29/5 Ultima lezione

- $y' = \dfrac{y^2}{x}$
- $y' = \dfrac{y^2-1}{x}$
- $y' = \dfrac{1-y^2}{y}\,\dfrac{1}{1-x^2}$

Variabili separabili

- $\begin{cases} y'' - 2y' + 2y = e^x \cos(x) \\ y(0)=1,\ y'(0)=0 \end{cases}$ ←

- $A(y) = y^2-1 \qquad B(x) = \dfrac{1}{x} \qquad \{x \neq 0\}$
- $y = \pm 1$ soluzioni costanti

$y' = \dfrac{y^2-1}{x}$

INTEGRIAMO L'EQ CON LA COND. $y(x_0)=y_0$
$(y_0 \neq \pm 1)$

$$\int_{y_0}^{y(x)} \frac{ds}{s^2-1} = \int_{x_0}^{x} \frac{dt}{t}$$

- $\frac{1}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) \quad \left(= \frac{1}{2}\,\frac{\cancel{s}+1-\cancel{s}+1}{s^2-1} = \frac{2}{2(s^2-1)} = \right).$

$$\int_{y_0}^{y} \frac{ds}{s^2-1} = \frac{1}{2}\int_{y_0}^{y}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) ds = \ln\left(\left|\frac{s-1}{s+1}\right|^{1/2}\right)\Bigg|_{y_0}^{y} =$$

$$\ln\left(\frac{\sqrt{\left|\frac{y-1}{y+1}\right|}}{\sqrt{\left|\frac{y_0-1}{y_0+1}\right|}}\right)$$

- a destro trovo $\displaystyle\int_{x_0}^{x} \frac{dt}{t} = \ln\left|\frac{x}{x_0}\right|$

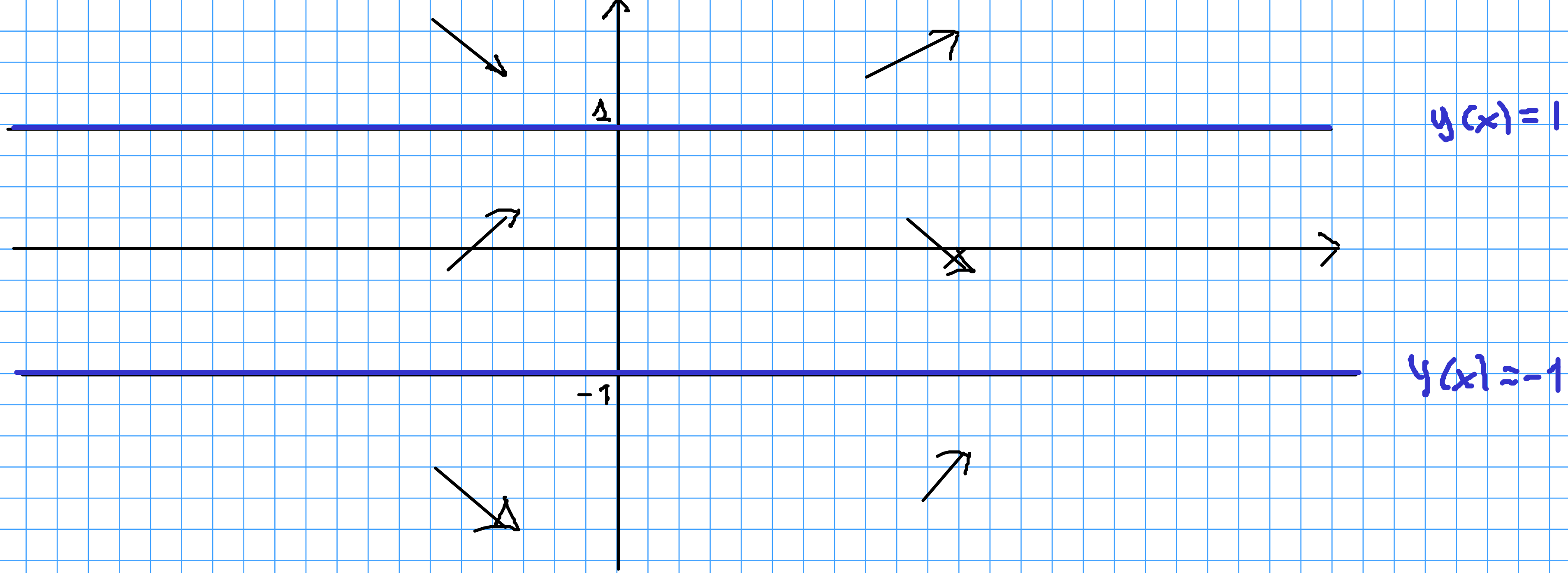
$\Rightarrow$ se uguaglio posso togliere i log.

