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Functional convergence of Snell envelopes: Applications to American options approximations

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Abstract. The main result of the paper is a stability theorem for the Snell envelope under convergence in distribution of the underlying processes: more precisely, we prove that if a sequence (X^n) of stochastic processes converges in distribution for the Skorokhod topology to a process X and satisfies some additional hypotheses, the sequence of Snell envelopes converges in distribution for the Meyer–Zheng topology to the Snell envelope of X (a brief account of this rather neglected topology is given in the appendix). When the Snell envelope of the limit process is continuous, the convergence is in fact in the Skorokhod sense.

This result is illustrated by several examples of approximations of the American options prices; we give moreover a kind of robustness of the optimal hedging portfolio for the American put in the Black and Scholes model.

Key words: American options, Snell envelopes, convergence in distribution, optimal stopping times

JEL classification: G13

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1 Introduction

If (S_t) is the risky asset price in the Black and Scholes model, it is well known (see e.g. [7] or [26]) that the discounted price at time t of the American put option with strike price K is given by

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$$J_t = \operatorname*{ess\,sup}_{t \leq \sigma \leq T} \mathbb{E}\left[\mathrm{e}^{-r\sigma} (K - S_\sigma)^+ \,|\mathscr{F}_t \right] ,$$

where σ is a stopping time with values in $[t, T]$, r is the istantaneous interest rate and the conditional expectation is with respect to the unique equivalent probability measure for which $\tilde{S}_t = \mathrm{e}^{-rt} S_t$ is a martingale. More precisely, the stochastic process J is the Snell envelope of the process $\mathrm{e}^{-rt} (K - S_t)^+$, i.e. the smallest supermartingale Z such that, for every t, $Z_t \geq \mathrm{e}^{-rt} (K - S_t)^+$.

It is natural to ask whether the Snell envelope is stable for a convergence of stochastic processes, i.e. if X^n converges to X in some sense, do the Snell envelopes of $f(t, X_t^n)$ converge to the Snell envelope of $f(t, X_t)$?

Discrete–time approximations to continuous-time models (see e.g. [10]) are frequently used in Mathematical Finance, and are based on stability results for convergence in distribution.

The celebrated Cox–Ross–Rubinstein simplified approach to the American Option price (see [8]) is justified by the convergence of Snell envelopes for the binomial approximation to the Black and Scholes model; paper [3] gives a stability theorem of Snell envelopes in the framework of diffusion processes and paper [6] investigates the same problem by non–standard techniques.

General results are contained in papers [21] and [22] of Lamberton and Pagès: they have proved that if X^n converges in distribution on the space $\mathbb{D} = \mathbb{D}([0, T]; \mathbb{R})$ (of regular right continuous paths) endowed with the Skorokhod topology, then (denoting by J^n and J the Snell envelopes of $f(t, X_t^n)$ and $f(t, X_t)$ respectively), under additional conditions J_0^n converges to J_0.

In this paper we extend the result of Lamberton and Pagès proving that, if X^n converges in distribution to X when $\mathbb{D}$ is endowed with the Skorokhod topology, under suitable additional conditions the stochastic processes J^n converge in distribution to J if $\mathbb{D}$ is endowed with a weaker topology, namely the Meyer-Zheng topology (a brief account of this topology will be given in the appendix).

The motivation for the use of the pseudo–paths or Meyer–Zheng topology, less commonly used than the well known Skorokhod topology, is essentially the convenience of the tightness/compactness criteria for laws of semimartingales.

We recall that a family X^i of stochastic processes with paths on the space $\mathbb{D}$ is said to be tight if the family of the probability distributions of the processes X^i is tight, that is if, for every $\varepsilon > 0$, there exists a compact K in $\mathbb{D}$ such that, for every i, $\mathbb{P}\left(X^i \in K\right) \geq 1 - \varepsilon$; so, by the theorem of Prokhorov, this family is relatively compact for convergence in distribution.

As it will be shown in Sect. 3, the sequence (J^n) of the Snell envelopes of a sequence of uniformly integrable stochastic processes is tight (and therefore relatively compact) for the Meyer–Zheng topology, while this result seems not to be true in general for the Skorokhod topology.

Naturally, when it is possible to show directly that the sequence J^n is relatively compact for the Skorokhod topology, our convergence theorem is valid also for this more habitual topology; this is the case, as we will show, if the Snell envelope of the limit process X is continuous.

The importance of obtaining a convergence result of the whole Snell envelope as a stochastic process will be evident in Sect. 6, where a kind of robustness of the optimal hedging portfolio will be proved for the Black and Scholes model.

2 Notations and preliminary results

Given a stochastic basis $(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$ with the usual assumptions, we will consider real valued adapted stochastic processes $Y = (Y_t)_{0 \le t \le T}$ with càdlàg (the French notation for "right continuous with left–hand limits") paths (we refer to [17] for definitions and properties concerning "General Theory of Stochastic Processes").

We indicate by $\mathbf{F}^Y$ the smallest (right continuous) filtration with respect to which Y is adapted, and by $\mathbb{P}^Y$ the distribution of Y on the canonical space $\mathbb{D} = \mathbb{D}([0,T];\mathbb{R})$ of càdlàg paths. If X is the canonical process of coordinate projections on $\mathbb{D}$, then $\mathbf{D} = (\mathscr{D}_t)_{0 \le t \le T}$ is the filtration generated by X and $\mathscr{D} = \mathscr{D}_T$.

We will consider on $\mathbb{D}$, besides the usual Skorokhod topology referred to as the S–topology ([17] is the best reference for our purposes) , the MZ–topology introduced by Meyer and Zheng in [27] . We refer to the Appendix for the definition and a brief account of the main properties of the MZ–topology.

We will use the notation $Y_n \Longrightarrow^S Y$ (respectively $Y_n \Longrightarrow^{MZ} Y$) to indicate that the probabilities $\mathbb{P}^{Y_n}$ converge strictly to $\mathbb{P}^Y$ when the space $\mathbb{D}$ is endowed with the S (respectively the MZ) –topology.

