Analisi Matematica 1 - Compito del 14/09/2009

1)
$$f(x) = \frac{\sin(|x|)}{x} \quad \text{se } x \neq 0 \qquad f(0) = 0$$

(a) $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin(x)}{x} = 1 \neq 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(-x)}{x} = -1 \neq 0$$

DUNQUE f $\boxed{\text{NON}}$ è continua

(b) Dato che f non è continua essa $\boxed{\text{NON}}$ ammette derivata

(c) Dato che f ha limite destro finito essa è limitata in $]0, \delta]$ per $\delta > 0$ piccolo. Dato che

$$\lim_{x \to +\infty} f(x) = 0$$

f è limitata su $[M, +\infty[$ per M grande. Dato che f è continua su $[\delta, M]$ essa è ivi limitata. Dunque f è limitata su $]0, +\infty[$ e per motivi analoghi lo è su $]-\infty, 0[$.

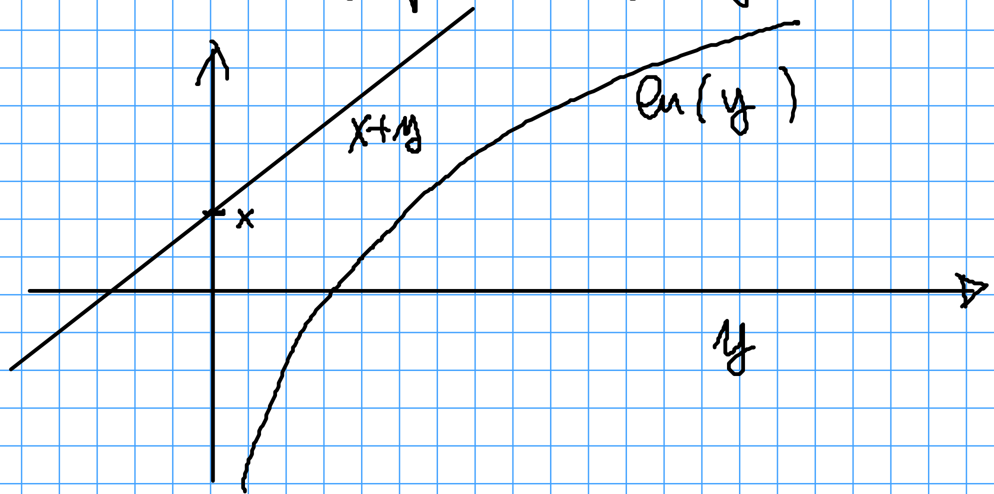
DUNQUE $\boxed{f \text{ È LIMITATA}}$

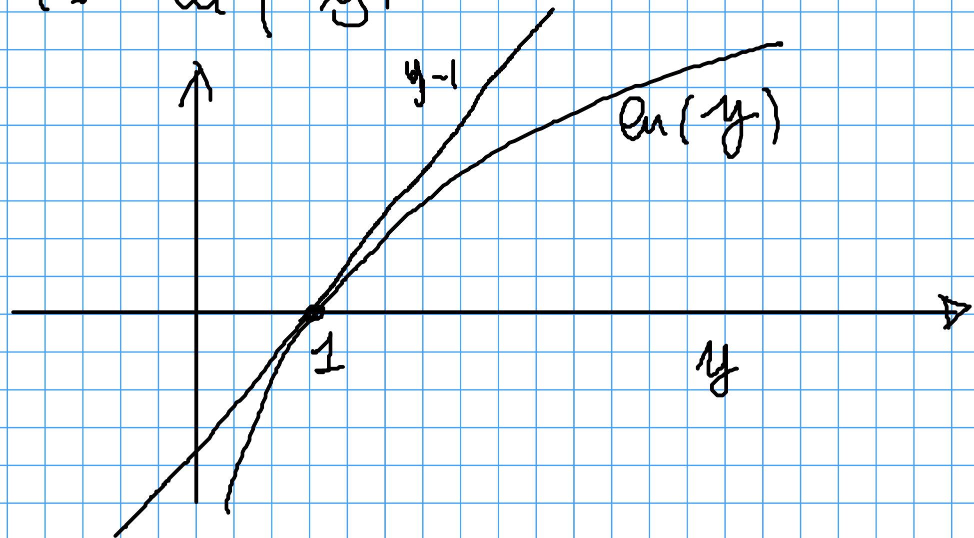
(d) È facile vedere che $f(-x) = \frac{\sin(|-x|)}{-x} = -\frac{\sin(|x|)}{x} = -f(x)$