$$\sqrt{\left|\frac{y-1}{y+1}\right|} = \sqrt{\left|\frac{y_0-1}{y_0+1}\right|}\left|\frac{x}{x_0}\right|$$

se rinuncio a seguire x_0 e y_0 posso scrivere

$$\boxed{\sqrt{\left|\frac{y(x)-1}{y(x)+1}\right|} = c\,|x|}$$

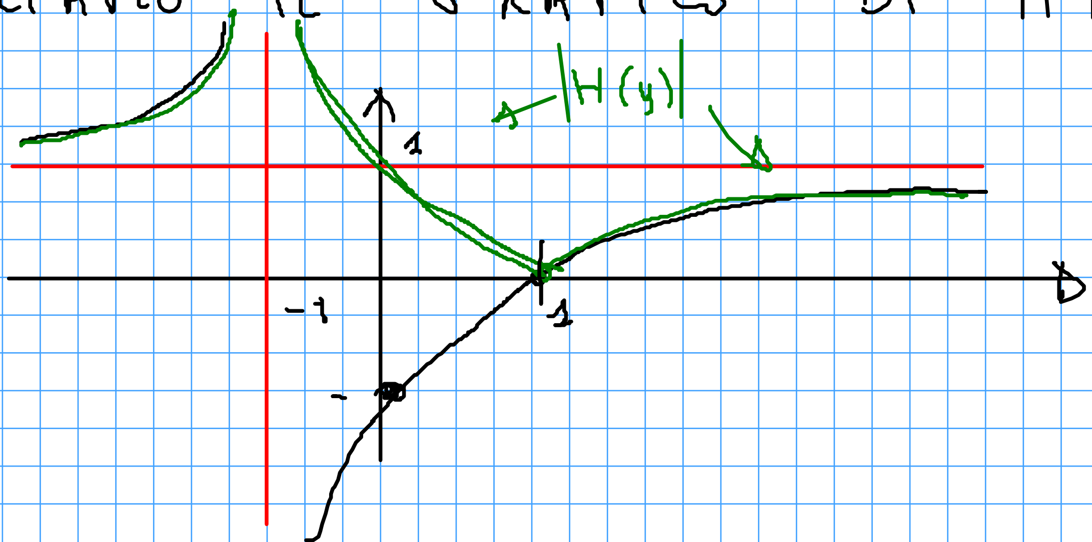
per c costante ≥ 0



$$\Longrightarrow \quad \left| \frac{y(x)-1}{y(x)+1} \right| = c^2 x^2$$

(trattandosi di termini ≥ 0 non introduco soluzioni)