Let Y be a positive stochastic process of class (D) (i.e. the random variables Y_τ , for every stopping time τ , are uniformly integrable) : the Snell envelope of Y is the smallest supermartingale J such that, for every t, $J_t \ge Y_t$. Since the paths of Y are right continuous, it is known (see, for instance, [11] or [23]) that, for every stopping time τ ,

$$J_\tau = \operatorname*{ess\,sup}_{\tau \le \sigma \le T} \mathbb{E}[Y_\sigma | \mathscr{F}_\tau] \tag{2.1}$$

where σ varies in the set of stopping times with values in $[\tau, T]$. Moreover one has, for every stopping time τ and $A \in \mathscr{F}_\tau$,

$$\int_A J_\tau \, d\mathbb{P} = \sup_{\tau \le \sigma \le T} \int_A Y_\sigma \, d\mathbb{P} \tag{2.2}$$

(see [23] page 258).

The following lemma guarantees that the law of the Snell envelope of Y with respect to the proper filtration $\mathbf{F}^Y$ depends only on the law of Y .

Lemma 2.3 *Let J be the Snell envelope of Y with respect to the filtration $\mathbf{F}^Y$ and $\tilde{J}$ be the Snell envelope of the canonical process X when $\mathbb{D}$ is provided with the probability $\mathbb{P}^Y$: then $J = \tilde{J} \circ Y$.*

Proof. It is easy to see that $\tilde{J} \circ Y$ is a supermartingale on $(\Omega, \mathscr{F}, \mathbf{F}^Y, \mathbb{P})$ such that $\tilde{J}_t \circ Y \geq X_t \circ Y = Y_t$ and so $\tilde{J}_t \circ Y \geq J_t$. Conversely, for every stopping time σ on $(\mathbb{D}, \mathbf{D})$ with $\sigma \geq t$, since $\sigma \circ Y$ is a stopping time on Ω with respect to the filtration $\mathbf{F}^Y$, we have

$$\mathbb{E}^{\mathbb{P}^Y}[X_\sigma] = \mathbb{E}^{\mathbb{P}}[X_\sigma \circ Y] = \mathbb{E}^{\mathbb{P}}[Y_{\sigma \circ Y}] \leq \mathbb{E}^{\mathbb{P}}[J_t].$$

Then

$$\mathbb{E}^{\mathbb{P}}[\tilde{J}_t \circ Y] = \mathbb{E}^{\mathbb{P}^Y}[\tilde{J}_t] = \sup_{t \leq \sigma \leq T} \mathbb{E}^{\mathbb{P}^Y}[X_\sigma] \leq \mathbb{E}^{\mathbb{P}}[J_t]$$

and hence, for every fixed t, $\tilde{J}_t \circ Y = J_t$. The equality between stochastic processes is a consequence of the right continuity of the paths.

The following hypothesis was introduced by Brémaud and Yor in the paper [5] under the name of hypothesis (H).

Definition 2.4 *Let Y be a stochastic process adapted to a filtration $\mathbf{F}$: the pair $(Y, \mathbf{F})$ satisfies hypothesis (H) if every $\mathbf{F}^Y$–martingale is an $\mathbf{F}$–martingale.*

This condition is studied in detail in [5] : it is shown, in particular, that if Y is Markovian with respect to the filtration $\mathbf{F}$, then $(Y, \mathbf{F})$ satisfies (H).

Lamberton and Pagès also use this condition ([22] p. 349) : they show that the following is a sufficient condition for (H).

(2.5) there exists a dense subset $\mathscr{T} \subset [0, T]$ such that, for every $n \geq 1$, $t_1, \dots,$ $t_n, t \in \mathscr{T}$ and for every bounded continuous function defined on $\mathbb{R}^n$, one has

$$\mathbb{E}\left[h(Y_{t_1}, \dots, Y_{t_n}) | \mathscr{F}_t\right] = \mathbb{E}\left[h(Y_{t_1}, \dots, Y_{t_n}) | \mathscr{F}_t^Y\right].$$

The following lemma has already been proved in [22] ; we will present the proof for the sake of completeness.

Lemma 2.6 *If $(Y, \mathbf{F})$ satisfies hypothesis (H), the Snell envelopes of Y with respect to the filtrations $\mathbf{F}^Y$ and $\mathbf{F}$ coincide.*

Proof. Let J and Z respectively be the Snell envelopes with respect to $\mathbf{F}^Y$ to $\mathbf{F}$: since J is an $\mathbf{F}$–supermartingale with $J_t \geq Y_t$, we have $J_t \geq Z_t$. Conversely, it is easy to check that $\mathbb{E}[Z_t | \mathscr{F}_t^Y]$ is an $\mathbf{F}^Y$–supermartingale bigger than $\mathbb{E}[Y_t | \mathscr{F}_t^Y] = Y_t$ and so $\mathbb{E}[Z_t | \mathscr{F}_t^Y] \geq J_t$: in particular $\mathbb{E}[Z_t] \geq \mathbb{E}[J_t]$. We have necessarily $Z_t = J_t$.

More precisely we will be interested in the Snell envelope of a stochastic process of the form $f(t, Y_t)$ with respect to the filtration generated by Y : it is immediate to modify Lemmas 2.3 and 2.6 to this situation.

Lemma 2.7 *If Y is a supermartingale of class (D) and $(Y, \mathbf{F})$ satisfies (H), the Doob–Meyer decompositions of Y with respect to $\mathbf{F}^Y$ and $\mathbf{F}$ coincide.*

Proof. Let us write Y in the form $Y = M - A$, where M is an $\mathbf{F}^Y$–martingale and A an increasing predictable process with $A_0 = 0$: since M is also an $\mathbf{F}$–martingale and A is predictable with respect to $\mathbf{F}$, the result is a consequence of the uniqueness of the Doob–Meyer decomposition.

3 The convergence result

In this section we consider a sequence (X^n) of positive stochastic processes, satisfying the following hypotheses:

(3.1) *the processes* (X^n) *are uniformly of class (D), i.e. the r.v.* (X^n_τ) *, for* $n \in \mathbb{N}$ *and* τ *stopping time for the filtration* $\mathbf{F}^{X^n}$ *, are uniformly integrable;*

(3.2) *for every* $\varepsilon > 0$ *, there exist* $n_0 \in \mathbb{N}$ *and* $\delta > 0$ *such that if* $n \geq n_0$ *,* $0 < s < \delta$ *and* τ *is an* $\mathscr{F}^{X^n}$*–stopping time, we have*

$$\mathbb{E}^n \left[| X^n_{\tau+s} - X^n_\tau | \right] < \varepsilon .$$

(we require that all stopping times take values in $[0, T]$, but it is technically convenient to regard each process X as extended to $[0, +\infty)$ by putting $X_t = X_T$ for $t \geq T$: this enables us to write $X^n_{\tau+s}$ instead of $X^n_{(\tau+s)\wedge T}$).