$\Rightarrow$ $\boxed{f \text{ È DISPARI}}$ (NOTA CHE $f(0)=0$)

② $A = \{x : \ln(1+y) > x+y \quad \forall y>0\}$

A è l'insieme delle intercette delle rette $y \mapsto x+y$ che stanno sopra la funzione $y \mapsto \ln(y)$





La retta più bassa possibile è quella con $\boxed{x=-1}$ (cioè la tangente in $y=1$)

③ (a) $\lim_{n\to\infty} \frac{n^9+2^n}{1-10^{\sqrt{n}}-n^6} = \boxed{-\infty}$

Infatti

$$\frac{n^9+2^n}{1-10^{\sqrt{n}}-n^6} = \frac{2^n}{10^{\sqrt{n}}}\left(\frac{\frac{n^9}{2^n}+1}{\frac{1}{10^{\sqrt{n}}}-1-\frac{n^6}{10^{\sqrt{n}}}}\right)$$

e $\cdot$ $$\frac{2^n}{10^{\sqrt{n}}} = \frac{e^{n\ln(2)}}{e^{\sqrt{n}\ln(10)}} = e^{n\ln(2)-\sqrt{n}\ln(10)} \to e^{+\infty} = +\infty$$

$$\frac{n^9}{2^n} = \frac{e^{9\ln(n)}}{e^{n\ln(2)}} = e^{9\ln(n)-\ln(2)\cdot n} \to e^{-\infty} = 0$$

$$\frac{n^6}{10^{\sqrt{n}}} = \frac{e^{6\ln(n)}}{e^{\sqrt{n}\ln(10)}} = e^{6\ln(n)-\ln(10)\sqrt{n}} \to e^{-\infty} = 0$$

(b) $$\lim_{n\to+\infty}\left(\frac{n^4+2n^2+1}{n^4-2n^2+1}\right)^{n^2} = \lim_{n\to+\infty}\left(1+\frac{4n^2}{n^4-2n^2+1}\right)^{n^2} =$$

$$\lim_{n\to\infty} e^{n^2\ln\left(1+\frac{4n^2}{n^4-2n^2+1}\right)} = e^{\lim_{n\to\infty}\frac{n^2 4n^2}{n^4-2n^2+1}} = \boxed{e^4}$$

(o anche $$\left(\frac{n^2+1}{n^2-1}\right)^{2n^2} = \left(1+\frac{2}{n^2-1}\right)^{2n^2} = e^{2n^2\ln\left(1+\frac{2}{n^2-1}\right)} \simeq e^{\frac{4n^2}{n^2-1}} \to e^4$$)

(4) $$\lim_{x\to 0} \frac{e^{-x}\sqrt{\frac{1+x}{1-x}}-1}{\ln(1-x)+x} = 0$$

Infatti
$$e^{-x} = 1 - x + \frac{x^2}{2} + o(x^2)$$

$$\frac{1+x}{1-x} = 1 + \frac{2x}{1-x} = 1 + 2x(1-x)^{-1} = 1 + 2x\left(1 + x + o(x)\right) =$$
$$1 + 2x + 2x^2 + o(x^2)$$

$$\sqrt{\frac{1+x}{1-x}} = \left(1 + 2x + 2x^2 + o(x^2)\right)^{1/2} = 1 + \frac{1}{2}\left(2x + 2x^2 + o(x^2)\right) - \frac{1}{8}\left(2x + o(x)\right)^2$$
$$+ o\left(O(x)^2\right) = 1 + x + x^2 - \frac{1}{2}x^2 + o(x^2) = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$e^{-x}\sqrt{\frac{1+x}{1-x}} = \left(1 + x + \frac{x^2}{2} + o(x^2)\right)\left(1 - x + \frac{x^2}{2} + o(x^2)\right) =$$
$$1 + x + \frac{x^2}{2} - x - x^2 + \frac{x^2}{2} + o(x^2) = 1 + o(x^2)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 + o(x^2)$$

$$\frac{e^{-x}\sqrt{\frac{1+x}{1-x}} - 1}{\ln(1-x) + x} = \frac{o(x^2)}{-\frac{1}{2}x^2 + o(x^2)} \longrightarrow 0$$