— FACCIAMO IL GRAFICO DI $H(y) = \frac{y-1}{y+1} = 1 - \frac{2}{y+1}$



Se faccio $|H(y)|$ trovo la linea verde

CASO 1 SOLUZIONI $-1 < y(x) < 1$

Devo invertire $|H(y)| = -H(y)$ tra -1 e 1

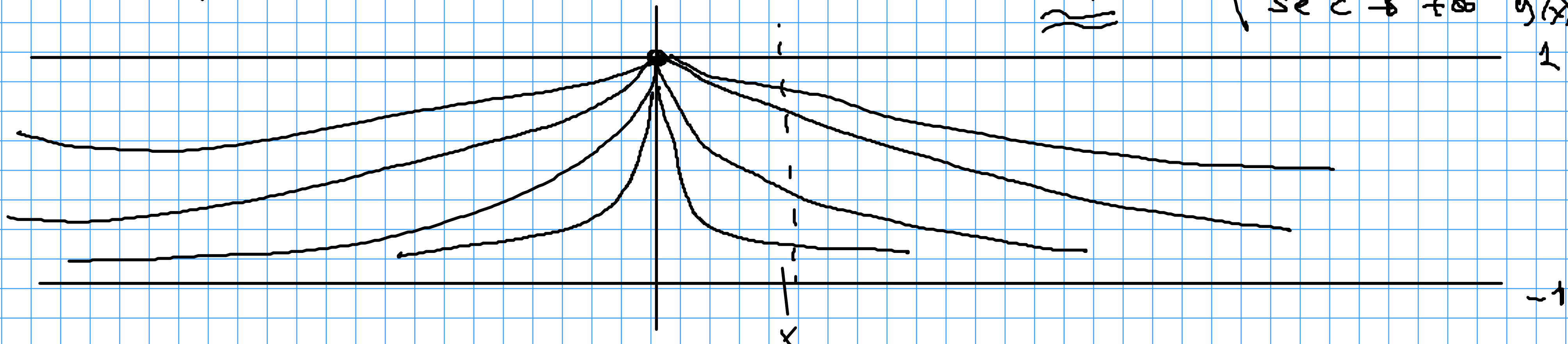
cioè devo risolvere $\frac{1-y}{1+y} = t \iff$

$1 - y = t + ty \iff 1 - t = (1+t)y \iff y = \frac{1-t}{1+t}$ $(t \geq 0)$

$$\Rightarrow \quad y(x) = \frac{1 - c^2x^2}{1 + c^2x^2} \quad \left(\in \,]-1, 1[\right)$$

Per quali x ha senso?? $\forall x$ dato che $c^2x^2 \geq 0$

e quindi $c^2x^2 \in$ DOMINIO DI H^{-1} $\left(\begin{array}{l} \text{se } c \to 0 \quad y(x) \to 1 \\ \text{se } c \to +\infty \quad y(x) \to -1 \end{array} \right)$



<u>NOTA</u> NON CI SONO SOLUZIONI CON $y(0) = y_0 \neq 1$

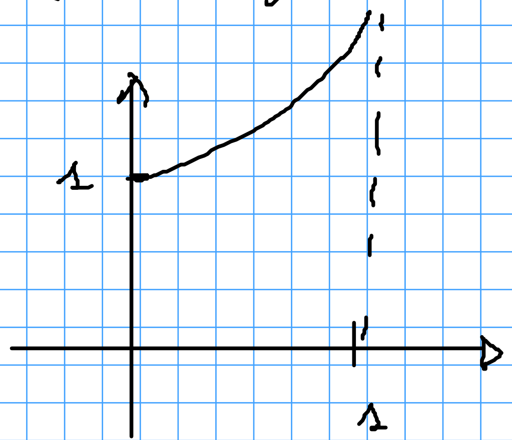
$x=0$ è singolare per l'eq.