Hypothesis (3.2) is known as the “Aldous tightness criterion” and (together with the condition that all r.v. (X^n_t) , for t fixed, are tight on $\mathbb{R}$, a condition which is obviously implied by (3.1)) guarantees the tightness of the laws $\mathbb{P}^{X^n}$ for the S–topology (see [1] page 365).

Lemma 3.3 *Given* $\varepsilon > 0$ *, there exist* $\gamma > 0$ *and* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$ *,* $0 < \delta < \gamma$ *and every stopping time* τ *one has*

$$\left| \mathbb{E}^n \left[X^n_\tau \right] - \mathbb{E}^n \left[\frac{1}{\delta} \int_\tau^{\tau+\delta} X^n_s \, ds \right] \right| \leq \varepsilon .$$

Proof. Due to the inequalities

$$\left| \mathbb{E}^n \left[X^n_\tau \right] - \mathbb{E}^n \left[\frac{1}{\delta} \int_\tau^{\tau+\delta} X^n_s \, ds \right] \right| = \left| \frac{1}{\delta} \int_0^\delta \mathbb{E}^n \left[X^n_{\tau+s} - X^n_\tau \right] \, ds \right|$$

$$\leq \frac{1}{\delta} \int_0^\delta \mathbb{E}^n \left[|X^n_{\tau+s} - X^n_\tau| \right] \, ds \ ;$$

the assertion is an evident consequence of (3.2).

Lemma 3.4 *Given* $\varepsilon > 0$ *, there exists a* $C > 0$ *such that, for every* $\delta > 0$ *,* $c > C$ *and every stopping time* τ *one has*

$$\mathbb{E}^n \left[\frac{1}{\delta} \int_\tau^{\tau+\delta} \left(X^n_s \wedge c \right) \, ds \right] \geq \mathbb{E}^n \left[\frac{1}{\delta} \int_\tau^{\tau+\delta} X^n_s \, ds \right] - \varepsilon .$$

Proof. By hypothesis (3.1), for every $n \in \mathbb{N}$ every stopping time ρ and c sufficiently large, we have

$$\mathbb{E}^n \left[X^n_\rho \wedge c \right] \geq E^n \left[X^n_\rho \right] - \varepsilon .$$

By considering that

$$\mathbb{E}^n\left[\frac{1}{\delta}\int_\tau^{\tau+\delta}\left(X^n_s\wedge c\right)\,ds\right] = \frac{1}{\delta}\int_0^\delta \mathbb{E}^n\left[X^n_{\tau+s}\wedge c\right]\,ds\ ,$$

one immediately concludes the proof.

Now we consider the pairs (X^n, J^n), where J^n is the Snell envelope of X^n for its natural filtration $\mathbf{F}^{X^n}$: by property (3.1), the positive supermartingales J^n are uniformly L^1–bounded. The sequence (X^n, J^n) is tight in $\mathbb{D}^2 = \mathbb{D}([0,T],\mathbb{R}^2)$ for the MZ–topology (see the Appendix) and so there exists a subsequence such that $(X^n, J^n) \Longrightarrow^{MZ} (X, J)$. (Here and in the sequel, for the sake of simplicity of notations, we will assume that indices have been renamed so that the whole sequence converges).

Let us now consider, for every n, a stopping time τ^n with respect to the filtration $\mathbf{F}^{X^n}$, and the law of (X^n, J^n, τ^n) on $\mathbb{D}^2\times[0,T]$: these laws are evidently tight for the product topology, if $\mathbb{D}^2$ is endowed with the MZ–topology.

Let (X, J, θ) be the canonical process on $\mathbb{D}^2\times[0,T]$ and $\mathbf{D}^\theta = .\mathscr{F}^{X,J,\theta}$ be the smallest right continuous filtration such that (X, J) are adapted and θ is a stopping time (i.e. $\mathscr{D}_t^\theta = \cap_{s>t}\,\sigma\left\{X_u, J_u, \{\theta\le u\}; u\le s\right\}$).

There exists a subsequence and a probability $\mathbb{Q}$ on $\mathbb{D}^2\times[0,T]$ such that

$$(X^n, J^n, \tau^n) \Longrightarrow^{MZ} (X, J, \tau)$$

(this notation means that the probabilities $\mathbb{P}^{X^n, J^n, \tau^n}$ converge strictly to $\mathbb{Q}$ for the product topology specified above).

Theorem 3.5 *Suppose that $X^n \Longrightarrow^S X$, that (3.1) and (3.2) are verified and that, for any stopping time τ^n for $\mathbf{F}^{X^n}$ and every limit law of (X^n, J^n, τ^n) on $\mathbb{D}^2\times[0,T]$, the pair $\left(X, \mathbf{F}^{X,J,\theta}\right)$ satisfies the hypothesis (H) of definition 2.2: then $(X^n, J^n) \Longrightarrow^{MZ} (X, J)$ where J is the Snell envelope of X.*

Proof. We can suppose that $(X^n, J^n) \Longrightarrow^{MZ} (X, J)$ (by considering if necessary a subsequence), and we begin by observing that, for every t,
$\mathbb{E}[J_t] \le \liminf_{n\to\infty}\mathbb{E}^n\left[J^n_t\right]$.

In fact, J being a positive supermartingale, we have

$$\mathbb{E}[J_t] = \sup_{c>0}\sup_{0<\delta<T-t}\mathbb{E}\left[\frac{1}{\delta}\int_t^{t+\delta}(J_s\wedge c)\,ds\right]\ ,$$

(and the same equality for every J^n): the functions $w\mapsto \frac{1}{\delta}\int_t^{t+\delta}(w(s)\wedge c)ds$ are continuous on $\mathbb{D}$ for the MZ–topology and consequently $J\mapsto\mathbb{E}[J_t]$ is lower–semicontinuous for MZ–convergence.