METODO SIMILE (forse più chiaro). Sempre usando Taylor

$$e^{-x} = 1 - x + \frac{x^2}{2} + o(x^2)$$

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)$$

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{8}x^2 + o(x^2)$$

$$\Downarrow$$

RICORDIAMO CHE

$$(1+x)^{\alpha} = \sum_{h=0}^{m} \binom{\alpha}{h} x^h + o(x^m)$$

dove $\binom{\alpha}{h} = \frac{\alpha(\alpha-1)\cdots(\alpha-h+1)}{h!}$

$$e^{-x}\sqrt{\frac{1+x}{1-x}} = \left(1 - x + \frac{x^2}{2} + o(x^2)\right)\left(1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)\right)\left(1 + \frac{x}{2} + \frac{3}{8}x^2 + o(x^2)\right)$$

$$= \left(1 - x + \frac{x^2}{2} + o(x^2) + \frac{x}{2} - \frac{x^2}{2} + o(x^2) - \frac{x^2}{8} + o(x^2)\right)\left(1 + \frac{x}{2} + \frac{3}{8}x^2 + o(x^2)\right) =$$

$$= \left(1 - \frac{x}{2} - \frac{x^2}{8} + o(x^2)\right)\left(1 + \frac{x}{2} + \frac{3}{8}x^2 + o(x^2)\right) =$$

$$\left(1 - \cancel{\frac{x}{2}} - \cancel{\frac{x^2}{8}} + o(x^2) + \cancel{\frac{x}{2}} - \cancel{\frac{x^2}{4}} + o(x^2) + \cancel{\frac{3}{8}x^2} + o(x^2)\right) = 1 + o(x^2)$$

E DA QUI SI CONCLUDE COME NELL'ALTRA PAGINA

(5) $y'' + 3y' = \cos(x)$

Polinomio caratteristico $= P(z) = z^2 + 3z$; RADICI $\left\langle \begin{array}{l} z=0 \\ z=-3 \end{array} \right.$

$\Rightarrow$ sol. dell'omogenea : $y(x) = A e^{-3x} + B$

Cerco una sol particolare di $y'' + 3y' = e^{ix}$ come $\tilde{y}(x) = c e^{ix}$

$\Rightarrow$ $\tilde{y}' = ic e^{ix}$ $\quad \tilde{y}'' = -c e^{ix}$ $\quad \tilde{y}'' + 3\tilde{y}' = c(-1+3i) e^{ix}$

e quindi $c = \dfrac{1}{-1+3i} = \dfrac{-1-3i}{1+9} = -\dfrac{1+3i}{10}$. e $\tilde{y}(x) = \dfrac{(-1-3i)}{10} e^{ix}$

Se prendo la parte di $\tilde{y}$ trovo una sol. particolare dell'eq. iniziale

$\rightarrow$ $\bar{y}(x) = \mathrm{Re}\left[\dfrac{(-1-3i)}{10}(\cos(x) + i\sin(x))\right] = -\dfrac{1}{10}\cos(x) + \dfrac{3}{10}\sin(x)$

Dunque la sol. generale è $y(x) = A e^{-3x} + B + \dfrac{3}{10}\sin(x) - \dfrac{1}{10}\cos(x)$

e $y'(x) = -3A e^{-3x} + \dfrac{3}{10}\cos(x) + \dfrac{1}{10}\sin(x)$ .. Imponendo le cond. in $x=0$

$0 = y(0) = A + B - \dfrac{1}{10}$

$0 = y'(0) = -3A + \dfrac{3}{10}$

$\Rightarrow$ $A = \dfrac{1}{10}$ $\quad B=0$; in definitiva

$$\boxed{y(x) = \frac{1}{10} e^{-3x} + \frac{3}{10}\sin(x) - \frac{1}{10}\cos(x)}$$