<u>CASO 2</u> SOLUZIONI IN $\{y>1\}$

Devo invertire $H(y) = H(y)$ su $]1, +\infty[$

cioè devo risolvere $\frac{y-1}{y+1} = t \iff y - 1 = ty + t$

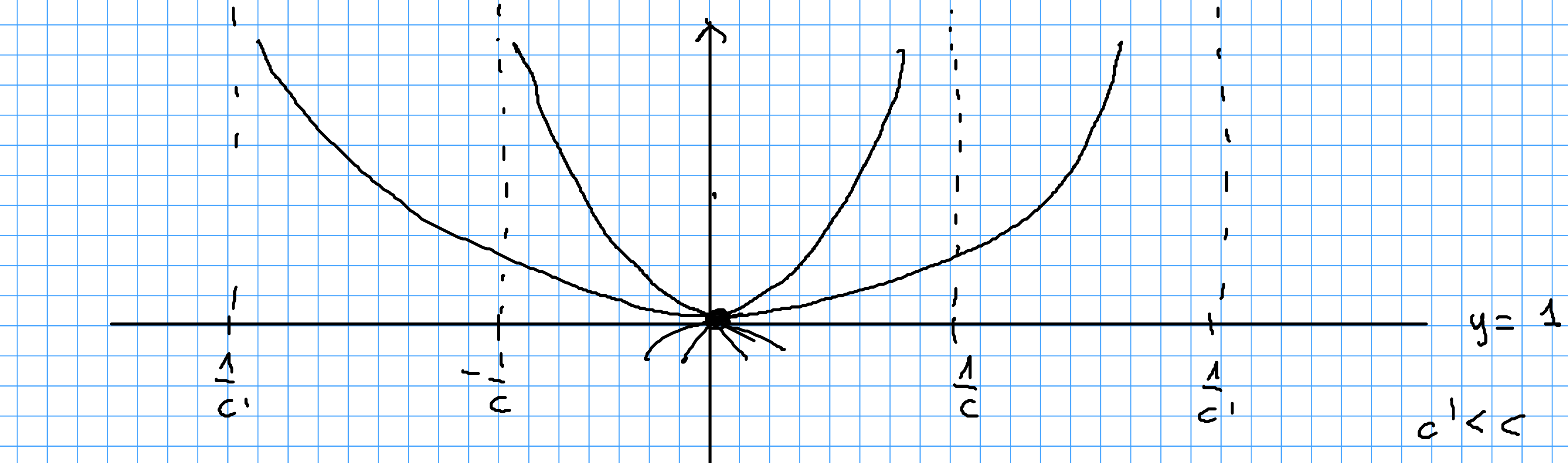
$(t-1)y = -1-t \iff y = \frac{1+t}{1-t}$ $\quad$ <u>$0 < t < 1$</u>