On $\mathbb{D}^2$, the pair $(X, \mathbf{F}^{X,J})$ satisfies hypothesis (H) (one can consider, for instance, the sequence of stopping times $\tau^n = T$): consequently the Snell envelope of X with respect to $\mathbf{F}^X$, denoted by Z, coincides with the one obtained with respect to the filtration $\mathbf{F}^{X,J}$.

J is an $\mathbf{F}^{X,J}$–supermartingale with $J_t \geq X_t$ for every t (and so $J_t \geq Z_t$): to complete the proof it is enough to prove that, for every t, $\mathbb{E}[Z_t] \geq \mathbb{E}[J_t]$.

Suppose this inequality is not true for a particular t, and let $\varepsilon = \dfrac{\mathbb{E}[J_t] - \mathbb{E}[Z_t]}{4} > 0$.

For every n, we can choose by (2.2) a stopping time τ^n with values in $[t, T]$ such that $\mathbb{E}^n\left[X^n_{\tau^n}\right] \geq \mathbb{E}^n\left[J^n_t\right] - \varepsilon$.

We can suppose that $(X^n, J^n, \tau^n) \Longrightarrow^{MZ} (X, J, \theta)$: we remark that, by 2.3 and 2.6, the Snell envelopes of X on $\mathbb{D}^2$ with respect to $\mathbf{F}^{X,J}$ and on $\mathbb{D}^2 \times [0, T]$ with respect to $\mathbf{F}^{X,J,\theta}$ have the same distribution.

By Lemmas 3.3 and 3.4, it is possible to determine $\gamma > 0$, $C > 0$ and $n_0 \in \mathbb{N}$ such that, for $0 < \delta < \gamma$, $c > C$ and $n > n_0$, one has

$$\mathbb{E}^n\left[\frac{1}{\delta}\int_{\tau^n}^{\tau^n+\delta}\left(X^n_s \wedge c\right)\, ds\right] \geq \mathbb{E}^n\left[X^n_{\tau^n}\right] - 2\varepsilon \geq \mathbb{E}^n\left[J^n_t\right] - 3\varepsilon\ ;$$

and, letting $n \to \infty$,

$$\mathbb{E}\left[\frac{1}{\delta}\int_{\theta}^{\theta+\delta}(X_s \wedge c)\, ds\right] \geq \mathbb{E}\left[J_t\right] - 3\varepsilon\ .$$

Letting $\delta \to 0$, one has $\mathbb{E}\left[X_\theta \wedge c\right] \geq \mathbb{E}[J_t] - 3\varepsilon$ and so $\mathbb{E}[Z_t] \geq \mathbb{E}[X_\theta] \geq \mathbb{E}[J_t] - 3\varepsilon$, which leads to a contradiction.

Remark 3.6 On checking the proof of the previous theorem, one can verify that the convergence $X^n \Longrightarrow^S X$ is not strictly necessary, but it is sufficient that $X^n \Longrightarrow^{MZ} X$ and that (3.1) and the assertion of Lemma 3.3 are satisfied. This slight extension doesn't seem to be very important since (3.2) is essentially always verified in applications.

Remark 3.7 Theorem 3.5 gives a general result which includes the most usual ways to approximate financial markets: e.g. the Cox–Ross–Rubinstein method is a particular case of Example 4.4 below. Also the results of the paper [3] could be viewed as a consequence of Theorem 3.5.

Remark 3.8 If the processes X^n are left continuous in expectation, for every t there exists a stopping time $\sigma^n \geq t$ such that $\mathbb{E}^n[J^n_t] = \mathbb{E}^n[X^n_{\sigma^n}]$ (see [11] Theorem 2.18): such a stopping time is called t–optimal. In this case one can verify that every limit law of (X^n, J^n, σ^n) is the law of (X, J, θ) where θ is a t–optimal stopping time for X.

Remark 3.9 In Theorem 3.5 we have considered the Snell envelope of X, but for the American put in the Black and Scholes model we need to consider the Snell envelope of $\mathrm{e}^{-rt}(K - S_t)^+$. More general processes are also considered: for instance, with a continuous cash flow rate $c(t, x)$ and a terminal payoff $H(t, x)$, the value of the American Option in a complete market is the Snell envelope of $\left(\int_0^t \frac{c(s,X_s)}{S_0(s)}ds + \frac{H(t,X_t)}{S_0(t)}\right)$, where $S_0(t)$ is the riskless asset.

One can easily extend the validity of Theorem 3.5 to these more general situations by considering $\mathbb{R}^2$–valued processes (X^n, S^n), and denoting by J^n the Snell envelope of S^n with respect to $\mathbf{F}^{X^n}$: we suppose that $(X^n, S^n) \Longrightarrow^S (X, S)$ and that S is $\mathbf{F}^X$–adapted.

If the processes S^n are positive and verify (3.1) and (3.2), and if, for every limit law of (X^n, S^n, J^n, τ^n) on $\mathbb{D}^3 \times [0,T]$, the pair $\left(X, \mathbf{F}^{X,S,J,\tau}\right)$ verifies hypothesis (H), then $(X^n, S^n, J^n) \Longrightarrow^{MZ} (X, S, J)$ and J is the Snell envelope of S.

4 Remarks on condition (H)

To apply Theorem 3.5, the most intriguing step is to verify for the limit process X that $(X, \mathbf{F}^{X,J,\theta})$ satisfies hypothesis (H).

This property is verified, for instance, if X is markovian with respect to the filtration $\mathbf{F}^{X,J,\theta}$: unfortunately, if every X^n is markovian with respect to $\mathbf{F}^{X^n,J^n,\tau^n} = \mathbf{F}^{X^n}$, this property doesn't hold for the limit for convergence in distribution.

Nevertheless it is shown in [21] that, if X^n is a Markov process and the transition probabilities satisfy some suitable regularity conditions, for every limit law of (X^n, τ^n) the canonical process X is markovian with respect to $\mathbf{F}^{X,\theta}$ (see [21] Lemma 4.1): their arguments may easily be extended to the situation that we are interested in.

We study another condition, suitable for processes with independent increments. Processes with independent increments are obviously markovian, but we don't require any regularity condition for transition probabilities.