Allora

(a) $\lim_{x \to -\infty} y(x) = +\infty \to$ $\boxed{NO}$

(b) $\lim_{x \to +\infty} y(x)$ NON ESISTE $\to$ $\boxed{NO}$

(c) $y(\pi) = \frac{1}{10} e^{-3\pi} + \frac{1}{10} \to$ $\boxed{SI'}$

(d) $y'(\pi) = -\frac{3}{10} e^{-3\pi} - \frac{3}{10} \to$ $\boxed{SI'}$

6) $\sum_{n=1}^{\infty} \frac{1+n^2}{n^\alpha + n^{1-5\alpha}}$. Sia $a_n = \frac{1+n^2}{n^\alpha + n^{1-5\alpha}}$

Allora $a_n \simeq \frac{n^2}{n^\alpha} = \frac{1}{n^{\alpha-2}}$ se $\alpha > 1-5\alpha$ $(\Leftrightarrow \alpha > 1/6)$

$a_n \simeq \frac{n^2}{n^{1-5\alpha}} = \frac{1}{n^{-1-5\alpha}}$ se $\alpha < 1-5\alpha$ $\left(\Leftrightarrow \alpha < \frac{1}{6}\right)$

$a_n \simeq \frac{n^2}{2n^{1/6}} = \frac{n^{11/6}}{2}$ se $\alpha = 1/6$

Chiaramente se $\alpha = \frac{1}{6}$ la serie diverge. Negli altri casi

$\alpha > \frac{1}{6}$ la serie conv. $\iff$ $\alpha - 2 > 1$ $\iff$ $\alpha > 3$

$\alpha < \frac{1}{6}$ la serie conv. $\iff$ $-1-5\alpha > 1$ $\iff$ $\alpha < -\frac{2}{5}$

In definitiva $\sum_{n=1}^{\infty} a_n$ conv. $\iff$ $\boxed{\alpha \in \left]-\infty, -\frac{2}{5}\right[\cup \left]3, +\infty\right[}$

7) $(*) = \int_0^{+\infty} \frac{x}{1+x^3}\, dx$. Cerco $\frac{x}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{x^2-x+1}$

$\Rightarrow$ $A(x^2-x+1) + B(x+x^2) + C(1+x) = x$

$$\begin{cases} A+B=0 \\ -A+B+C=1 \\ A+C=0 \end{cases} \quad \begin{cases} B=-A \\ -2A+C=1 \\ A+C=0 \end{cases} \quad \begin{cases} B=-A \\ C=-A \\ -3A=1 \end{cases} \quad \begin{cases} A=-1/3 \\ B=1/3 \\ C=1/3 \end{cases} \Rightarrow$$

$$\underset{(*)}{\overset{\parallel}{\frac{1}{3}}} \int_0^{+\infty} \left(-\frac{1}{x+1} + \frac{x+1}{x^2-x+1} \right) dx = \frac{1}{3} \int_0^{+\infty} \left(-\frac{1}{x+1} + \frac{x-1/2}{x^2-x+1} + \frac{3/2}{x^2-x+1} \right) dx =$$

$$= \frac{1}{3}\int_0^{+\infty}\left(-\frac{1}{x+1}+\frac{1}{2}\frac{\frac{d}{dx}(x^2-x+1)}{x^2-x+1}+\frac{3}{2}\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right)dx =$$

$$\frac{1}{3}\int_0^{+\infty}\left(\frac{1}{2}\frac{\frac{d}{dx}(x^2-x+1)}{x^2-x+1}-\frac{1}{x+1}\right)dx+\frac{1}{2}\,\frac{4}{3}\int_0^{+\infty}\frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} =$$

$$\frac{1}{3}\left[\ln\frac{\sqrt{x^2-x+1}}{x+1}\right]_0^{+\infty}+\frac{2}{3}\left[\frac{\sqrt{3}}{2}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^{+\infty} =$$

$$\parallel$$

$$0-0+\frac{\sqrt{3}}{3}\left(\frac{\pi}{2}+\operatorname{arctg}\frac{1}{\sqrt{3}}\right)=\frac{\sqrt{3}}{3}\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\frac{4\sqrt{3}\pi}{6\cdot 3}=$$

$$\frac{2\sqrt{3}}{9}\pi$$

8) $(x+1)\,y' = 3y + \dfrac{2x^2+4x+2}{x+2}, \quad y(0)=y_0 \qquad x > -1$

(a) In forma normale:

