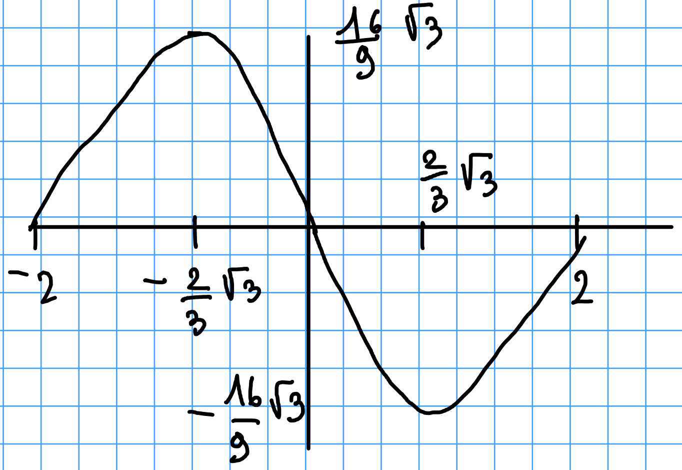
1) $f(x) = x^3 - 4x \qquad f(-2) = 0 \quad f(2) = 0$

$f'(x) = 3x^2 - 4 \;;\; f'(x) = 0 \iff x = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$



$f\left(\pm\frac{2}{\sqrt{3}}\right) = \mp\frac{16}{9}\sqrt{3} = \pm\frac{2}{\sqrt{3}}\left(\frac{4-12}{3}\right.$

SI VEDE ALLORA CHE LE RISPOSTE SONO

(a) $\boxed{\text{NO}}$ ci sono due minimi rel. $\left(0 \text{ e } -\frac{16}{3}\sqrt{3}\right)$

(b) $\boxed{\text{NO}}$ f non è pari (è addirittura dispari)

(c) $\boxed{\text{SI}}$ i pti di max rel. sono $-\frac{2}{3}\sqrt{3}$ e 2

(d) $\boxed{\text{NO}}$ f ha solo due pti stazionari $\left(\pm\frac{2\sqrt{3}}{2}\right)$

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2) Si vede facilmente che $a_n \leq 1 \;\forall n$ e se $n = 2k$

$a_{2k} = \frac{16k^4}{2+16k^4} \xrightarrow{k \to \infty} 1 \Rightarrow \sup a_n = 1$ $\boxed{b}$

3 (a) $$\frac{3^n - n!}{2^n - n^2} = \frac{n!}{2^n}\underbrace{\left(\frac{3^n/n! - 1}{1 - n^2/2^n}\right)}_{\to -1}$$

$\frac{n!}{2^n} \to +\infty$

DATO CHE (LIM. NOTEVOLI)

$$\frac{3^n}{n!} \to 0 \; ; \; \frac{n^2}{2^n} \to 0$$

$$\frac{n!}{2^n} \to +\infty$$

e quindi $$\lim_{n\to\infty} \frac{3^n - n!}{2^n - n^2} = \boxed{-\infty}$$

(b) $$\frac{\ln(1+2^n)}{n+1} = \frac{\ln(2^n(2^{-n}+1))}{n+1} = \frac{n\ln(2)}{n+1} + \frac{\ln(1+2^{-n})}{n+1}.$$

$\frac{n\ln(2)}{n+1} \to \ln(2)$, $\frac{\ln(1+2^{-n})}{n+1} \to 0$

Dato che $2^{-n} \to 0 \Rightarrow \ln(1+2^{-n}) \to \ln(1) = 0$;

$$\frac{n}{n+1} \to 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\ln(1+2^n)}{n+1} = \boxed{\ln(2)}$$

(c) $$\underbrace{\frac{n^2 + \sin(n) - n^3}{n^2+1}}_{(*)} \sin\left(\frac{1}{n}\right) = \frac{\cancel{n^3}}{\cancel{n^3}} \underbrace{\frac{\frac{1}{n} + \frac{1}{n^3}\sin(n) - 1}{1 + \frac{1}{n^2}}}_{\to -1} \, n\sin\left(\frac{1}{n}\right)$$

$n\sin\left(\frac{1}{n}\right) \to 1$

in fatti $\frac{\sin(n)}{n^3} \to 0$ (limitato × infinitesimo)

$n\sin\left(\frac{1}{n}\right) \to 1$ (limite notevole) $\Rightarrow \lim_{n\to\infty} (*) = \boxed{-1}$

(d) $n\sqrt[3]{n^3+6n+4}-n^2 = n^2\left(\sqrt[3]{1+\frac{6}{n^2}+\frac{4}{n^3}}-1\right) =$