Theorem 4.1 *Let (X^n, J^n) be stochastic processes and τ_n be stopping times: we suppose that X^n and τ^n are $\mathbf{F}^{X^n}$–adapted and that $(X^n, J^n, \tau^n) \Longrightarrow^{MZ} (X, J, \theta)$. If, for every n and $s < t$, $(X^n_t - X^n_s)$ is independent of $\mathscr{F}_s^{X^n,J^n,\tau^n} = \mathscr{F}^{X^n}$, then $(X_t - X_s)$ is independent of $\mathscr{F}_s^{X,J,\theta}$.*

Proof. We begin by observing that to every stopping time τ we may associate a stochastic process defined as the indicator function of the stochastic interval $[\![\tau, T]\!]$, that is $Z_s(\omega) = I_{\{s \geq \tau(\omega)\}}$.

The paths of this process are of the type $I_{[a,T]}(t)$ (with $0 \leq a \leq T$) and (since the functions $I_{[a_n,T]}$ converge in measure to $I_{[a,T]}$ if and only if $\lim_{n\to\infty} a_n = a$) it is obvious to conclude that τ_n converges in distribution to θ iff $Z^n = I_{[\![\tau^n,T]\!]}$ converges in distribution to $Z = I_{[\![\theta,T]\!]}$ on the space $\mathbb{D}$ endowed with the MZ–topology. So, if we consider the processes Z^n in place of the stopping times τ^n, we have $(X^n, J^n, Z^n) \Longrightarrow^{MZ} (X, J, Z)$ on the space $\mathbb{D}^3$.

Let us consider a countable dense subset $\mathscr{T} \subset [0,T]$ and a subsequence such that the finite dimensional distributions of $(X^n_t, J^n_t, Z^n_t)_{t\in\mathscr{T}}$ converge to those of

$(X_t, J_t, Z_t)_{t\in\mathscr{T}}$. Fix $t_1 < \ldots < t_k < s < t \in \mathscr{T}$ and a bounded continuous function g defined on $\mathbb{R}^{3k}$: we have, for every n,

$$\mathbb{E}^n\left[\exp(iu(X^n_t - X^n_s))\, g(X^n_{t_1}, \ldots, X^n_{t_k}, J^n_{t_1}, \ldots, J^n_{t_k}, Z^n_{t_1}, \ldots, Z^n_{t_k})\right] =$$

$$\mathbb{E}^n\left[\exp(iu(X^n_t - X^n_s))\right] . \mathbb{E}^n\left[g(X^n_{t_1}, \ldots, X^n_{t_k}, J^n_{t_1}, \ldots, J^n_{t_k}, Z^n_{t_1}, \ldots, Z^n_{t_k})\right]$$

and this equality continues to hold if we let n tend to infinity. Using a monotone class argument, we deduce that

$$\mathbb{E}\left[\exp(iu(X_t - X_s))\middle|\mathscr{G}_s\right] = \mathbb{E}\left[\exp(iu(X_t - X_s))\right] ,$$

where $\mathscr{G}_s$ is the σ–field generated by the random variables (X_r, J_r, Z_r) with $r < s$ (i.e. $(X_t - X_s)$ is independent from $\mathscr{G}_s$).

Considering the equality $\mathscr{F}_s^{X,J,Z} = \cap_{r>s}\mathscr{G}_r$ and the right continuity of the paths, we conclude the proof.

Remark 4.2 In many methods of discretization, $(X^n_t - X^n_s)$ is not independent of $\mathscr{F}_s^{X^n}$ but, for every $\varepsilon > 0$, there exists n_0 such that for $n > n_0$, $(X^n_t - X^n_s)$ is independent of $\mathscr{F}_{s-\varepsilon}^{X^n}$. It is obvious that in this case also, the conclusion of Theorem 4.1 holds.

In general the stochastic process S, which represents the value of the risky asset, does not have independent increments. Nevertheless we present some examples in which it is possible to apply Theorem 4.1.

Example 4.3 The process S satisfies the equation $dS_t = S_{t-}\, dX_t$, where X is a semimartingale with independent increments and $\Delta X_t > -1$ (see [13]) : the filtrations generated by X and by S coincide (observe that X satisfies the equation $dX_t = (S_{t-})^{-1}\, dS_t$).

Taking a sequence X^n of stochastic processes such that $X^n \overset{S}{\Longrightarrow} X$, it is well known that $S^n \overset{S}{\Longrightarrow} S$. If the increments of the processes (X^n) are independent (or also asymptotically independent in the sense of Remark 4.2), and if the sequence (S^n) satisfies (3.1) and (3.2), then Theorem (3.5) holds.

Example 4.4 Consider a market in which the risky asset satisfies the stochastic differential equation

$$dS_t = S_{t-}(\mu dt + \sigma dW_t + \varphi dN_t) \tag{4.5}$$

where $\sigma > 0$, $\varphi > -1$, W is a Wiener process and N an independent Poisson process with intensity λ (when $\varphi = 0$, the model reduces to the Black and Scholes model).

If $\varphi \neq 0$, the market is not complete, but in [18] it is shown that, considering another asset which satisfies a stochastic differential equation similar to (4.5), there exists an equivalent martingale probability $\mathbb{P}^*$ with respect to which the new market is complete. If we consider an American put on the asset (S_t) with strike price K, the option price at time t is given by $U_t = u(t, S_t)$, where $u(t,x) = \sup_{t\leq\tau\leq T} \mathbb{E}^* \left[e^{r(t-\tau)} (K - S_\tau)^+ \middle| S_t = x \right]$, τ being a stopping time.

We pose, for $n \in \mathbb{N}$ and for $\frac{k}{n}T \leq t < \frac{k+1}{n}T$

$$W_t^n = \left(\sum_{j\leq k} X_j \right) \cdot \sqrt{\frac{T}{n}} \qquad \text{and} \qquad N_t^n = \sum_{j\leq k} Y_j$$

where X_j are independent Bernoulli r.v. (with mean 0 and variance 1) and Y_j are independent Poisson r.v. with parameter $\lambda\frac{T}{n}$.

It is easy to check that $(W^n, N^n) \stackrel{S}{\Longrightarrow} (W, N)$ and so $S^n \stackrel{S}{\Longrightarrow} S$ (S^n satisfies (4.5) with W and N replaced by W^n and N^n). The filtration generated by S coincides with the one generated by W and N, and (by Remark 4.2) we can apply Theorem 3.5 (the validity of (3.1) and (3.2) for the sequence S^n is straightforward).