$$y' = \frac{3}{x+1}\,y + \frac{2(x+1)^2}{(x+2)(x+1)} = \frac{3}{x+1}\,y + \frac{2(x+1)}{(x+2)}$$

$$\Rightarrow \quad y(x) = (x+1)^3\left(y_0 + 2\int_0^x \frac{(t+1)}{(t+2)(t+1)^3}\,dt\right) =$$

$$(x+1)^3\left(y_0 + 2\int_0^x \frac{dt}{(t+2)(t+1)^3}\right) =$$

$$(x+1)^3\left(y_0 + 2\int_0^x \left\{\frac{A}{(t+2)} + \frac{B}{t+1} + \frac{C}{(t+1)^2}\right\} dt\right) \qquad \text{dove}$$

$$A(t+1)^2 + B(t+2)(t+1) + C(t+2) = 1 \iff$$

$$A(t^2+2t+1) + B(t^2+3t+2) + C(t+2) = 1 \iff$$

$$\begin{cases} A+B = 0 \\ 2A+3B+C = 0 \\ A+2B+2C = 1 \end{cases} \iff \begin{cases} B = -A \\ -A+C = 0 \\ -A+2C = 1 \end{cases} \iff \begin{cases} B = -A \\ C = A \\ A = 1 \end{cases} \iff \begin{cases} A = 1 \\ B = -1 \\ C = 1 \end{cases} \Longrightarrow$$

$$y(x) = (x+1)^3\left(y_0 + 2\left[\ln\left|\frac{t+2}{t+1}\right| - \frac{1}{t+1}\right]_0^x\right) =$$

$$(x+1)^3\left(c + 2\ln\left(\frac{x+2}{x+1}\right) - \frac{2}{x+1}\right)$$

dove $c = y_0 + 2 - \ln(4)$

(b)

$$\lim_{x\to 1^+} y(x) = \lim_{x\to 1^+} (x+1)^2\left(c(x+1) - 2(x+1)\ln\left(\frac{x+1}{x+2}\right) - 2\right) = 0^-$$

$$\lim_{x\to+\infty} y(x) = \begin{cases} +\infty & \text{se } c>0 \iff y_0 > \ln(4)-2 =: \bar{y} \\ -\infty & \text{se } c\leq 0 \iff y_0 < \bar{y} \end{cases}$$

Nel caso $c=0$ si può usare Taylor:

$$2(x+1)^3\left(-\frac{1}{x+1} + \ln\left(\frac{x+2}{x+1}\right)\right) = -2(x+1)^3\left(\frac{1}{x+1} - \ln\left(1+\frac{1}{x+1}\right)\right) =$$

$$-2(x+1)^3\left(\cancel{\frac{1}{x+1}} - \cancel{\frac{1}{x+1}} + \frac{1}{2}\frac{1}{(x+1)^2} + o\left(\left(\frac{1}{x+1}\right)^2\right)\right) = -(x+1) + o(x+1) \to -\infty$$

(c) Poniamo $F(x,y) = \dfrac{3y}{x+1} + \dfrac{2(x+1)}{x+2}$ $\left(\Rightarrow y' = F(x,y) \right)$

Allora $F(x,y) = 0 \iff y = -\dfrac{2}{3}\dfrac{(x+1)^2}{x+2} =: g(x)$

(e analogamente $F(x,y) \gtrless 0 \iff y \gtrless g(x)$.

