$$A^n = \left(\frac{1}{A^{-1}}\right)^n = \frac{1}{(A^{-1})^n} \to 0$$

$$A^{-1} > 1 \Rightarrow (A^{-1})^n \to +\infty$$

(caso $0 < A < 1$)

se $\sqrt[n]{a_n} \to l$

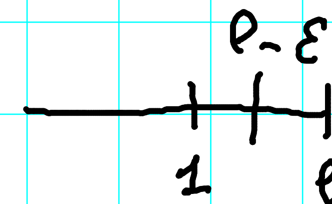
$l > 1 \Rightarrow a_n \to +\infty$

Dim. Fisso $\varepsilon > 0$ so che $\exists\, n_1 : \forall n \geq n_1$

$$l-\varepsilon < \sqrt[n]{a_n} < l+\varepsilon \quad \forall n \geq n_1$$

Scelgo $l-\varepsilon = \frac{l+1}{2} > 1$ (punto medio tra l e 1)

ε IN MODO CHE $\varepsilon = \frac{l-1}{2}$.



$\Rightarrow \forall n \geq n_1$

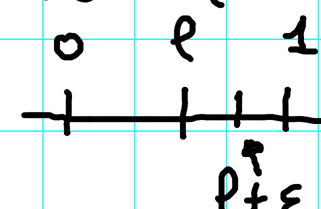
$$\boxed{\sqrt[n]{a_n} \geq l-\varepsilon} \iff a_n \geq (l-\varepsilon)^n \quad \forall n \geq n_1$$

Dato che $l-\varepsilon > 1 \Rightarrow (l-\varepsilon)^n \to +\infty$

$\Rightarrow a_n \to +\infty$

Se $0 \leq l < 1$ prendo $\varepsilon = \frac{1-l}{2} > 0$

$\Rightarrow l+\varepsilon < 1$



$\Rightarrow$ per $n \geq n_1$ $\sqrt[n]{a_n} \leq l+\varepsilon \Rightarrow a_n \leq \underbrace{(l+\varepsilon)^n}_{\to 0}$

$\Rightarrow a_n \to 0$

$$\sqrt[n]{n} \longrightarrow 1$$

Dalla def. Fisso $\varepsilon > 0$.

Cerco n_1: $\forall n \geq n_1$ $\underbrace{\sqrt[n]{n} < 1+\varepsilon}_{(*)}$

$1-\varepsilon \leq 1 \leq$

$(*) \iff n \leq (1+\varepsilon)^n$ per $n \geq n_1$

se uso la formula del Binomio di Newton

$$(1+\varepsilon)^n = 1 + \varepsilon n + \binom{n}{2}\varepsilon^2 + \underbrace{\cdots\cdots}_{\geq 0} \geq$$

$$\frac{n!}{2!\,(n-2)!}\varepsilon^2 = \frac{n(n-1)}{2}\varepsilon^2$$

Allora mi basta: $\boxed{\frac{n(n-1)}{2}\varepsilon^2 \geq n}$ $\forall n \geq n_1$

$$\frac{\varepsilon^2}{2}n^2 - \frac{n}{2}\varepsilon^2 - n \geq 0$$ per n grande

VERO

$n \geq$ seconda radice di $\frac{\varepsilon^2}{2}x^2 - \left(\frac{\varepsilon^2}{2}+1\right)x$

IN REALTÀ

(Se conosco CESARO)

$$\lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} \frac{n+1}{n} = 1$$

FINE

Se poi c'è anche K

$$\sqrt[n]{n^K} = \left(\sqrt[n]{n}\right)^K \to 1^K = 1$$

$$A > 1 \Rightarrow \frac{A^n}{n^k} \to +\infty$$

dim. $a_n = \frac{A^n}{n^k}$

$a_n \to +\infty$

$\Uparrow$

calcoliamo $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = A > 1$

infatti:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{A^{n+1}}{(n+1)^k}}{\frac{A^n}{n^k}} = \frac{A^{n+1}}{A^n} \frac{n^k}{(n+1)^k} =$$

$$A\left(\frac{n}{n+1}\right)^k = A\left(\frac{\cancel{n}}{\cancel{n}} \frac{1}{1+1/n}\right)^k \to A$$

CASO $0 < A < 1$ $\qquad A^{-1} > 1$

$$n^k A^n = \frac{n^k}{(A^{-1})^n} = 1 \Big/ \frac{(A^{-1})^n}{n^k} \to \frac{1}{+\infty} = 0^+$$