The processes S^n are in fact finite–time stochastic processes and therefore their Snell envelopes can be explicitly computed: we recall that if $(Z_i)_{i=0,1,\dots,N}$ is a stochastic process adapted to a finite filtration $\mathbf{F} = (\mathscr{F}_i)_{i=0,\dots,N}$, its Snell envelope (U_i) is given by the formula:

$$\begin{cases} U_N &= Z_N \\ U_i &= \max\{Z_i, \mathbb{E}[U_{i+1}|\mathscr{F}_i]\} \end{cases}$$

So one finds an explicit formula for the approximating option prices: this formula contains a series which can be truncated with the required precision. For the case $\varphi = 0$ this formula is exposed in [8] ; for the general case [25] illustrates a similar approximation, but less convenient in practical computations.

Example 4.6 In the same situation as in the previous example, we consider a sequence of stochastic process S^n satisfying the equation

$$dS_t^n = S_{t-}^n \left(\mu dt + \frac{\sigma d\tilde{N}_t^n}{\sqrt{n}} + \varepsilon dN_t \right) \tag{4.7}$$

where N is a standard Poisson process and $\tilde{N}^n$ a compensated Poisson process with intensity n.

In a similar way as in Example 4.4, one proves that $S^n \stackrel{S}{\Longrightarrow} S$ and that Theorem 3.5 holds; for the practical computations of the Snell envelope of a process of the form $f(t, S_t^n)$, one can utilize a discretization procedure as in 4.4.

Example 4.8 If we consider the Black and Scholes model and a sequence of approximating process S^n where the Wiener process is approximated by Markov chains, it is shown in [20] that the sequence of critical prices $s_n(t)$ corresponding to the approximating puts, converges uniformly to $s(t)$, the critical price in the Black and Scholes model.

Since also in the case in which S satisfies an equation of the form (4.5) the critical price is smooth, it is possible to extend the method of [20] and prove (by utilizing Remark 3.8) that, also in Examples 4.4 and 4.6, the sequence of critical prices for the approximating models converges uniformly to the critical price of the limit model.

5 Conditions for Skorokhod convergence

In this section we examine additional conditions which guarantee that the convergence of Snell envelopes is with respect to the S–topology on $\mathbb{D}$.

We consider a sequence (X^n) of stochastic processes satisfying the hypothesis of Theorem 3.5: let, for every n, $J^n = M^n - A^n$ the Doob–Meyer decomposition of the Snell envelope J^n with respect to $\mathbf{F}^{X^n}$. We point out that, by property (2.2), the processes (J^n) are uniformly of class (D), and consequently also the r.v. A^n_T are uniformly integrable (see [9]. chap. VII Theorem 16): so the sequence (J^n, M^n, A^n) is tight in $\mathbb{D}^3$ for the MZ–topology (see the Appendix) and there exists a subsequence such that $(J^n, M^n, A^n) \overset{MZ}{\Longrightarrow} (J, M, A)$, where M is a martingale and A an increasing process.

Lemma 5.1 *The sequence* (A^n) *is tight for the S–topology and therefore* $A^n \overset{S}{\Longrightarrow} A$.

Proof. Let τ be a stopping time for the filtration $\mathbf{F}^{X^n}$ and $s > 0$: since $J^n - A^n$ is a martingale, one has

$$\mathbb{E}^n\left[A^n_{\tau+s} - A^n_\tau\right] = \mathbb{E}^n\left[J^n_\tau - J^n_{\tau+s}\right].$$

Given $\varepsilon > 0$, there exists, by (2.2), a stopping time $\sigma \ge \tau$ such that $\mathbb{E}^n\left[J^n_\tau\right] \le \mathbb{E}^n\left[X^n_\sigma\right] + \varepsilon$. Consequently

$$\mathbb{E}^n\left[A^n_{\tau+s} - A^n_\tau\right] \le \mathbb{E}^n\left[X^n_\sigma\right] - \sup_{\rho \ge \tau+s} \mathbb{E}^n\left[X^n_\rho\right] + \varepsilon$$

$$\le \mathbb{E}^n\left[X^n_\sigma\right] - \mathbb{E}^n\left[X^n_{\sigma+s}\right] + \varepsilon \le \mathbb{E}^n\left[\left|X^n_\sigma - X^n_{\sigma+s}\right|\right] + \varepsilon .$$

Since (X^n) satisfies (3.2), it is evident that (A^n) also satisfies (3.2) and (3.1) is satisfied since the r.v. A^n_T are uniformly integrable. Therefore the sequence (A^n) is tight for the S–topology.

Proposition 5.2 *The limit process A is continuous.*

Proof. We begin by observing that, for every $\eta > 0$,

$$\lim_{n\to\infty} \sup_{\tau \in \mathscr{P}^n} \mathbb{P}^n\left\{\Delta A^n_\tau > \eta\right\} = 0\,, \tag{5.3}$$

where $\mathscr{P}^n$ is the set of all predictable strictly positive stopping times for the filtration $\mathbf{F}^{X^n}$. Suppose it is possible that (5.3) were false: there exists a subsequence n_k, two positive numbers ε and η and, for every k, a predictable stopping time τ^k such that $\mathbb{P}^{n_k}\left\{\Delta A^{n_k}_{\tau^k} > \eta\right\} > 2\varepsilon$.

Let ρ^k be a stopping time such that $\rho^k < \tau^k$ and $\mathbb{P}^{n_k}\left\{\rho^k + \frac{1}{k} \geq \tau^k\right\} > 1 - \varepsilon$ (the existence of such a stopping time is due to the predictability of τ^k, see e.g. [17] page 17). We remark that

$$\left\{\Delta A^{n_k}_{\tau^k} > \eta\,,\ \rho^k + \frac{1}{k} \geq \tau^k\right\} \subseteq \left\{\left(A^{n_k}_{\rho^k+\frac{1}{k}} - A^{n_k}_{\rho^k}\right) > \eta\right\}$$

and so $\mathbb{P}^{n_k}\left\{\left(A^{n_k}_{\rho^k+\frac{1}{k}} - A^{n_k}_{\rho^k}\right) > \eta\right\} > \varepsilon$, which is in contradiction with the property (3.2) for the sequence (A^n).