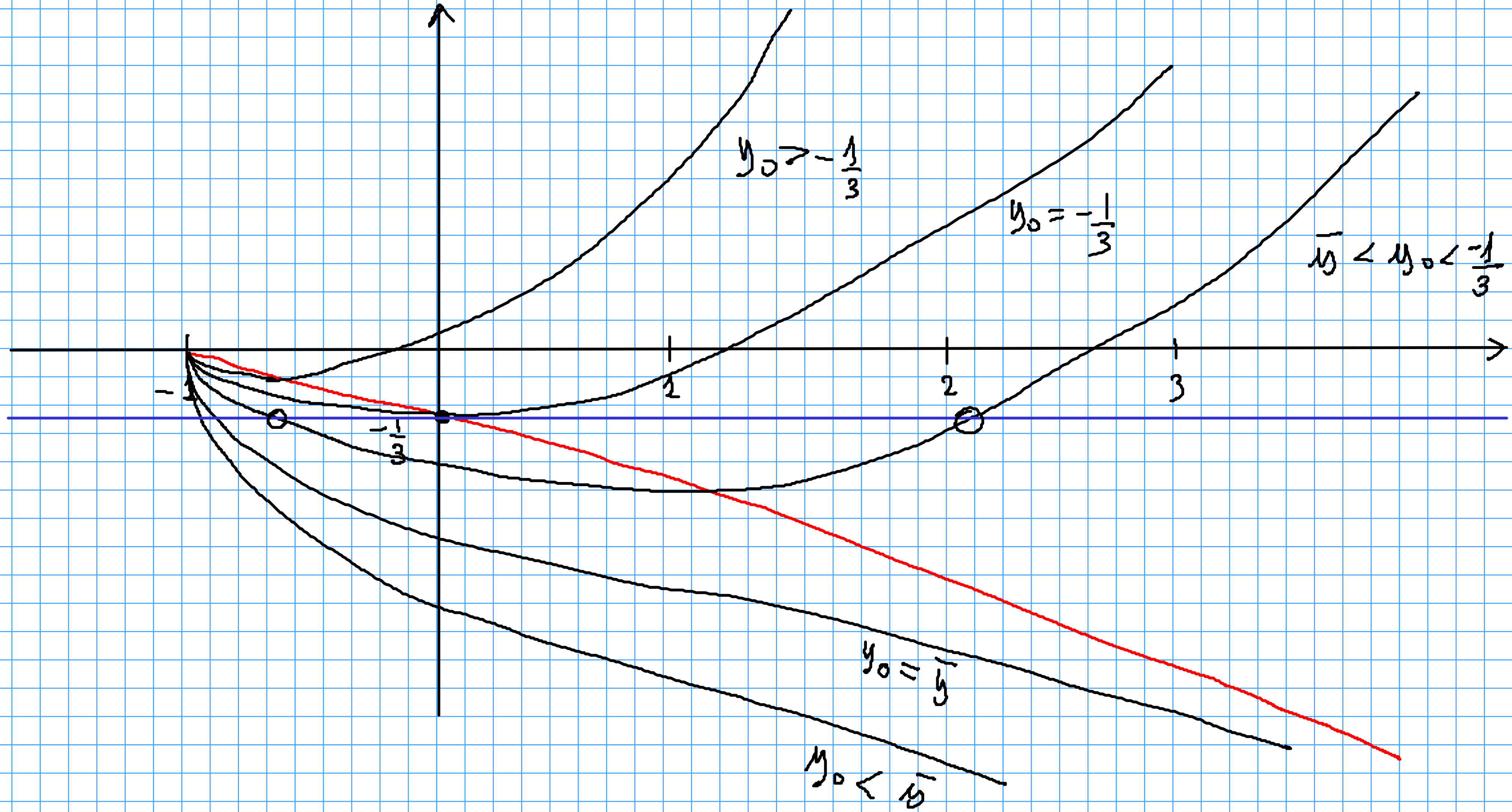
Studiamo $g(x)$ su $[-1, +\infty[$.

$g(-1) = 0$, $\lim_{x \to +\infty} g(x) = -\infty$; $g'(x) = -\dfrac{2}{3}\dfrac{2(x+1)(x+2) - (x+1)^2}{(x+2)^2} =$

$-\dfrac{2}{3}\dfrac{(x+1)}{(x+2)^2}\left(2x+4-x-1\right) = -\dfrac{2}{3}\dfrac{(x+1)}{(x+2)^2}\left(x+3\right)$ sempre < 0 su $]1, +\infty[$.

Ne risulta il grafico di g indicato in rosso $\left(\text{NOTA } g(0) = -\dfrac{1}{3}\right)$ nella figura (pagina successiva)

(d) Dai grafici si deduce che $y(x)$ attraversa due volte la retta $y = -\dfrac{1}{3}$ se e solo se $\bar{y} < y_0 < -\dfrac{1}{3}$



$y_0 > -\frac{1}{3}$
$y_0 = -\frac{1}{3}$
$\bar{y} < y_0 < -\frac{1}{3}$
$y_0 = \bar{y}$
$y_0 < \bar{y}$
-1
$-\frac{1}{3}$
1
2
3