H^{-1} $\Rightarrow y(x) = \frac{1+c^2x^2}{1-c^2x^2}$

per le x tali che $0 < c^2x^2 < 1$

$\iff |x| < \frac{1}{c}$ $\quad -\frac{1}{c} < x < \frac{1}{c}$



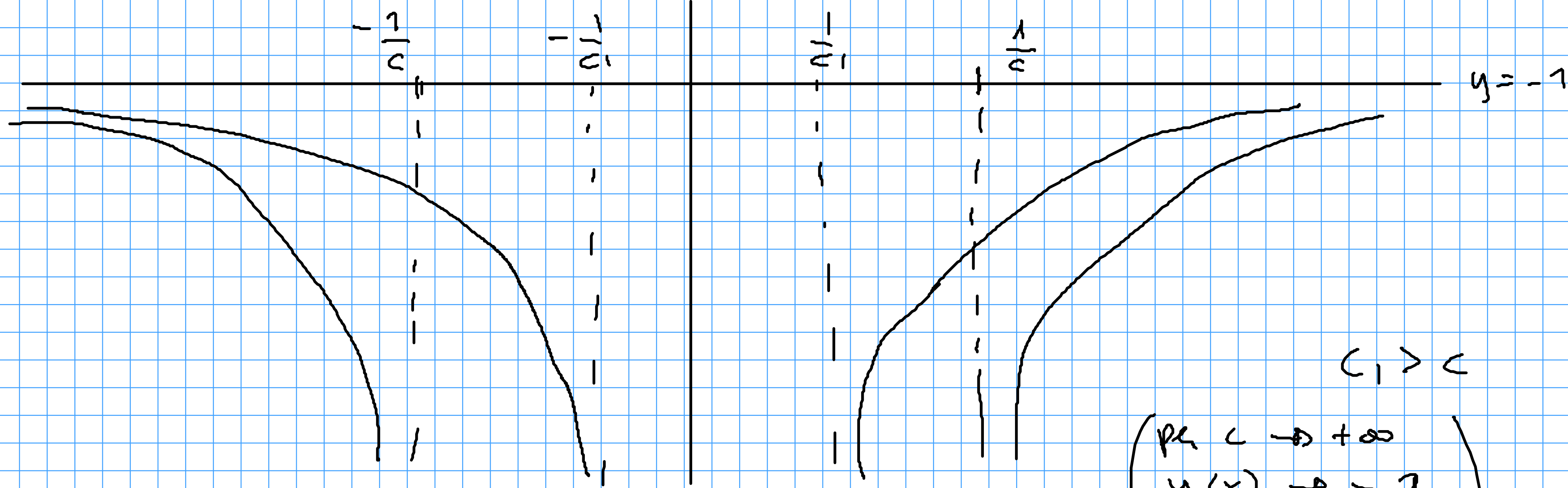
CASO 3 SOLUZIONI in $\{y < -1\}$

Devo invertire $|H(y)| = H(y)$ su $]-\infty, -1[$

$$H^{-1}(t) = \frac{1+t}{1-t} = -\frac{1+t}{t-1} \qquad t > 1$$

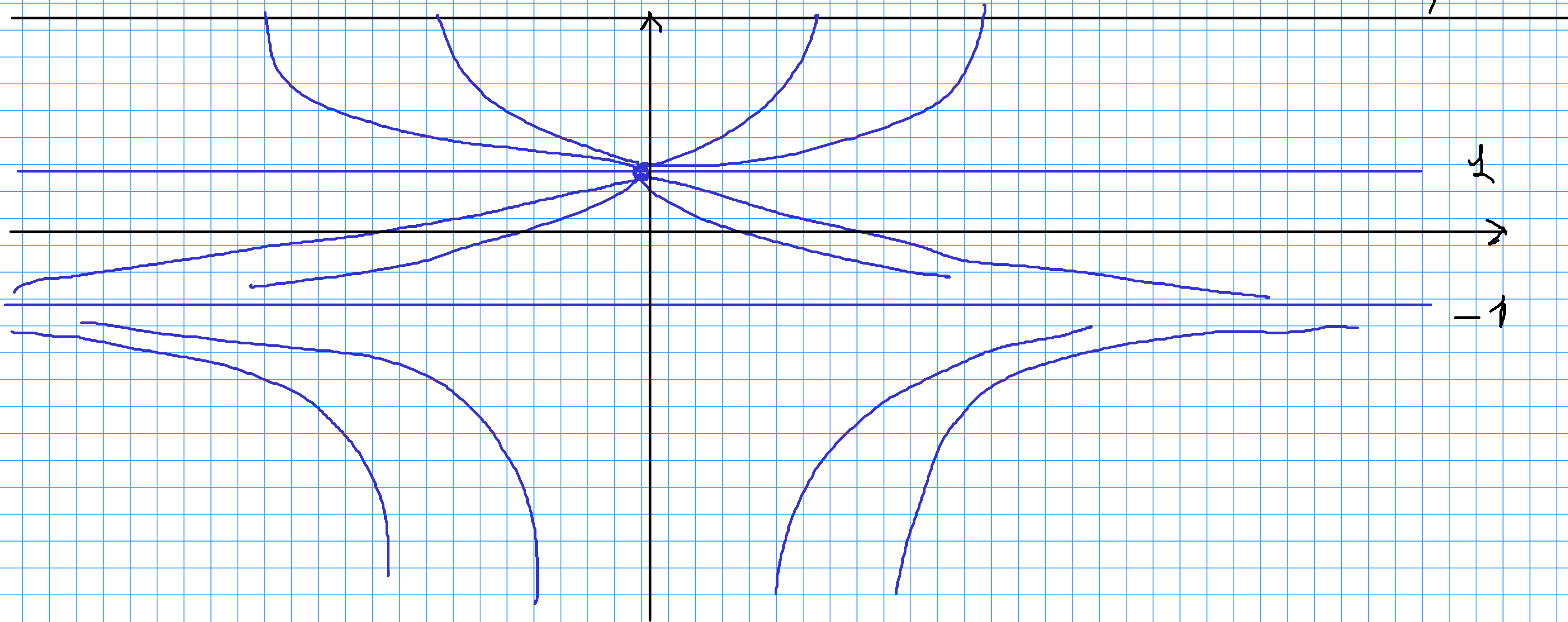
$$y(x) = -\frac{1+c^2x^2}{c^2x^2-1}$$ sulle x tali che $c^2x^2 > 1$