$n^2\left(1+\frac{1}{3}\left(\frac{6}{n^2}+\frac{4}{n^3}\right)+o\left(\frac{6}{n^2}+\frac{4}{n^3}\right)-1\right) =$

$n^2\left(1+\frac{1}{3}\frac{6}{n^2}+o\left(\frac{1}{n^2}\right)-1\right) = n^2\left(\frac{2}{n^2}+o\left(\frac{1}{n^2}\right)\right) =$

$2+o(1) \longrightarrow \boxed{2}$

---

4 • $\cos(2x) = 1-\frac{(2x)^2}{2}+\frac{(2x^2)^2}{24}+o((2x^2)^2) =$

$1-2x^2+\frac{16}{24}x^4+o(x^4) = 1-2x^2+\frac{2}{3}x^4+o(x^4)$

VEDI IN FONDO PER UNO SVOLGIMENTO CON DE L'HÔPITAL

• $\sqrt[3]{1+y} = 1+\frac{1}{3}y+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2}y^2+o(y^2) = 1+\frac{y}{3}-\frac{y^2}{9}+o(y^2)$

$\Rightarrow \sqrt[3]{1-6x^2} = 1+\frac{1}{3}(-6x^2)-\frac{1}{9}(-6x^2)^2+o((-6x^2)^2) =$

$1-2x^2-\frac{36}{9}x^4+o(x^4) = 1-2x^2-4x^4+o(x^4)$

$\Rightarrow \frac{\cos(2x)-\sqrt[3]{1-6x^2}}{x^4} = \frac{\frac{2}{3}x^4+4x^4+o(x^4)}{x^4} = \frac{\frac{14}{3}x^4+o(x^4)}{x^4} \longrightarrow \boxed{\frac{14}{3}}$

5) (a) $a_n := (-1)^n \frac{\ln(1+n^2)}{n^2} \Rightarrow |a_n| = \frac{\ln(1+n^2)}{n^2} =$

$= \frac{1}{n^{3/2}} \frac{\ln(1+n^2)}{\sqrt{n}}$. Dato che $\frac{\ln(1+n^2)}{\sqrt{n}} \xrightarrow{n\to\infty} 0$ si ha che

$\frac{\ln(1+n^2)}{\sqrt{n}} \leq C$ per un'opportuna costante e allora

$|a_n| \leq \frac{C}{n^{3/2}}$. Dato che $\frac{3}{2} > 1$ $\sum_{n=1}^{\infty} \frac{C}{n^{3/2}} < +\infty$

$\Rightarrow \sum |a_n| < +\infty$ (teor. del confronto) $\Rightarrow \sum a_n$ $\boxed{A.C}$

(b) $a_n := (-1) \frac{n}{\ln(n)}$; dato che $\frac{n}{\ln(n)} \to +\infty$ $a_n \not\to 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{NC}$

(c) $a_n := (-1)^n \frac{n!\,2^n}{n^n}$; passiamo ai valori assoluti $|a_n| = \frac{n!\,2^n}{n^n}$

$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!\,2^{n+1}\,n^n}{n!\,2^n\,(n+1)^n} = 2\left(\frac{n}{n+1}\right)^n \to \frac{2}{e} < 1$

$\Rightarrow \sum_{n=1}^{\infty} |a_n| < +\infty$ (crit. del rapporto) $\Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{AC}$

(d) $a_n := (-1)^n \frac{\ln(n)}{n}$; $|a_n| = \frac{\ln(n)}{n} \geq \frac{1}{n}$. Dato che

$$\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty \Rightarrow \sum_{n=1}^{+\infty} |a_n| = +\infty \Rightarrow \sum a_n \text{ NON CONV. ASS.}$$

Però $|a_n| = \frac{\ln(n)}{n}$ è decrescente (si vede facilmente) $\Rightarrow$

$$\sum_{n=1}^{\infty} a_n \text{ converge (per Leibniz)} \Rightarrow \sum_{n=1}^{\infty} a_n \quad \boxed{C}$$

6) $f(x) := 3\operatorname{arctg}(x) + e^{2x}$. Allora $f(0) = 1$

$\Leftrightarrow f^{-1}(1) = 0$. Inoltre $f'(x) = \frac{3}{1+x^2} + 2e^{2x}$

$\Rightarrow f'(0) = 3 + 2 = 5$. Per cui $(f^{-1})'(1) = \frac{1}{5} \rightarrow \boxed{C}$

