $\mathcal{E}$ is perfect, we have:

$$\mathsf{Map}_{/X}(\mathsf{Spec}(A), \mathbb{V}(\mathcal{E})) \simeq \mathsf{Map}_{A\text{-}\mathsf{Mod}}(x^*\mathcal{E}, A) \simeq \mathsf{Map}_{A\text{-}\mathsf{Mod}}(A, x^*(\mathcal{E}^\vee)) .$$

Observe now that we can identify $\mathsf{Map}_{A\text{-}\mathsf{Mod}}(A, x^*(\mathcal{E}^\vee))$ with the underlying complex of abelian groups of $x^*(\mathcal{E}^\vee)$. In other words, it coincides by definition with $U(\mathcal{E}^\vee)(\mathsf{Spec}(A))$. □

Let now $\mathcal{F} \in \mathsf{Perf}^{[-1,0]}(X)$ be a perfect complex in tor-amplitude $[-1,0]$. Then $\mathcal{F}[-1] \in \mathsf{Perf}^{\geqslant 0}(X)$ and therefore the above proposition supplies us with an equivalence

$$\mathbb{V}(\mathcal{F}[-1]) \simeq U(\mathcal{F}^\vee[1]) .$$

PROPOSITION A.2.5. *The stack* $U(\mathcal{F}^\vee[1])$ *coincides with the so-called vector bundle stack* $(h^1/h^0)(\mathcal{F}^\vee)$ *of* [BF97].

PROOF. This follows tautologically if one believes to the claim at the beginning of [BF97, §2] that $(h^1/h^0)(\mathcal{F}^\vee)$ coincides with the construction $\mathsf{ch}(-)$ performed in [SGA73, Exposé XVIII, §1.4]. □

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