$\iff x^2 > \frac{1}{c^2}$ $\quad |x| > \frac{1}{c}$ $\quad x \geq \frac{1}{c}$ oppure $x \leq -\frac{1}{c}$



$c_1 > c$

$\left(\begin{array}{l} \text{pr } c \to +\infty \\ y(x) \to -1 \end{array} \right)$



$$\begin{cases} y'' - 2y' + 2y = e^{x}\cos(x) \\ y(0) = 1 \quad y'(0) = 0 \end{cases}$$

- eq. II° ordine
- coeff costanti
- non omogenea

• $p(z) = z^2 - 2z + 2$

radici $1 \pm \sqrt{1-2} = 1 \pm i$ (complesse coniugate)

$\Rightarrow$ sol. omogenea $y(x) = e^{x}\left(c_1 \cos(x) + c_2 \sin(x)\right)$

• Sol. particolare. Conviene passare a

$$y'' + 2y' + 2y = e^{(1+i)x} \quad \left(e^{x}\cos(x) = \mathrm{Re}\left(e^{(1+i)x}\right)\right)$$

<u>Dato che</u> $e^{(1+i)x}$ è sol <u>dell'omogenea</u>

cerco $\bar{y}(x) = \gamma\, x\, e^{(1+i)x}$

Ne segue $\bar{y}'(x) = \gamma\left(e^{(1+i)x} + x(1+i)\,e^{(1+i)x}\right)$

$$\bar{y}''(x) = \gamma\left((1+i)\,e^{(1+i)x} + (1+i)\,e^{(1+i)x} + x\,\underbrace{(1+i)^2}_{=2i}\,e^{(1+i)x}\right) =$$

$$\gamma\left(2(1+i)\,e^{(1+i)x} + 2ix\,e^{(1+i)x}\right).$$

DUNQUE

$$\bar{y}'' - 2\bar{y}' + 2y = \gamma\,e^{(1+i)x}\left(2(\cancel{1}+i) + \cancel{2i}x - \cancel{2} - 2(\cancel{1}+\cancel{i})x + \cancel{2x}\right)$$

$$= \gamma\,e^{(1+i)x}\,2i$$

$$\Rightarrow \quad \gamma = \frac{1}{2i} = \frac{-i}{2} \quad \text{e quindi}$$

$$\bar{y}(x) = -\frac{i}{2}\,x\,e^{x}\left(\cos(x) + i\sin(x)\right) =$$

$$\frac{x}{2}\,e^{x}\left(-i\cos(x) + \sin(x)\right)$$ $\leftarrow$ sol. del problema con $e^{(1+i)x}$

La parte reale mi dà $\boxed{\dfrac{x\,e^{x}\sin(x)}{2}}$ (sol. del problema iniziale)

⇒ SOL. COMPLETA

$$y(x) = e^{x}\left(c_1 \cos(x) + c_2 \sin(x)\right) + \frac{x e^{x}}{2} \sin(x)$$

$$y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$$

$$y'(x) = e^{x}\left(\cos(x) + c_2 \sin(x)\right) + e^{x}\left(-\sin(x) + c_2 \cos(x)\right) + \frac{e^{x}}{2} \sin(x) + \frac{x}{2} e^{x} \sin(x) + \frac{x e^{x}}{2} \cos(x)$$

$$y'(0) = 1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -1$$

LA SOL. CERCATA VALE

$$e^{x}\left(\cos(x) - \sin(x) + \frac{x}{2} \sin(x)\right) =$$

$$e^{x}\left(\cos(x) + \left(\frac{x-2}{2}\right) \sin(x)\right)$$