Now let, for $u > 0$, $\tau^n(u)$ be the predictable stopping time

$$\tau^n(u) = \inf\left\{s > 0\ :\ \Delta A^n_s > u\right\} \wedge T\,.$$

This stopping time is strictly positive because $\Delta A^n_0 = A^n_0 = 0$, and by (5.3) it is evident that $(\tau^n(u))$ converges in distribution to T.

By Theorem 5.12 of [16], for u in a dense subset of $(0, +\infty)$, $\left(\tau^n(u)\,,\ \Delta A^n_{\tau^n(u)}\right)$ converges in distribution to $\left(\tau(u)\,,\ \Delta A_{\tau(u)}\right)$ (where $\tau(u)$ is defined for A in the same way as $\tau^n(u)$ for A^n) and this implies the continuity of the limit process A.

Remark 5.4 Since the process A is continuous, it is predictable with respect to the canonical filtration on $\mathbb{D}^3$, and therefore $J = M - A$ is the Doob–Meyer decomposition of J for the filtration $\mathbf{F}^{J,M,A}$. If the pair $(J, \mathbf{F}^{J,M,A})$ satisfies property (H) of (2.4), then it coincides with the decomposition with respect to $\mathbf{F}^J$ (see 2.7).

Theorem 5.5 *Let (X^n) be a sequence of stochastic processes satisfying the hypothesis of Theorem 3.5 and suppose that the Snell envelope J of the limit process X is continuous: then $(X^n, J^n) \Longrightarrow^S (X, J)$.*

Proof. Given the decomposition $J = M - A$, M is a continuous martingale and $M^n \Longrightarrow^{MZ} M$.

Aldous showed that, if the finite dimensional distributions of (M^n) converge to those of M and M is a continuous martingale, then $M^n \Longrightarrow^S M$ (see [2] Proposition 1.2).

The MZ–convergence implies the convergence of finite dimensional distributions only for a subset $\mathscr{T} \subset [0, T]$ of full Lebesgue measure (see the Appendix), but a careful analysis of the proof of [2] shows that, in fact, the author makes use only of the property that (M^n) converges in distribution to M for the topology of convergence in measure (i.e. the MZ-topology) on $\mathbb{D}$.

Therefore $M^n \Longrightarrow^S M$ and, since the limit processes J, M and A are continuous, $J^n \Longrightarrow^S J$ and $(X^n, J^n) \Longrightarrow^S (X, J)$.

The last theorem applies in particular to the Black and Scholes model: in this case the approximations we have proposed in Examples 4.4 and 4.6 converge in fact for the S–topology.

6 Stability of the optimal hedging strategy for the Black and Scholes model

Let (S_t^0, S_t) be the asset prices in the Black and Scholes model, more precisely the risky asset (S_t) satisfies the equation

$$\begin{cases} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_o &= x \end{cases}$$

and let $S_t^0 = \mathrm{e}^{rt}$ be the riskless asset. Let $\beta_t = (S_t^0)^{-1}$ be the *discounting coefficient* at time t. If J_t is the discounted price at time t of the American put option with strike price K, then J satisfies the equation (see [7] or [26]):

$$J_t = x + \int_0^t P_x(u, S_u)\, d\tilde{S}_u \;-\; \int_0^t \mathrm{e}^{-ru} rKI_{\{S_u < s(u)\}}\, du\,,$$

where $\tilde{S}_u = S_u\beta_u$, $s(u)$ is the "critical price" at time u, $P(t,x)$ is the American put value function and the derivative $P_x(t,x)$ is the "delta" of the American put.

In this formula, the process $\left(x + \int_0^t P_x(u, S_u)\, d\tilde{S}_u\right)_{0\le t\le T}$ represents the optimal discounted portfolio and $\left(\int_0^t \mathrm{e}^{-ru} rKI_{\{S_u<s(u)\}}\, du\right)_{0\le t\le T}$ the consumption. So this equality gives a formula for the optimal hedging strategy.

We consider a sequence (S^n) of stochastic processes and let $\tilde{S}_u^n = S_u^n\beta_u$: we suppose that on $(\Omega^n, \mathscr{F}^n, \mathbb{P}^n)$, there exists an equivalent probability such that, when Ω^n is endowed with this probability, $S^n \Longrightarrow^S S$. It is evident that $(S^n, \tilde{S}^n) \Longrightarrow^S (S, \tilde{S})$.

Let J^n be the Snell envelope of $\beta_t(K - S_t^n)^+$ (the discounted value of the American put with respect to S^n) and consider, for every n, the Doob–Meyer decomposition $J^n = M^n - A^n$ relative to the filtration $\mathbf{F}^{S^n}$. We suppose that the hypotheses of Theorem 3.5 are satisfied and so, by Theorem 5.5

$$(S^n, J^n, M^n, A^n) \Longrightarrow^S (S, J, M, A)$$

where $J = M - A$ is the Doob–Meyer decomposition of J.

Theorem 6.1 *Suppose that for the limit law of (S^n, J^n, M^n, A^n) on the canonical space $\mathbb{D}^4$ the pair $(S, \mathbf{F}^{S,J,M,A})$ satisfies condition (H) of (2.4). Then:*
(a) if, for every n, the market represented by S^n is complete, and if V^n is the optimal (discounted) hedging portfolio for S^n, then

$$\left(V_t^n - x - \int_0^t P_x(u, S_{u-}^n)\, d\tilde{S}_u^n\right)_{0\le t\le T} \Longrightarrow^S 0\,;$$

(b) further, in general

$$\left(x + \int_0^t P_x(u, S_{u-}^n)\, d\tilde{S}_u^n - J_t^n\right)_{0\le t\le T} \Longrightarrow^S \left(\int_0^t \mathrm{e}^{-ru} rKI_{\{S_u<s(u)\}}\, du\right)_{0\le t\le T}.$$

This means that the optimal hedging strategy for the Black and Scholes model gives, to the limit, the optimal strategy in the case of complete markets and a hedging strategy in the case of more general markets.

Proof. Since $J_t = \mathrm{e}^{-rt} P(t, S_t)$ and P_x is strictly positive (see [26]), the filtrations generated by J and S coincide: so the pair $(J, \mathbf{F}^{J,M,A})$ satisfies condition (H). By Lemma 2.7, $J = M - A$ is the Doob–Meyer decomposition of J with respect to $\mathbf{F}^S$.