7
$$\int_0^{+\infty} \frac{dx}{(4+x^2)^2} = \frac{1}{4}\int_0^{+\infty} \frac{4+x^2-x^2}{(4+x^2)^2}\,dx = \frac{1}{4}\int_0^{+\infty} \frac{dx}{4+x^2} - \frac{1}{4}\int_0^{+\infty} \frac{x^2}{(4+x^2)^2}\,dx$$

$$= \frac{1}{16}\int_0^{+\infty} \frac{dx}{1+\left(\frac{x}{2}\right)^2} + \frac{1}{8}\int_0^{+\infty} x\left(-\frac{2x}{(4+x^2)^2}\right)dx =$$

(INTEGRO IL SECONDO PER PARTI: NOTA CHE $\frac{d}{dx}\frac{1}{4+x^2} = \frac{-2x}{4+x^2}$)

$$= \frac{1}{8}\left[\operatorname{arctg}\left(\frac{x}{2}\right)\right]_0^{+\infty} + \frac{1}{8}\left[x\,\frac{1}{4+x^2}\right]_0^{+\infty} - \frac{1}{8}\int_0^{+\infty} \frac{dx}{4+x^2} =$$

$$\frac{1}{8}\left(\frac{\pi}{2}-0\right)+\frac{1}{8}\left(0-0\right)-\frac{1}{32}\int_0^{+\infty}\frac{dx}{1+\left(\frac{x}{2}\right)^2}=\frac{\pi}{16}-\frac{1}{16}\left[\arctan\left(\frac{x}{2}\right)\right]_0^{+\infty}$$

$$=\frac{\pi}{16}-\frac{1}{16}\left(\frac{\pi}{2}-0\right)=\boxed{\frac{\pi}{32}}$$

8

$$y'=\frac{2}{x+2}y-\frac{x+2}{1-x^2} \qquad -1<x<1$$

L'equazione è del tipo $y'=a(x)y-b(x)$. Allora

$$A(x)=\int_0^x a(t)\,dt=\int_0^x\frac{2}{t+2}dt=2\ln(t+2)\Big|_0^x=2\ln\left(\frac{x+2}{2}\right)$$

$$\Rightarrow\quad y(x)=e^{A(x)}\left(y(0)-\int_0^x e^{-A(t)}b(t)\,dt\right)=$$

$$\frac{(x+2)^2}{4}\left(y_0-\int_0^x\frac{4}{(t+2)^2}\frac{t+2}{1-t^2}dt\right)=(x+2)^2\left(\frac{y_0}{4}+\int_0^x\frac{dt}{(t+2)(t-1)(t+1)}\right)=$$

$$(x+2)^2\left(\frac{y_0}{4}+\int_0^x\left(\frac{1/3}{t+2}+\frac{1/6}{t-1}+\frac{-\frac{1}{2}}{t+1}\right)dt\right)=$$

$$(x+2)^2\left(c+\frac{1}{6}\ln\left(\frac{(x+2)^2(1-x)}{(x+1)^3}\right)\right) \qquad \text{dove } c=\frac{y_0}{4}-\frac{1}{3}\ln(2)$$

Allora $\lim_{x\to -1^+} y(x) = +\infty$; $\lim_{x\to 1^-} y(x) = -\infty$

Per studiare la monotonia di y introduciamo

$$F(x,y) := \frac{2y}{x+2} - \frac{x+2}{1-x^2}. \quad F(x,y)=0 \iff y = \frac{(x+2)^2}{2(1-x^2)}.$$

Poniamo $g(x) := \dfrac{(x+2)^2}{2(1-x^2)}$ e studiamo g tra -1 e 1

$$\lim_{x\to -1^+} g(x) = \lim_{x\to 1^-} g(x) = +\infty \; ; \; g'(x) = \frac{2(x+2)(1-x^2) - (x+2)^2(-2x)}{2(1-x^2)^2} =$$

$$\frac{(x+2)\left(1-x^2+(x+2)x\right)}{(1-x^2)^2} = \frac{(x+2)(1+2x)}{(1-x^2)^2}. \quad g'(x)=0 \iff x = -\frac{1}{2}$$