- $n! \to +\infty$ (dato che $n! \geq n$)
- $n^n \to +\infty$ (anche $n^n \geq n$)
- $\frac{A^n}{n!} \to 0$; come prima $a_n := \frac{A^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{A^{n+1}}{(n+1)!}}{\frac{A^n}{n!}} = \frac{A^{n+1}}{A^n} \underbrace{\frac{n!}{(n+1)!}}_{(n+1)! = n!\,(n+1)} =$$

$$= \frac{A}{n+1} \xrightarrow{n \to +\infty} 0$$

Se $\frac{a_{n+1}}{a_n} \to 0 \Rightarrow a_n \to 0$

ALLORA $\frac{n!}{A^n} \to +\infty \quad \forall A$

$n!$ <u>VINCE</u> su A^n

A^n AMMAZZA qualunque n^k

n^n VINCE SU $n!$

$$\frac{n^n}{n!} \to +\infty \qquad a_n := \frac{n^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{n+1}}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{n^n} =$$

$$\frac{(n+1)^{\cancel{n+1}}}{n^n \cancel{(n+1)}} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n \to e > 1 \Rightarrow a_n \to +\infty$$

<u>NOTA</u>: si poteva fare:

$$\frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} \geq n \to +\infty$$

- IL PRIMO METODO DICE "PIÙ O MENO"

$$\frac{n^n}{n!} \simeq e^n \quad (\underline{\text{DA PRECISARE}})$$

$$\left(\sqrt[n]{\frac{n^n}{n!}} \to e\right)$$

Dim. (di Cesaro) $a_n > 0 \quad \forall n$

So CHE $\frac{a_{n+1}}{a_n} \to \ell$

ALLORA $\sqrt[n]{a_n} \to \ell$.

(CASO $\ell \in \mathbb{R}$). Fisso $\varepsilon > 0$ trovo che $\exists\, n_1$ tale che

$$\ell - \varepsilon < \frac{a_{n+1}}{a_n} < \ell + \varepsilon \qquad \forall n \geq n_1$$

$$\ell-\varepsilon < \frac{a_{n+1}}{a_n} < \ell+\varepsilon \qquad \forall n \geq n_1$$

Allora $\forall n \geq n_1$

$$\boxed{(\ell-\varepsilon)a_n \leq a_{n+1} \leq a_n(\ell+\varepsilon)}$$

se anche $n-1 \geq n_1$ so che
$n \geq n_1+1$

$$(\ell-\varepsilon)\,a_{n-1} \leq a_n \leq a_{n-1}(\ell+\varepsilon) \quad \text{da cui}$$

$$a_{n-1}(\ell-\varepsilon)^2 \leq a_{n+1} \leq a_{n-1}(\ell+\varepsilon)^2$$

Se $n \geq n_1+2$ . . . . .

$$a_{n-2}(\ell-\varepsilon)^3 \leq a_{n+1} \leq a_{n-2}(\ell+\varepsilon)^3$$

. . . . . Se $n \geq n+k$

$$a_{n-k}(\ell-\varepsilon)^{k+1} \leq a_{n+1} \leq a_{n-k}(\ell+\varepsilon)^{k+1}$$

Se scelgo $k = n-n_1 \Rightarrow \forall n \geq n_1$

$$a_{n_1}(\ell-\varepsilon)^{n+1-n_1} \leq a_{n+1} \leq a_{n_1}(\ell+\varepsilon)^{n+1-n_1}$$

allora $\forall n \geq n_1$

$$\frac{a_{n_1}}{(\ell-\varepsilon)^{n_1}} (\ell-\varepsilon)^{n+1} \leq a_{n+1} \leq \frac{a_{n_1}}{(\ell+\varepsilon)^{n_1}} (\ell+\varepsilon)^{n+1}$$

FACCIO LA RADICE $n+1 \Rightarrow \forall n \geq n_1$

$$\underbrace{\sqrt[n+1]{\frac{a_{n_1}}{(\ell-\varepsilon)^{n_1}}} (\ell-\varepsilon)}_{\to \ell-\varepsilon} \leq \sqrt[n+1]{a_{n+1}} \leq \underbrace{\sqrt[n+1]{\frac{a_{n_1}}{(\ell+\varepsilon)^{n_1}}} (\ell+\varepsilon)}_{\to \ell+\varepsilon}$$

(SO CHE $\sqrt[n+1]{A} \to 1 \quad \forall A$)

$\Rightarrow$ trovo n_2 GRANDE : $\forall n \geq n_2$ $(\geq n_1)$
(permanenza del segno!)

$$\ell - 2\varepsilon < \sqrt[n+1]{a_{n+1}} < \ell + 2\varepsilon$$

$\Rightarrow$ HO VERIFICATO CHE

$$\sqrt[n+1]{a_{n+1}} \to \ell$$

(se ora ottenuti trovo

$\sqrt[n]{a_n} \to \ell$)

FINE LEZIONE !!