Hence the following relations hold:

$$M_t = x + \int_0^t P_x(u, S_u)\, d\tilde{S}_u \ ;$$

$$A_t = \int_0^t \mathrm{e}^{-ru} rKI_{\{S_u < s(u)\}}\, du \ .$$

Moreover, the following property holds:

$$\left(S_t \, , \, x + \int_0^t P_x(u, S^n_{u-})\, d\tilde{S}^n_u\right)_{0 \le t \le T} \Longrightarrow^S \left(S_t \, , \, x + \int_0^t P_x(u, S_u)\, d\tilde{S}_u\right)_{0 \le t \le T} .$$

The last equation is, for instance, a consequence of the converging result presented in [24] (taking into account that $(S^n, \tilde{S}^n) \Longrightarrow^S (S, \tilde{S})$, that $P_x(t, .)$ is continuous and bounded and that the martingales $\tilde{S}^n$ satisfy condition c) of Proposition 3.2 in [24]).

We put $P^n_t = x + \int_0^t P_x(u, S^n_{u-})\, d\tilde{S}^n_u$ and $P_t = x + \int_0^t P_x(u, S_u)\, d\tilde{S}_u$, and we observe that $M = P$.

Since all the limit processes are continuous, we have

$$(S^n, J^n, M^n, A^n, P^n) \Longrightarrow^S (S, J, M, A, P).$$

Property (b) follows from $(P^n - J^n) \Longrightarrow^S (P - J) = A$.

Property (a) follows on taking into account the fact that, if the markets are complete, the martingale M^n represents the optimal hedging portfolio for the approximating model and that $(M^n - P^n) \Longrightarrow^S (M - P) = 0$.

Example 6.2 Let us consider the sequence of processes S^n defined by the equation

$$dS^n_t = S^n_{t-}\, (\mu dt + \sigma dW_t + \varepsilon_n dN_t)$$

where N is a Poisson process and $\varepsilon_n \to 0$. It is easy to check that $S^n \Longrightarrow^S S$ and that condition (H) is satisfied: so we may apply part (b) of Theorem 6.1.

Further, it is possible (see [18]) to consider a second asset R^n such that the new market is complete. So

$$\left(V^n_t - x - \int_0^t P_x(u, S^n_{u-})\, d\tilde{S}^n_u\right)_{0 \le t \le T} \Longrightarrow^S 0$$

where V^n is the optimal portfolio based on (S^n, R^n).

Remark 6.3 The "delta" $P_x(t,x)$ is not known explicitly, and so numerical approximations are needed: see for instance [7] p. 93 for an efficient analytic approximation.

Theorem 6.1 should be compared with the deep results of the paper [13]; nevertheless, we remark that the methods of [13] cannot be applied to a "perturbation" of the Black and Scholes model with a jump–diffusion as in Example 6.2.

7 Conclusions

The Snell envelope is stable for convergence in distribution, provided that a further hypothesis is satisfied by the limit process: nevertheless this condition (which has been stated in Sect. 2 under the name of hypothesis (H)) is usually satisfied in the approximation–discretization methods considered in the literature.

Therefore a general convergence theorem is available (Theorem 3.5) which furnishes a unified approach to several approximation results (for instance the Cox–Ross–Rubinstein approach [8], or the Amin–Khanna results [3]).

Appendix

The MZ-topology

Let λ be the normalized Lebesgue probability measure on $[0,T]$, and let $w(t)$ be a real Borel function defined on $[0,T]$: the pseudo–path of w is the image measure of λ under the mapping $t \mapsto (t, w(t))$.

The mapping which associates to a path w its pseudo–path, restricted to $\mathbb{D}$, is injective and provides an imbedding of $\mathbb{D}$ into the compact space of all probabilities defined on the compact set $[0,T]\times[-\infty,+\infty]$: the induced topology on $\mathbb{D}$ is the pseudo–path or Meyer–Zheng topology. The Borel σ–field on $\mathbb{D}$ for the MZ–topology, as for the S–topology, coincides with the canonical σ–field $\mathscr{D}$.

Endowed with the MZ–topology, $\mathbb{D}$ is a metric (but not a Polish) space: so (contrary to the case of S–topology) for a family of stochastic processes with paths on $\mathbb{D}$, the tightness condition is only a sufficient condition for relative compactness for convergence in distribution.

The MZ–topology on $\mathbb{D}$ is in fact the topology of convergence in measure, much weaker than the S–topology ([27], Lemma 1), but the original definition is better adapted to give tightness conditions for supermartingales and quasimartingales.

The following result has been used in Sect. 3 (see [27] page 262–263):

(A.1) *Let (X^n) be a sequence of positive supermartingales uniformly L^1–bounded (that is $\sup_n \mathbb{E}^n\left[X_0^n\right] = C < +\infty$). Then the family of distributions $\mathbb{P}^{X^n}$ is relatively compact on $\mathbb{D}$ for the MZ–topology, and for every limit law $\mathbb{P}$, the canonical process X is a positive supermartingale such that $\mathbb{E}\left[X_0\right] \leq C$.*

A similar result holds for a sequence (M^n) of uniformly L^1–bounded martingales such that, for every fixed t, the r.v. (M_t^n) are uniformly integrable ([27] page 368), and for a sequence (A^n) of positive increasing processes which are uniformly L^1–bounded ([27] page 367).

Contrary to the case of S–topology, the MZ–topology on the product space $\mathbb{D}^k = \mathbb{D}\left([0,T],\mathbb{R}^k\right)$ is the product topology: so, given a family $(X^i)_{i\in I} = (X_1^i,\dots,X_k^i)_{i\in I}$ of $\mathbb{R}^k$–valued stochastic processes, if each component $(X_h^i)_{i\in I}$ is tight for $h = 1,\dots,k$; then the vector stochastic processes $(X^i)_{i\in I}$ are also tight.

Concerning the convergence of finite dimensional distributions, we have the following result ([27] Theorem 6):

(A.2) *Let (X^n) be a sequence of stochastic processes such that $X^n \overset{MZ}{\Longrightarrow} X$: then there exists a subsequence X^{n_k} and a subset $\mathscr{T} \subset [0,T]$ of full Lebesgue measure such that the finite dimensional distributions of $(X_t^{n_k})_{t\in\mathscr{T}}$ converge to those of $(X_t)_{t\in\mathscr{T}}$.*

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