$$g(-1/2) = 3/2$$

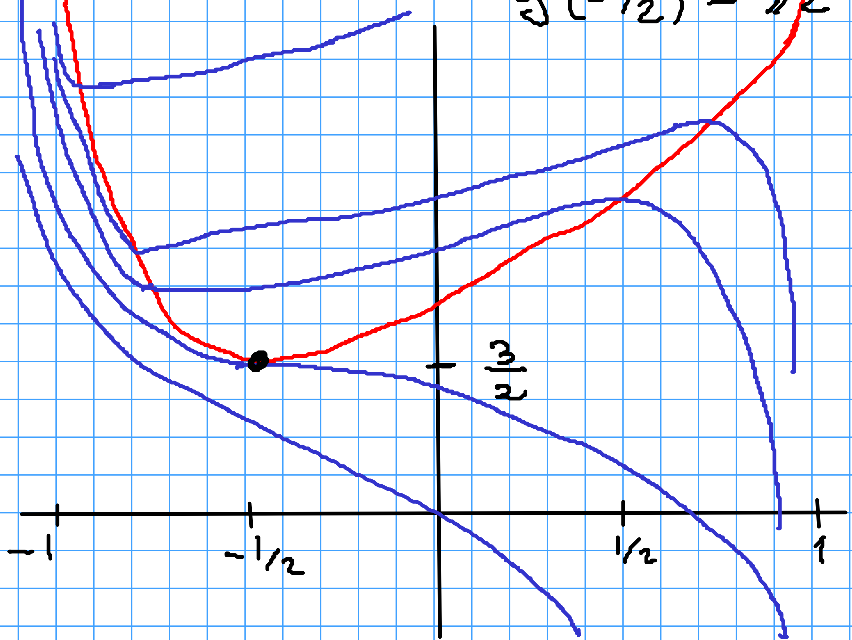
Ricordando che

$y' > 0 \iff y > g(x)$

$y' < 0 \iff y < g(x)$

$y' = 0 \iff y = g(x)$

si ottengono i grafici $\longrightarrow$



La curva che "divide" i due comportamenti è quella che in $-\frac{1}{2}$ vale $\frac{3}{2}$. Tale curva si ottiene imponendo

$$\frac{3}{2} = \left(-\frac{1}{2}+2\right)^2\left(c+\frac{1}{6}\ln\left(\frac{\left(-\frac{1}{2}+2\right)^2\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^3}\right)\right) \iff$$

$$\frac{3}{2} = \left(\frac{3}{2}\right)^2\left(c+\frac{1}{6}\ln\left(\frac{\left(\frac{3}{2}\right)^2\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3}\right)\right) \iff$$

$$1 = \frac{3}{2}\left(c+\frac{1}{6}\ln\left(3^3\right)\right) \iff \frac{2}{3} = c+\frac{1}{2}\ln(3) \iff c = \frac{2}{3}-\frac{1}{2}\ln(3)$$

$$\iff \frac{y_0}{4}-\frac{1}{3}\ln(2) = \frac{2}{3}-\frac{1}{2}\ln(3) \iff y_0 = \frac{8}{3}+\frac{4}{3}\ln(2)-2\ln(3)$$

Dunque y è strettamente decrescente $\iff$

$$y_0 \leq \frac{8}{3}+\frac{4}{3}\ln(2)-2\ln(3)$$

Per completezza riportiamo lo svolgimento del limite (es. 4) mediante de l'Hôpital

$$\lim_{x\to 0}\frac{\cos(2x)-\sqrt[3]{1-6x^2}}{x^4} \underset{(H)}{=} \lim_{x\to 0}\frac{-2\sin(2x)-\frac{1}{3}(1-6x^2)^{-\frac{2}{3}}\cdot(-12x)}{4x^3}$$

$$=\lim_{x\to 0}\frac{-2\sin(2x)+4x(1-6x^2)^{-\frac{2}{3}}}{4x^3} = (H)$$

$$=\lim_{x\to 0}\frac{-4\cos(2x)+4(1-6x^2)^{-\frac{2}{3}}+4x\left(-\frac{2}{3}\right)(1-6x^2)^{-\frac{5}{3}}(-12x)}{12x^2} =$$

$$=\lim_{x\to 0}\frac{-4\cos(2x)+4(1-6x^2)^{-\frac{2}{3}}+32x^2(1-6x^2)^{-\frac{5}{3}}}{12x^2} = (H)$$

$$=\lim_{x\to 0}\left(8\sin(2x)+4\left(-\frac{2}{3}\right)(1-6x^2)^{-\frac{5}{3}}(-12x)+64x(1-6x^2)^{-\frac{5}{3}}+\right.$$

$$\left.+32x^2\left(-\frac{5}{3}\right)(1-6x^2)^{-\frac{8}{3}}(-12x)\right)\frac{1}{24x} =$$

$$\lim_{x\to 0}\frac{8\sin(2x)+96x(1-6x^2)^{-\frac{5}{3}}+640x^3(1-6x^2)^{-\frac{8}{3}}}{24x} =$$

$$\frac{1}{3}\lim_{x\to 0}\frac{\sin(2x)}{x}+4\lim_{x\to 0}(1-6x^2)^{-\frac{5}{3}}+\frac{80}{3}\lim_{x\to 0}x^2(1-6x^2)^{-\frac{8}{3}} = \frac{2}{3}+4+0=\frac{14}{3}$$

$$\boxed{\text{MOLTO MEGLIO CON TAYLOR}}$$

NON VALE LA PENA
DI FARE UN ALTRO HÔPITAL